CRUX MATHEMATICORUM CHALLENGES - VIII

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4689. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ x + y + z = 3 \\ x^{x^2} \cdot y^{y^2} \cdot z^{z^2} = \frac{1}{(x^2)^{xz} \cdot (y^2)^{yz} \cdot (z^2)^{zy}} \end{cases}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.
Let be $f : (0, \infty) \to \mathbb{R}; f(x) = \ln x$.

$$f'(x) = \frac{1}{x}; f''(x) = -\frac{1}{x^2} < 0; f \text{ - concave}$$

By Jensen's inequality:

$$f\Big(\lambda_1 \cdot \frac{1}{x} + \lambda_2 \cdot \frac{1}{y} + \lambda_3 \cdot \frac{1}{z}\Big) \geq \lambda_1 f\Big(\frac{1}{x}\Big) + \lambda_2 f\Big(\frac{1}{y}\Big) + \lambda_3 f\Big(\frac{1}{z}\Big);$$

$$\lambda_1, \lambda_2, \lambda_3 > 0; \lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\text{Let be: } \lambda_1 = \frac{x^2 + 2xz}{(x+y+z)^2}; \lambda_2 = \frac{y^2 + 2yx}{(x+y+z)^2}; \lambda_3 = \frac{z^2 + 2zy}{(x+y+z)^2}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{x^2 + 2xz + y^2 + 2yx + z^2 + 2zy}{(x+y+z)^2} = \frac{(x+y+z)^2}{(x+y+z)^2} = 1$$

$$\ln\Big(\frac{x^2 + 2xz}{(x+y+z)^2} \cdot \frac{1}{x} + \frac{y^2 + 2yx}{(x+y+z)^2} \cdot \frac{1}{y} + \frac{z^2 + 2zy}{(x+y+z)^2} \cdot \frac{1}{z}\Big) \geq$$

$$\geq \frac{x^2 + 2xz}{(x+y+z)^2} \ln \frac{1}{x} + \frac{y^2 + 2yx}{(x+y+z)^2} \ln \frac{1}{y} + \frac{z^2 + 2zy}{(x+y+z)^2} \ln \frac{1}{z}$$

$$\ln\Big(\frac{x + 2z}{9} + \frac{y + 2x}{9} + \frac{z + 2y}{9}\Big) \geq$$

$$\geq -\frac{x^2 + 2xz}{9} \cdot \ln x - \frac{y^2 + 2yx}{9} \cdot \ln y - \frac{z^2 + 2zy}{9} \cdot \ln z$$

$$\ln\Big(\frac{3(x+y+z)}{9}\Big) \geq -\frac{1}{9} \ln\Big(x^{x^2 + 2xz} \cdot y^{y^2 + 2yx} \cdot z^{z^2 + 2zy}\Big)$$

$$-9 \ln\Big(\frac{3 \cdot 3}{9}\Big) \leq \ln\Big(x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot x^{2xz} \cdot y^{2yx} \cdot z^{2zy}\Big)$$

$$-9 \ln 1 = 0 \leq \ln\Big(x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot x^{2xz} \cdot y^{2yx} \cdot z^{2zy}\Big)$$

$$\ln(x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot (x^2)^{xz} \cdot (y^2)^{yx} \cdot (z^2)^{zy}\Big) \geq \ln 1$$

$$x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot (x^2)^{xz} \cdot (y^2)^{yz} \cdot (z^2)^{zy} \geq 1$$

$$x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \geq \frac{1}{(x^2)^{xz} \cdot (y^2)^{yz} \cdot (z^2)^{zy}}$$

Equality holds for $x = y = z = 1$. □

Solution 2 by Editorial Board.
We show that the only solution is the obvious on: $(x, y, z) = (1, 1, 1)$.
Let $f(t) = \log t + \frac{1}{t} - 1$ for $t > 0$. Then $f'(t) = \frac{t-1}{t^2}$ is negative for $0 < t < 1$ and positive for $t > 1$, so f has a unique global maximum point (0,1). Suppose $x, y, z > 0$ with $x + y + z = 3$ and $(x, y, z) \neq (1, 1, 1)$. Then

$$\begin{aligned} 0 &= (x+y+z)^2 - 3(x+y+z) \\ &= (x^2+2xz)\Big(1-\frac{1}{x}\Big) + (y^2+2yx)\Big(1-\frac{1}{y}\Big) + (z^2+2zy)\Big(1-\frac{1}{z}\Big) \\ &< (x^2+2xz)\log x + (y^2+2yx)\log y + (z^2+2zy)\log z \\ &= \log(x^{x^2+2xz} \cdot y^{y^2+2yx} \cdot z^{z^2+2zy}) \end{aligned}$$

Hence,

$$x^{x^2+2xz} \cdot y^{y^2+2yx} \cdot z^{z^2+2zy} > 1, \quad \text{or} \quad x^{x^2} \cdot y^{y^2} \cdot z^{z^2} > \frac{1}{(x^2)^{xz} \cdot (y^2)^{yx} \cdot (z^2)^{zy}}$$

So (x, y, z) can't be a solution, completing the proof.

