4809

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If $a, b > 0$ then find:

$$\Omega(a,b) = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \left(\int_0^1 \frac{x^k}{ax+b}dx\right)^{-1} \left(\int_0^1 \frac{x^k}{bx+a}dx\right)^{-1}$$

Solution 1 by proposer.

$$a, b > 0, x \in [0,1] \Rightarrow ax + b > 0 \tag{1}$$

$$0 \le x \le 1 \Rightarrow x^{k+1} \le x^k \overset{(1)}{\Rightarrow} \frac{x^{k+1}}{ax+b} \le \frac{x^k}{ax+b}$$

$$\int_0^1 \frac{x^{k+1}}{ax+b}dx \le \int_0^1 \frac{x^k}{ax+b} \tag{2}$$

Denote: $I_k = \int_0^1 \frac{x^k}{ax+b}dx$ and $J_k = \int_0^1 \frac{x^k}{bx+a}dx$, by (2), we get:

$$I_{k+1} \le I_k \tag{3}$$

$$aI_{k+1} + bI_k = a\int_0^1 \frac{x^{k+1}}{ax+b}dx + b\int_0^1 \frac{x^k}{ax+b}dx =$$

$$= \int_0^1 \frac{x^k(ax+b)}{ax+b}dx = \int_0^1 x^k dx = \frac{1}{k+1}$$

$$aI_{k+1} + bI_k = \frac{1}{k+1} \tag{4}$$

$$a_{k+1} = \frac{1}{k+1} - bI_k \Rightarrow I_{k+1} = \frac{1}{a(k+1)} - \frac{b}{a}I_k$$

Replace I_{k+1} in (3): $\frac{1}{a(k+1)} - \frac{b}{a}I_k \le I_k$

$$\frac{1}{a(k+1)} \le \left(1 + \frac{b}{a}\right)I_k \Leftrightarrow \frac{1}{a(k+1)} \le \frac{a+b}{a}I_k$$

$$I_k \ge \frac{1}{(a+b)(k+1)} \tag{5}$$

By (4): $bI_k = \frac{1}{k+1} - aI_{k+1} \Rightarrow I_k = \frac{1}{b(k+1)} - \frac{a}{b}I_{k+1}$

Replace I_k in (3): $I_{k+1} \le \frac{1}{b(k+1)} - \frac{a}{b}I_{k+1}$

$$\left(1 + \frac{a}{b}\right)I_{k+1} \le \frac{1}{b(k+1)}$$

$$\frac{a+b}{b}I_{k+1} \le \frac{1}{b(k+1)}$$

$$I_{k+1} \le \frac{1}{(a+b)(k+1)}$$

(6) $$I_k \le \frac{1}{(a+b)k}$$

(7) By (5) and (6): $$\frac{1}{(a+b)(k+1)} \le I_k \le \frac{1}{(a+b)k}$$

Analogous: $J_{k+1} \le J_k; bJ_{k+1} + aJ_k = \dfrac{1}{k+1}$

(8) $$\frac{1}{(a+b)(k+1)} \le J_k \le \frac{1}{(a+b)k}$$

By multiplying (7), (8):

$$\frac{1}{(a+b)^2(k+1)^2} \le I_k J_k \le \frac{1}{(a+b)^2 k^2}$$

$$(a+b)^2 k^2 \le (I_k J_k)^{-1} \le (a+b)^2(k+1)^2$$

$$(a+b)^2 \sum_{k=1}^{n} k^2 \le \sum_{k=1}^{n} I_k^{-1} J_k^{-1} \le (a+b)^2 \sum_{k=1}^{n} (k+1)^2$$

$$(a+b)^2 \cdot \frac{n(n+1)(2n+1)}{6} \le \sum_{k=1}^{n} I_k^{-1} J_k^{-1} \le \left(\frac{(n+1)(n+2)(2n+3)}{6} - 1\right)(a+b)^2$$

$$(a+b)^2 \cdot \frac{n(n+1)(2n+1)}{6n^3} \le \frac{1}{n^3}\sum_{k=1}^{n} I_k^{-1} J_k^{-1} \le \left(\frac{(n+1)(n+2)(2n+3)}{6n^3} - \frac{1}{n^3}\right)(a+b)^2$$

$$\frac{2(a+b)^2}{6} \le \Omega(a,b) \le \frac{2(a+b)^2}{6}$$

$$\Omega(a,b) = \frac{(a+b)^2}{3}$$

□

Solution 2 by Yunyong Zhang - China.
For $x \in [0,1]$,

$$(a+b)x \le (ax+b) \le (a+b), \qquad (a+b)x \le (bx+a) \le (a+b)$$

implying that

$$\frac{1}{(a+b)(k+1)} = \int_0^1 \frac{x^k}{a+b}dx \le \int_0^1 \frac{x^k}{ax+b}dx \le \int_0^1 \frac{x^{k-1}}{a+b}dx = \frac{1}{(a+b)k}$$

and

$$\frac{1}{(a+b)(k+1)} = \int_0^1 \frac{x^k}{a+b}dx \le \int_0^1 \frac{x^k}{bx+a} \le \int_0^1 \frac{x^{k-1}}{a+b}dx = \frac{1}{(a+b)k}$$

Multiplying gives

$$(a+b)^2 k^2 \le \frac{1}{(\int_0^1 \frac{x^k}{ax+b}dx)(\int_0^1 \frac{x^k}{bx+a}dx)} \le (a+b)^2(k+1)^2$$

and summing then gives

$$\frac{(a+b)^2}{6}(2n^3+3n^2+n) < \sum_{k=1}^{n} \frac{1}{(\int_0^1 \frac{x^k}{ax+b}dx)(\int_0^1 \frac{x^k}{bx+a}dx)} < \frac{(a+b)^2}{6}(2n^3+9n^2+13n).$$

We thus have

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \left(\int_0^1 \frac{x^k}{ax+b}dx\right)^{-1} \left(\int_0^1 \frac{x^k}{bx+a}dx\right)^{-1} = \frac{(a+b)^2}{3}.$$

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