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If $0 < a \leq b < 1$ then:

$$\exp\left(\int_a^b \int_a^b \frac{x+y+2}{1-xy}dxdy\right) \leq \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

Solution 1 by proposer.
First we prove that if $x, y \in (0,1)$ then:

$$\frac{x+y+2}{1-xy} \leq \frac{1}{1-x} + \frac{1}{1-y} \tag{1}$$

$$\frac{x+y+2}{1-xy} \leq \frac{1-y+1-x}{(1-x)(1-y)} \Leftrightarrow \frac{x+y+2}{1-xy} \leq \frac{2-x-y}{1-x-y+xy}$$

$$(x+y+2)(1-x-y+xy) \leq (2-x-y)(1-xy)$$

$$x - x^2 - xy + x^2y + y - yx - y^2 + xy^2 + 2 - 2x - 2y + 2xy \leq$$

$$\leq 2 - 2xy - x + x^2y - y + xy^2$$

$$-x^2 - y^2 \leq -2xy \Leftrightarrow x^2 + y^2 - 2xy \geq 0 \Leftrightarrow (x-y)^2 \geq 0$$

Integrating in (1):

$$\int_a^b \int_a^b \frac{x+y+2}{1-xy}dxdy \leq \int_a^b \int_a^b \frac{1}{1-x}dxdy + \int_a^b \int_a^b \frac{1}{1-y}dxdy =$$

$$= \int_a^b \frac{1}{1-x}dx \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{1}{1-y}dy =$$

$$= -2(b-a)(\ln(1-b) - \ln(1-a)) = \ln\left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

$$\int_a^b \int_a^b \frac{x+y+2}{1-xy}dxdy \leq \ln\left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

$$\exp\left(\int_a^b \int_a^b \frac{x+y+2}{1-xy}dxdy\right) \leq \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

Equality holds for $a = b$. $\square$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morroco and UClan Cyprus Problem Solving Group.

First we prove that for $x, y \in (0,1)$,

$$\frac{x+y+2}{1-xy} \leq \frac{2-x-y}{(1-x)(1-y)} = \frac{1}{1-x} + \frac{1}{1-y}$$

That holds since, after expanding we have

$$(2-x-y)(1-xy) - (x+y+2)(1-x)(1-y) = (x-y)^2 \geq 0$$

Integrating, we have:

$$\int_a^b \int_a^b \frac{x+y+2}{1-xy} dxdy \leq \int_a^b \int_a^b \frac{1}{1-x} dxdy + \int_a^b \int_a^b \frac{1}{1-y} dxdy$$
$$= \int_a^b \frac{1}{1-x} dx \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{1}{1-y} dy$$
$$= -2(b-a)(\ln(1-b) - \ln(1-a)) = \ln\left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

Hence

$$\exp\left(\int_a^b \int_a^b \frac{x+y+2}{1-xy} dxdy\right) \leq \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

Equality holds for $a = b$.

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