□

4704. If $a, b, c, d \in [0, 1)$ then:

$$\frac{1}{a^6-1} + \frac{1}{b^6-1} + \frac{1}{c^6-1} + \frac{1}{d^2-1} \leq \frac{2}{(abc)^2-1} + \frac{2}{abcd-1}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.
If $x, y, z, t \geq 0$ then:

$$\begin{aligned} x^6+y^6+z^6+t^2 &\overset{\text{AM-GM}}{\geq} 3\sqrt[3]{x^6y^6z^6} + t^2 = \\ &= 3x^2y^2z^2 + t^2 = 2x^2y^2z^2 + x^2y^2z^2 + t^2 \geq \\ &\overset{\text{AM-GM}}{\geq} 2x^2y^2z^2 + 2\sqrt{(x^2y^2z^2)\cdot t^2} = \\ &= 2x^2y^2z^2 + 2xyzt \end{aligned}$$

$$x^6+y^6+z^6+t^2 \geq 2x^2y^2z^2 + 2xyzt \tag{1}$$

In (1) we take $x = a^n; y = b^n; z = c^n; t = d^n$

$$a^{6n} + b^{6n} + c^{6n} + d^{2n} \geq 2(abc)^{2n} + 2(abcd)^n$$

$$\sum_{n=0}^{\infty} a^{6n} + \sum_{n=0}^{\infty} b^{6n} + \sum_{n=0}^{\infty} c^{6n} + \sum_{n=0}^{\infty} d^{2n} \geq$$

$$\geq 2\sum_{n=0}^{\infty}(abc)^{2n} + 2\sum_{n=0}^{\infty}(abcd)^n$$

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^2} \geq \frac{2}{1-(abc)^2} + \frac{2}{1-abcd} \tag{2}$$

We use the fact: $a, b, c, d \in [0, 1) \Rightarrow a^\infty = b^\infty = c^\infty = d^\infty = 0$.
By multiplying (2) with "-1":

$$\frac{1}{a^6-1} + \frac{1}{b^6-1} + \frac{1}{c^6-1} + \frac{1}{d^2-1} \leq \frac{2}{(abc)^2-1} + \frac{2}{abcd-1}$$

Equality holds for:

$$b = a; c = a; d = a^3$$

□

Solution 2 by Mohamed Amine Ben Ajiba - Tanger - Morocco.
By the AM-GM inequality, we have for all $x, y \in [0, 1)$ that

$$\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{\sqrt{(1-x^2)(1+y^2)}} = \frac{2}{\sqrt{(1-xy)^2-(x-y)^2}} \geq \frac{2}{1-xy}$$

with equality when $x = y$. Using this inequality, we have

$$(1) \qquad \frac{1}{1-a^6} + \frac{1}{1-b^6} \geq \frac{2}{1-(ab)^3}$$

$$(2) \qquad \frac{1}{1-c^6} + \frac{1}{1-(abc)^2} \geq \frac{2}{1-abc^4}$$

$$(3) \qquad \frac{1}{1-d^2} + \frac{1}{1-(abc)^2} \geq \frac{2}{1-abcd}$$

$$(4) \qquad 2\left(\frac{1}{1-(ab)^3} + \frac{1}{1-abc^4}\right) \geq \frac{4}{1-(abc)^2}$$

Adding $(1)-(4)$, we get:

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^6} + \frac{2}{1-(abc)^2} + \frac{2}{1-(ab)^3} + \frac{2}{1-abc^4}$$
$$\geq \frac{2}{1-(ab)^3} + \frac{2}{1-abc^4} + \frac{2}{1-abcd} + \frac{4}{1-(abc)^2},$$

which simplifies to

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^6} \geq \frac{2}{1-(abc)^2} + \frac{2}{1-abcd},$$

completing the proof. Equality holds if and only if $a = b = c = \lambda, d = \lambda^3$ for some $\lambda \in [0, 1)$ □

Solution 3 by Marian Dincă.
Using the AM-GM inequality repeatedly, we obtain
(5)

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} \geq \sqrt[3]{\frac{1}{1-a^6} \cdot \frac{1}{1-b^6} \cdot \frac{1}{1-c^6}} = \frac{3}{\sqrt[3]{(1-a^6)(1-b^6)(1-c^6)}}$$

and

$$\sqrt[3]{(1-a^6)(1-b^6)(1-c^6)} \leq \frac{(1-a^6)+(1-b^6)+(1-c^6)}{3} = 1 - \frac{a^6+b^6+c^6}{3}$$

$$(6) \qquad \leq 1 - \sqrt[3]{a^6b^6c^6} = 1-(abc)^2$$

From (5) and (6), we have

$$(7) \qquad \frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} \geq \frac{3}{1-(abc)^2} = \frac{2}{1-(abc)^2} + \frac{1}{1-(abc)^2}$$

Next,

$$\frac{1}{1-(abc)^2}+\frac{1}{1-d^2}\geq 2\sqrt{\frac{1}{1-(abc)^2}\cdot\frac{1}{1-d^2}}$$
$$=\frac{2}{\sqrt{1-(abc)^2(1-d^2)}}\geq\frac{2}{\frac{1-(abc)^2+1-d^2}{2}}$$
$$=\frac{2}{1-\frac{(abc)^2+d^2}{2}}\geq\frac{2}{1-\sqrt{(abc)^2d^2}}=\frac{2}{1-abcd} \tag{8}$$

Finally, from (7) and (8), the conclusion follows. □

4724. Find $x,y>0$ such that:

$$\frac{1}{(x+1)^8}+\frac{1}{(y+1)^8}=\frac{1}{8(xy+1)^4}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

Solution 1 by proposers.

$$(xy+1)\left(\frac{x}{y}+1\right)=((\sqrt{xy})^2+1^2)\left(\left(\sqrt{\frac{x}{y}}\right)^2+1^2\right)\overset{\text{CBS}}{\geq}$$
$$\geq\left(\sqrt{xy}\cdot\sqrt{\frac{x}{y}}+1\cdot 1\right)^2=(x+1)^2$$
$$\frac{1}{(x+1)^2}\geq\frac{1}{(xy+1)(\frac{x}{y}+1)}=\frac{y}{(x+y)(xy+1)} \tag{1}$$

Analogous:

$$\frac{1}{(y+1)^2}\geq\frac{x}{(x+y)(xy+1)} \tag{2}$$

By adding $(1);(2)$:

$$\frac{1}{(x+1)^2}+\frac{1}{(y+1)^2}\geq\frac{x+y}{(x+y)(1+xy)}=\frac{1}{xy+1} \tag{3}$$

$$\frac{1}{(x+1)^8}+\frac{1}{(y+1)^8}=\frac{\left(\frac{1}{(x+1)^2}\right)^4}{1^3}+\frac{\left(\frac{1}{(y+1)^2}\right)^4}{1^3}\geq$$
$$\overset{\text{RADON}}{\geq}\frac{\left(\frac{1}{(x+1)^2}+\frac{1}{(y+1)^2}\right)^4}{(1+1)^3}\overset{(3)}{\geq}\frac{\left(\frac{1}{xy+1}\right)^4}{8}=\frac{1}{8(xy+1)^4} \tag{4}$$

Equality in (4) holds for $x=y=1$ □

Solution 2 by Didier Pinchon - France and Titu Zvonaru - Romania.
For two real numbers a and b, the following inequality is well-known and easily verified

$$a^2+b^2\geq\frac{1}{2}(a+b)^2$$

with equality if and only if $a = b$. Therefore, choosing first $a = \frac{1}{(x+1)^4}, b = \frac{1}{(x+1)^4}$ and then $a = \frac{1}{(x+1)^2}, b = \frac{1}{(x+1)^2}$,

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} \geq \frac{1}{2}\left(\frac{1}{(x+1)^4} + \frac{1}{(y+1)^4}\right)^2 \geq \frac{1}{8}\left(\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}\right)^4$$

with equality in each case if and only if $x = y$.
Now

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \geq \frac{1}{xy+1}$$

because

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} - \frac{1}{xy+1} = \frac{xy(x-y)^2 + (xy-1)^2}{(x+1)^2(y+1)^2(xy+1)} \geq 0,$$

and, for $x, y > 0$, the equality holds if and only if $x = y$ and $xy = 1$, i.e. $x = 1$ and $y = 1$.
As a result,

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} \geq \frac{1}{8(xy+1)^4},$$

with equality if and only if $x = 1$ and $y = 1$, which is the only solution for the proposed equation.

□

4726. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k}}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer. Let be:

$$A(n) = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos x)^n \sin x dx; n \in \mathbb{N}$$

$$1 - \cos x = t; \sin x dx = dt$$

$$x = \frac{\pi}{3} \Rightarrow t = \frac{1}{2}; x = \frac{\pi}{2} \Rightarrow t = 1$$

(1) $$A(n) = \int_{\frac{1}{2}}^{1} t^n dt = \frac{t^{n+1}}{n+1}\Big|_{\frac{1}{2}}^{1} = \frac{1}{n+1}\left(1 - \frac{1}{2^{n+1}}\right)$$

$$A(n) = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos x)^n \sin x dx =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cos^k x\right) \cdot \sin x dx =$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cdot \cos^k x dx =$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot \frac{(-\cos^{k+1} x)}{k+1} \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= -\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \binom{n}{k} \cdot \Big(0 - \frac{1}{2^{k+1}}\Big) =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \frac{1}{2^{k+1}} \cdot \binom{n}{k}$$

$$A(n) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \frac{1}{2^{n+1}} \cdot \binom{n}{k} \tag{2}$$

By (1); (2):

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \frac{1}{2^{k+1}} \cdot \binom{n}{k} = \frac{1}{n+1}\Big(1 - \frac{1}{2^{n+1}}\Big)$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k} = \frac{2^{n+1}-1}{n+1}$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{2^{n+1}-1}{n+1}} \overset{\text{CAUCHY-D'ALEMBERT}}{=}$$

$$= \lim_{n\to\infty} \frac{2^{n+2}-1}{n+2} \cdot \frac{n+1}{2^{n+1}-1} =$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} \cdot \lim_{n\to\infty} \frac{2^{n+2}-1}{2^{n+1}-1} = 1 \cdot 2 = 2$$

□

Solution 2 by Prithwijit De - India and Kee - Wai Lau - Hong Kong.
By the binomia theorem, we have

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

Integrating from 0 to $\frac{1}{2}$, we obtain

$$\int_0^{\frac{1}{2}} (1-x)^n dx = -\frac{(1-x)^{n+1}}{n+1}\Big|_0^{\frac{1}{2}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^{k+1}}{k+1}\Big|_0^{\frac{1}{2}}$$

or

$$\frac{-1}{n+1}\left(\Big(\frac{1}{2}\Big)^{n+1} - 1\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1}\Big(\frac{1}{2}\Big)^{n+1}$$

so

$$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1} 2^{n-k}$$

Hence

$$\lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k}} = \lim_{n\to\infty} \Big(\frac{2^{n+1}}{n}\Big)^{\frac{1}{2}} \Big(\frac{1 - \frac{1}{2}^{n+1}}{1 + \frac{1}{n}}\Big)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{2^{1+\frac{1}{n}}}{n^{\frac{1}{n}}}\right)\left(\frac{1-\frac{1}{2}^{n+1}}{1+\frac{1}{n}}\right)^{\frac{1}{n}}$$

$$= 2\lim_{n\to\infty}\left(\frac{2}{n}\right)^{\frac{1}{n}} = 2$$

□

4740. If $0 < a \le b < 1$ then:

$$\exp\left(\int_a^b\int_a^b \frac{x+y+2}{1-xy}dxdy\right) \le \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.
First we prove that if $x, y \in (0,1)$ then:

(1) $$\frac{x+y+2}{1-xy} \le \frac{1}{1-x} + \frac{1}{1-y}$$

$$\frac{x+y+2}{1-xy} \le \frac{1-y+1-x}{(1-x)(1-y)} \Leftrightarrow \frac{x+y+2}{1-xy} \le \frac{2-x-y}{1-x-y+xy}$$

$$(x+y+2)(1-x-y+xy) \le (2-x-y)(1-xy)$$

$$x - x^2 - xy + x^2y + y - yx - y^2 + xy^2 + 2 - 2x - 2y + 2xy \le$$

$$\le 2 - 2xy - x + x^2y - y + xy^2$$

$$-x^2 - y^2 \le -2xy \Leftrightarrow x^2 + y^2 - 2xy \ge 0 \Leftrightarrow (x-y)^2 \ge 0$$

Integrating in (1):

$$\int_a^b\int_a^b \frac{x+y+2}{1-xy}dxdy \le \int_a^b\int_a^b \frac{1}{1-x}dxdy + \int_a^b\int_a^b \frac{1}{1-y}dxdy =$$

$$= \int_a^b \frac{1}{1-x}dx \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{1}{1-y}dy =$$

$$= -2(b-a)(\ln(1-b) - \ln(1-a)) = \ln\left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

$$\int_a^b\int_a^b \frac{x+y+2}{1-xy}dxdy \le \ln\left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

$$\exp\left(\int_a^b\int_a^b \frac{x+y+2}{1-xy}dxdy\right) \le \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

Equality holds for $a = b$. □

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morroco and UClan Cyprus Problem Solving Group.

First we prove that for $x, y \in (0,1)$,

$$\frac{x+y+2}{1-xy} \le \frac{2-x-y}{(1-x)(1-y)} = \frac{1}{1-x} + \frac{1}{1-y}$$

That holds since, after expanding we have

$$(2-x-y)(1-xy) - (x+y+2)(1-x)(1-y) = (x-y)^2 \ge 0$$

Integrating, we have:

$$\int_a^b \int_a^b \frac{x+y+2}{1-xy} dxdy \le \int_a^b \int_a^b \frac{1}{1-x} dxdy + \int_a^b \int_a^b \frac{1}{1-y} dxdy$$

$$= \int_a^b \frac{1}{1-x} dx \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{1}{1-y} dy$$

$$= -2(b-a)(\ln(1-b) - \ln(1-a)) = \ln\Big(\frac{1-a}{1-b}\Big)^{2(b-a)}$$

Hence

$$\exp\left(\int_a^b \int_a^b \frac{x+y+2}{1-xy} dxdy\right) \le \Big(\frac{1-a}{1-b}\Big)^{2(b-a)}$$

Equality holds for $a = b$.

□

4756. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} [\log_k n];$$

$[*]$ - the greatest integer function.

Proposed by Daniel Sitaru - Romania

Solution by proposer.

$$\sum_{k=2}^{n^n} [\log_k n] =$$

$$= \sum_{2\le k<n} [\log_n k] + \sum_{n\le k<n^2} [\log_n k] + \sum_{n^2\le k<n^3} [\log_n k] + \dots$$

$$\dots + \sum_{n^{n-1}\le k<n^n} [\log_n k] + [\log_n n^n] =$$

$$= (n-1)\cdot 0 + (n^2-n)\cdot 1 + (n^3-n^2)\cdot 2 + (n^4-n^3)\cdot 3 + \dots + (n^n - n^{n-1})(n-1) + n =$$

$$= -n - n^2 - n^3 - \dots - n^{n-1} + (n-1)n^n + n =$$

$$= -\frac{n(n^n-1)}{n-1} + (n-1)n^n + n$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} [\log_n k] =$$

$$= \lim_{n\to\infty} \Big(\frac{n^{n+1}-n^n}{n^{n+1}} + \frac{1}{n^n} - \frac{n^n-1}{n^n(n-1)}\Big) =$$

$$= 1 + \frac{1}{\infty} - \frac{1}{\infty} = 1 + 0 - 0 = 1$$

□

Solution 2 by Henry Ricardo - New York - USA.

$$\lim_{n\to\infty}\frac{1}{n^{n+1}}\sum_{k=2}^{n^n}\log_n k$$

We have

$$\frac{1}{n^{n+1}}\sum_{k=2}^{n^n}\log_n k=\frac{1}{n^{n+1}\ln n}\sum_{k=2}^{n^n}\ln k=\frac{1}{n^{n+1}\ln n}\ln(n^n!)$$

Now we apply Stirling's approximation in the form

$$\ln n!=\frac{1}{2}\ln 2\pi+\frac{2n+1}{2}\ln n-n+O\Big(\frac{1}{n}\Big)$$

to obtain

$$\frac{\ln(n^n!)}{n^{n+1}\ln n}=\frac{\frac{1}{2}\ln 2\pi+(\frac{2n^n+1}{2})\cdot n\ln n-n^n+O(\frac{1}{n^n})}{n^{n+1}\ln n}$$

$$=1+\frac{\ln 2\pi}{2n^{n+1}\ln n}+\frac{1}{2n^n}-\frac{1}{n\ln n}+O\Big(\frac{1}{n^{2n+1}\ln n}\Big)\to 1 \text{ as } n\to\infty$$

□

4809. If $a,b>0$ then find:

$$\Omega(a,b)=\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\left(\int_0^1\frac{x^k}{ax+b}dx\right)^{-1}\left(\int_0^1\frac{x^k}{bx+a}dx\right)^{-1}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.

(1) $$a,b>0,x\in[0,1]\Rightarrow ax+b>0$$

$$0\le x\le 1\Rightarrow x^{k+1}\le x^k\overset{(1)}{\Rightarrow}\frac{x^{k+1}}{ax+b}\le\frac{x^k}{ax+b}$$

(2) $$\int_0^1\frac{x^{k+1}}{ax+b}dx\le\int_0^1\frac{x^k}{ax+b}$$

Denote: $I_k=\int_0^1\frac{x^k}{ax+b}dx$ and $J_k=\int_0^1\frac{x^k}{bx+a}dx$, by (2), we get:

(3) $$I_{k+1}\le I_k$$

$$aI_{k+1}+bI_k=a\int_0^1\frac{x^{k+1}}{ax+b}dx+b\int_0^1\frac{x^k}{ax+b}dx=$$

$$=\int_0^1\frac{x^k(ax+b)}{ax+b}dx=\int_0^1 x^k dx=\frac{1}{k+1}$$

(4) $$aI_{k+1}+bI_k=\frac{1}{k+1}$$

$$a_{k+1}=\frac{1}{k+1}-bI_k\Rightarrow I_{k+1}=\frac{1}{a(k+1)}-\frac{b}{a}I_k$$

Replace I_{k+1} in (3): $\frac{1}{a(k+1)}-\frac{b}{a}I_k\le I_k$

$$\frac{1}{a(k+1)} \leq \left(1+\frac{b}{a}\right) I_k \Leftrightarrow \frac{1}{a(k+1)} \leq \frac{a+b}{a} I_k$$

(5) $$I_k \geq \frac{1}{(a+b)(k+1)}$$

$$\text{By (4): } bI_k = \frac{1}{k+1} - aI_{k+1} \Rightarrow I_k = \frac{1}{b(k+1)} - \frac{a}{b} I_{k+1}$$

$$\text{Replace } I_k \text{ in (3): } I_{k+1} \leq \frac{1}{b(k+1)} - \frac{a}{b} I_{k+1}$$

$$\left(1+\frac{a}{b}\right) I_{k+1} \leq \frac{1}{b(k+1)}$$

$$\frac{a+b}{b} I_{k+1} \leq \frac{1}{b(k+1)}$$

$$I_{k+1} \leq \frac{1}{(a+b)(k+1)}$$

(6) $$I_k \leq \frac{1}{(a+b)k}$$

(7) $$\text{By (5) and (6): } \frac{1}{(a+b)(k+1)} \leq I_k \leq \frac{1}{(a+b)k}$$

$$\text{Analogous: } J_{k+1} \leq J_k; bJ_{k+1} + aJ_k = \frac{1}{k+1}$$

(8) $$\frac{1}{(a+b)(k+1)} \leq J_k \leq \frac{1}{(a+b)k}$$

By multiplying (7), (8):

$$\frac{1}{(a+b)^2(k+1)^2} \leq I_k J_k \leq \frac{1}{(a+b)^2 k^2}$$

$$(a+b)^2 k^2 \leq (I_k J_k)^{-1} \leq (a+b)^2 (k+1)^2$$

$$(a+b)^2 \sum_{k=1}^{n} k^2 \leq \sum_{k=1}^{n} I_k^{-1} J_k^{-1} \leq (a+b)^2 \sum_{k=1}^{n} (k+1)^2$$

$$(a+b)^2 \cdot \frac{n(n+1)(2n+1)}{6} \leq \sum_{k=1}^{n} I_k^{-1} J_k^{-1} \leq \left(\frac{(n+1)(n+2)(2n+3)}{6} - 1\right)(a+b)^2$$

$$(a+b)^2 \cdot \frac{n(n+1)(2n+1)}{6n^3} \leq \frac{1}{n^3} \sum_{k=1}^{n} I_k^{-1} J_k^{-1} \leq \left(\frac{(n+1)(n+2)(2n+3)}{6n^3} - \frac{1}{n^3}\right)(a+b)^2$$

$$\frac{2(a+b)^2}{6} \leq \Omega(a,b) \leq \frac{2(a+b)^2}{6}$$

$$\Omega(a,b) = \frac{(a+b)^2}{3}$$

□

Solution 2 by Yunyong Zhang - China.
For $x \in [0,1]$,

$$(a+b)x \leq (ax+b) \leq (a+b), \qquad (a+b)x \leq (bx+a) \leq (a+b)$$

implying that

$$\frac{1}{(a+b)(k+1)} = \int_0^1 \frac{x^k}{a+b}dx \leq \int_0^1 \frac{x^k}{ax+b}dx \leq \int_0^1 \frac{x^{k-1}}{a+b}dx = \frac{1}{(a+b)k}$$

and

$$\frac{1}{(a+b)(k+1)} = \int_0^1 \frac{x^k}{a+b}dx \leq \int_0^1 \frac{x^k}{bx+a} \leq \int_0^1 \frac{x^{k-1}}{a+b}dx = \frac{1}{(a+b)k}$$

Multiplying gives

$$(a+b)^2k^2 \leq \frac{1}{(\int_0^1 \frac{x^k}{ax+b}dx)(\int_0^1 \frac{x^k}{bx+a}dx)} \leq (a+b)^2(k+1)^2$$

and summing then gives

$$\frac{(a+b)^2}{6}(2n^3+3n^2+n) < \sum_{k=1}^{n} \frac{1}{(\int_0^1 \frac{x^k}{ax+b}dx)(\int_0^1 \frac{x^k}{bx+a}dx)} < \frac{(a+b)^2}{6}(2n^3+9n^2+13n).$$

We thus have

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \left(\int_0^1 \frac{x^k}{ax+b}dx\right)^{-1} \left(\int_0^1 \frac{x^k}{bx+a}dx\right)^{-1} = \frac{(a+b)^2}{3}.$$

□

4819. Let be $f : [0,1] \to [0,1], f$ - continuous function and $0 < a \leq b < 1$.
Prove that:

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \geq \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt \left(\int_0^{\frac{a+b}{2}} f(t)dt + \int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.

$$\text{Let be } F : [0,1] \to \mathbb{R}, F(x) = \left(\int_0^x f(t)dt\right)^2 - \int_0^x tf(t)dt$$

$$F'(x) = 2\left(\int_0^x f(t)dt\right)'\left(\int_0^x f(t)dt\right) - 2\left(\int_0^x tf(t)dt\right)' =$$

$$= 2f(x)\int_0^x f(t)dt - 2xf(x) = 2f(x)\left(\int_0^x f(t)dt - x\right) =$$

$$= 2f(x)\left(\int_0^x f(t)dt0 - \int_0^x dt\right) = 2f(x)\int_0^x (f(t)-1)dt \leq 0,$$

because $f(x) \geq 0, f(x) \leq 1, \forall x \in [0,1], F$ - decreasing

$$\frac{2ab}{a+b} \overset{\text{AM-HM}}{\leq} \frac{a+b}{2} \Rightarrow F\Big(\frac{2ab}{a+b}\Big) \geq F\Big(\frac{a+b}{2}\Big)$$

$$\left(\int_0^{\frac{2ab}{a+b}} f(t)dt\right)^2 - 2\int_0^{\frac{2ab}{a+b}} tf(t)dt \geq \left(\int_0^{\frac{a+b}{2}} f(t)dt\right)^2 - 2\int_0^{\frac{a+b}{2}} tf(t)dt$$

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \geq \left(\int_0^{\frac{a+b}{2}} f(t)dt - \int_0^{\frac{2ab}{a+b}} f(t)dt\right)\left(\int_0^{\frac{a+b}{2}} f(t)dt + \int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \geq \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt + \int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

Equality holds for $a = b$ or $f \equiv 0$. □

Solution 2 by Editorial Board of Crux.

Let $x = 2ab/(a+b)$ and $y = (a+b)/2$. Clearly $0 < x \leq y < 1$. The desired inequality is equivalent to

$$2\left(\int_0^y tf(t)dt - \int_0^x tf(t)dt\right) \geq \left(\int_0^y f(t)dt - \int_0^x f(t)dt\right)\left(\int_0^y f(t)dt + \int_0^x f(t)dt\right)$$

Thus, it suffices to show that

$$\left(\int_0^x f(t)dt\right)^2 - 2\int_0^x tf(t)dt \geq \left(\int_0^y f(t)dt\right)^2 - 2\int_0^y tf(t)dt.$$

Let

$$g(z) = \left(\int_0^z f(t)dt\right)^2 - 2\int_0^z tf(t)dt,$$

with $z \in (0,1)$. Since f is continuous, the Fundamental Theorem of Calculus gives

$$g'(z) = 2f(z)\left(\int_0^z f(t)dt - z\right) \leq 0$$

since $f(t) \in [0,1]$. Thus, $g(x) \geq g(y)$, as required.

□

B131. Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n}\Big(\frac{1}{i} - \frac{1}{i+1}\Big) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\sum_{i=1}^{n}\frac{1}{(i+1)(i+2)}=\sum_{i=1}^{n}\Big(\frac{1}{i+1}-\frac{1}{i+2}\Big)=\frac{1}{2}-\frac{1}{3}=\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n+1}-\frac{1}{n+2}$$
$$=\frac{1}{2}-\frac{1}{n+2}$$
$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}=\frac{1}{2}$$
$$\sum_{i=1}^{n}\frac{1}{i(i+2)}=\frac{1}{2}\sum_{i=1}^{n}\frac{2}{i(i+2)}=\frac{1}{2}\sum_{i=1}^{n}\frac{(i+2)-i}{i(i+2)}=\frac{1}{2}\sum_{i=1}^{n}\Big(\frac{1}{i}-\frac{1}{i+2}\Big)=$$
$$=\frac{1}{2}\Big(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\ldots+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n+2}\Big)=$$
$$=\frac{1}{2}\Big(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\Big)$$
$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\frac{1}{2}\Big(1+\frac{1}{2}\Big)=\frac{3}{4}$$
$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=\lim_{n\to\infty}\Big(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n+1)^2}-\frac{1}{1^2}\Big)=\frac{\pi^2}{6}-1$$
$$\Omega=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}=$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m-n}{mn(m+1)(n+1)}\cdot\frac{m-n}{(m+1)(m+2)(n+1)(n+2)}=$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m(n+1)-n(m+1)}{mn(m+1)(n+1)}\cdot\frac{(m+1)(n+2)-(m+2)(n+1)}{(m+1)(m+2)(n+1)(n+2)}=$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(\frac{1}{n(m+1)}-\frac{1}{m(n+1)}\Big)\Big(\frac{1}{(n+1)(m+2)}-\frac{1}{(m+1)(n+2)}\Big)\overset{\text{Binet-Cauchy}}{=}$$
$$=2\left(\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\right)\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\right)-2\left(\sum_{n=1}^{\infty}\frac{1}{n(n+2)}\right)\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right)=$$
$$=2\cdot 1\cdot\frac{1}{2}-2\cdot\frac{3}{4}\Big(\frac{\pi^2}{6}-1\Big)=1-\frac{3}{2}\cdot\frac{\pi^2}{6}+\frac{3}{2}=\frac{5}{2}-\frac{\pi^2}{4}=\frac{10-\pi^2}{4}$$

Observation.

Binet-Cauchy's identity:

If $a_i, b_i, x_i, y_i > 0, i=\overline{1,n}$ then:
$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)(x_iy_j-x_jy_i)=$$
$$=2\left(\sum_{i=1}^{n}a_ix_i\right)\left(\sum_{i=1}^{n}b_iy_i\right)-2\left(\sum_{i=1}^{n}a_iy_i\right)\left(\sum_{i=1}^{n}b_ix_i\right)$$

For $a_i=\frac{1}{i}, b_i=\frac{1}{i+1}, x_i=\frac{1}{i+1}, y_i=\frac{1}{i+2}$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{1}{i(j+1)}-\frac{1}{j(i+1)}\Big)\Big(\frac{1}{(i+1)(j+2)}-\frac{1}{(j+1)(i+2)}\Big)=$$

$$= 2\left(\sum_{i=1}^{n}\frac{1}{i(i+1)}\right)\left(\sum_{i=1}^{n}\frac{1}{(i+1)(i+2)}\right)-2\left(\sum_{i=1}^{n}\frac{1}{i(i+2)}\right)\left(\sum_{i=1}^{n}\frac{1}{(i+2)^2}\right)$$

□

Solution 2 by Rana Ranino - Setif - Algerie.

$$\Omega = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

$$\Omega = \sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{m}{(m+1)^2(m+2)}+$$

$$+\sum_{n=1}^{\infty}\frac{n}{(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{1}{m(m+1)^2(m+2)}-$$

$$-2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{1}{(m+1)^2(m+2)}$$

$$\text{By Symmetry: } \Omega = 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{m}{(m+1)^2(m+2)}-$$

$$-2\Big(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)}\Big)^2$$

$$\Omega_1 = \sum_{m=1}^{\infty}\frac{m}{(m+1)^2(m+2)} = 2\sum_{m=1}^{\infty}\Big(\frac{1}{m+1}-\frac{1}{m+2}\Big)-$$

$$-\sum_{m=1}^{\infty}\frac{1}{(m+1)^2} = 2-\frac{\pi^2}{6}$$

$$\Omega_2 = \sum_{n=1}^{\infty}\frac{2}{n(n+1)^2(n+2)} = \sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+2}\Big)-$$

$$-2\sum_{m=1}^{\infty}\frac{1}{(n+1)^2} = \frac{3}{2}+2-\frac{\pi^2}{3} = \frac{7}{2}-\frac{\pi^2}{3}$$

$$\Omega_3 = \sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)} = \sum_{n=1}^{\infty}\Big(\frac{1}{n+2}-\frac{1}{n+1}\Big)+$$

$$+\sum_{m=1}^{\infty}\frac{1}{(n+1)^2} = -\frac{1}{2}+\frac{\pi^2}{6}-1 = \frac{\pi^2}{6}-\frac{3}{2}$$

$$\Omega = \Big(2-\frac{\pi^2}{6}\Big)\Big(\frac{7}{2}-\frac{\pi^2}{3}\Big)-2\Big(\frac{\pi^2}{6}-\frac{3}{2}\Big)^2 = \frac{5}{2}-\frac{\pi^2}{4}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)} = \frac{5}{2}-\frac{\pi^2}{4}$$

□

Solution 3 by Ankush Kumar Parcha - India.

$$\text{We have } \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{(n-m)^2}{nm(n+1)^2(m+1)^2(n+2)(m+2)}$$

$$\overset{\therefore\text{ symmetry}}{\Longrightarrow} 2\sum_{n,m\in\mathbb{N}}\frac{n}{m(n+1)^2(m+1)^2(n+2)(m+2)} - 2\left(\sum_{n\in\mathbb{N}}\frac{1}{(n+1)^2(n+2)}\right)^2$$

$$\underset{(1)}{\overset{\text{Note section}}{\Longrightarrow}} 2\int_0^\infty \frac{1}{e^x-1}\mathcal{L}_n^{-1}\Big\{\frac{n}{(n+1)^2(n+2)}\Big\}(x)dx\cdot$$

$$\cdot\int_0^\infty \frac{1}{e^y-1}\mathcal{L}_m^{-1}\Big\{\frac{1}{m(m+1)^2(m+2)}\Big\}(y)dy-$$

$$-2\left(\int_0^\infty \frac{1}{e^z-1}\mathcal{L}_n^{-1}\Big\{\frac{1}{(n+1)^2(n+2)}\Big\}(z)dx\right)^2$$

$$\underset{(2)}{\overset{\text{Note section}}{\Longrightarrow}} 2\int_0^\infty \frac{xe^{-2x}+2e^{-3x}-2e^{-2x}}{1-e^{-x}}dx\int_0^\infty \frac{1}{1-e^{-y}}\Big(ye^{-2y}+\frac{e^{-3y}}{2}-\frac{e^{-y}}{2}\Big)dy-$$

$$-2\left(\int_0^\infty \frac{ze^{-2z}+e^{-3z}-e^{-2z}}{1-e^{-z}}dx\right)^2$$

$$\underset{(3)}{\overset{\text{Note section}}{\Longrightarrow}} 2\left(\zeta(2,2)-2\int_0^\infty e^{-2x}dx\right)\left(\zeta(2,2)+\int_0^\infty d\Big(\frac{e^{-2y}}{4}+\frac{e^{-y}}{2}\Big)\right)$$

$$-2\left(\zeta(2,2)-\int_0^\infty e^{-2z}dx\right)\Rightarrow 2(\zeta(2)-2)\Big(\zeta(2)-\frac{7}{4}\Big)-2\Big(\zeta(2)-\frac{3}{2}\Big)^2$$

$$\overset{\therefore\zeta(2)=\pi^2/6}{\Longrightarrow} \frac{\pi^4}{18}-\frac{5\pi^2}{4}+7-\frac{\pi^4}{18}+\pi^2-\frac{9}{2}$$

$$\Rightarrow \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{(n-m)^2}{nm(n+1)^2(m+1)^2(n+2)(m+2)} = \frac{5}{2}-\frac{\pi^2}{4}$$

Note Section

1. $\displaystyle\sum_{n\in\mathbb{N}}\mathcal{L}_t\{f(t)\}(n) = \int_0^\infty \frac{f(t)}{e^t-1}dt$ (Maz. sum identity)
2. $\displaystyle\mathcal{L}_x\{x^n e^{ax}\}(s) = \frac{n!}{(s-a)^{n+1}}, n\in\mathbb{N}\wedge\Re(s)>\Re(a)$
3. $\displaystyle\int_0^\infty \frac{t^{n-1}e^{-mt}}{1-e^{-t}}dt = \Gamma(n)\cdot\zeta(n,m), \Re(n)>1\wedge\Re(m)>0$

□

Solution 4 by Ravi Prakash - New Delhi - India.

$$\text{Let } a_{m,n} = \frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

$$= \frac{(m+1)^2+(n+1)^2-2(m+1)(n+1)}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

$$= \frac{1}{mn(n+1)^2(m+2)(n+2)} + \frac{1}{mn(m+1)^2(m+2)(n+2)} -$$

$$-\frac{2}{(mn(m+1)(n+1)(m+2)(n+2)}$$

$$S=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}$$

$$=2\sum_{m=1}^{\infty}\frac{1}{m(m+2)}\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}-2\left[\sum_{m=1}^{n}\frac{1}{m(m+1)(m+2)}\right]^2$$

$$=\left[\sum_{m=1}^{\infty}\Big(\frac{1}{m}-\frac{1}{m+2}\Big)\right]\left[\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}\right]-2\left[\sum_{m=1}^{\infty}\frac{1}{m(m+1)(m+2)}\right]^2$$

$$=\frac{3}{2}A_1-2A_2^2$$

where

$$A_1=\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}=\sum_{n=1}^{\infty}\frac{1}{(n+1)^2[(n+1)^2-1]}$$

$$=\sum_{n=1}^{\infty}\Big[\frac{1}{(n+1)^2-1}-\frac{1}{(n+1)^2}\Big]$$

$$=\sum_{n=1}^{\infty}\Big[\frac{1}{2n}-\frac{1}{2(n+2)}-\frac{1}{(n+1)^2}\Big]$$

$$=\frac{1}{2}\Big(1+\frac{1}{2}\Big)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$$

$$=\frac{3}{4}-\Big(\frac{\pi^2}{6}-1\Big)=\frac{7}{4}-\frac{\pi^2}{6}$$

Next,

$$A_2=\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\Big)$$

$$=\frac{1}{2}\left[\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+1}\Big)-\sum_{n=1}^{\infty}\Big(\frac{1}{n+1}-\frac{1}{n+2}\Big)\right]$$

$$=\frac{1}{2}\Big[1-\frac{1}{2}\Big]=\frac{1}{4}$$

Thus,

$$S=\frac{3}{2}\Big(\frac{7}{4}-\frac{\pi^2}{6}\Big)-2\Big(\frac{1}{4}\Big)^2$$

$$=\frac{21}{8}-\frac{1}{8}-\frac{\pi^2}{4}=\frac{5}{2}-\frac{\pi^2}{4}$$

□

B135. If $A,B,C\in M_2(\mathbb{R});\det A>0;\det B>0;\det C>0,\det(ABC)=64$ then:

$$\det(A+B+C)+\det(-A+B+C)+\det(A-B+C)+\det(A+B-C)\geq 48$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer.
Lemma: If $A, B \in M_2(\mathbb{R})$ then:

$$\det(A+B)+\det(A-B)=2(\det A+\det B)$$

Proof.

$$\text{Let be } A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}; B=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\det(A+B)+\det(A-B)=$$
$$=\begin{vmatrix} a+e & b+f \\ c+g & d+h \end{vmatrix}+\begin{vmatrix} a-e & b-f \\ c-g & d-h \end{vmatrix}=$$
$$=(a+e)(d+h)-(b+f)(c+g)+(a-e)(d-h)-(c-g)(b-f)=$$
$$=ad+ah+de+eh-bc-bg-cf-fg+$$
$$+ad-ah-de+eh-cb+cf+bg-gf=$$
$$=2ad-2bc+2eh-2gf=$$
$$=2(ad-bc)+2(eh-gf)=$$
$$=2\det A+2\det B=2(\det A+\det B)$$

□

Back to the problem:

$$\det(A+B+C)+\det(-A+B+C)+\det(A-B+C)+\det(A+B-C)=$$
$$=(\det((A+B)+C)+\det((A+B)-C)+$$
$$+\det(C+(A-B))+\det(C-(A-B)))\overset{\text{Lemma}}{=}$$
$$=2(\det(A+B)+\det C)+2(\det(C)+\det(A-B))=$$
$$=4\det C+2(\det(A+B)+\det(A-B))=$$
$$\overset{\text{Lemma}}{=}4\det C+2\cdot 2(\det A+\det B)=$$
$$=4(\det A+\det B+\det C)\overset{\text{AM-GM}}{\geq}$$
$$\geq 4\cdot 3\sqrt[3]{\det A\cdot\det B\cdot\det C}=$$
$$=12\cdot\sqrt[3]{\det(ABC)}=12\cdot\sqrt[3]{64}=12\cdot 4=48$$

Equality holds for:

$$\det A=\det B=\det C=4$$

□

Solution 2 by Ravi Prakash - New Delhi - India.

$$\text{Let } A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B=\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ and } C=\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$\det(A+B)=\begin{vmatrix} a1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{vmatrix}=$$
$$=\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}+\begin{vmatrix} a_1 & b_2 \\ a_3 & b_4 \end{vmatrix}+\begin{vmatrix} b_1 & a_2 \\ b_3 & a_4 \end{vmatrix}+\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}$$
$$=\det(A_1|A_2)+\det(A_1|B_2)+\det(B_1|A_2)+\det(B_1|B_2)$$
$$\text{where } A_1=\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, A_2=\begin{bmatrix} a_3 \\ a_4 \end{bmatrix}, \text{ etc}$$

$$\text{and } \det(A_1|A_2) = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}, \text{ etc}$$

Similarly,

$$\det(A-B) = \det(A_1|A_2) - \det(A_1|B_2) - \det(B_1|A_2) + \det(B_1|B_2)$$

Thus,

$$\begin{aligned} \det(A+B) + \det(A-B) &= 2\det(A_1|A_2) + 2\det(B_1|B_2) \\ &= 2\det(A) + 2\det(B) \end{aligned}$$

Thus,

$$\begin{aligned} \det(A+B+C) + \det(-A+B+C) + \det(A-B+C) + \det(A+B-C) = \\ = 2\det(B+C) + 2\det(A) + 2\det(B-C) + 2\det(A) \\ = 2[2\det(B) + 2\det(C)] + 4\det(A) = \\ = 4[\det(A) + \det(B) + \det(C)] \\ \geq (4)(3)[(\det(A))(\det(B))(\det(C))]^{\frac{1}{3}} \\ = 12[\det(ABC)]^{\frac{1}{3}} = (12)(4^3)^{\frac{1}{3}} \\ = 48 \end{aligned}$$

□

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