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Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k}}$$

Solution 1 by proposer. Let be:

$$A(n) = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos x)^n \sin x dx; n \in \mathbb{N}$$

$$1-\cos x = t; \sin x dx = dt$$

$$x = \frac{\pi}{3} \Rightarrow t = \frac{1}{2}; x = \frac{\pi}{2} \Rightarrow t = 1$$

(1)
$$A(n) = \int_{\frac{1}{2}}^{1} t^n dt = \frac{t^{n+1}}{n+1}\Big|_{\frac{1}{2}}^{1} = \frac{1}{n+1}\Big(1-\frac{1}{2^{n+1}}\Big)$$

$$A(n) = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos x)^n \sin x dx =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cos^k x\right) \cdot \sin x dx =$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cdot \cos^k x dx =$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot \frac{(-\cos^{k+1} x)}{k+1}\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= -\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \binom{n}{k} \cdot \Big(0 - \frac{1}{2^{k+1}}\Big) =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \frac{1}{2^{k+1}} \cdot \binom{n}{k}$$

(2)
$$A(n) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \frac{1}{2^{n+1}} \cdot \binom{n}{k}$$

By (1); (2):

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot \frac{1}{2^{k+1}} \cdot \binom{n}{k} = \frac{1}{n+1}\Big(1-\frac{1}{2^{n+1}}\Big)$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k} = \frac{2^{n+1}-1}{n+1}$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{2^{n+1}-1}{n+1}} \overset{\text{CAUCHY-D'ALEMBERT}}{=}$$

$$= \lim_{n\to\infty} \frac{2^{n+2}-1}{n+2} \cdot \frac{n+1}{2^{n+1}-1} =$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} \cdot \lim_{n\to\infty} \frac{2^{n+2}-1}{2^{n+1}-1} = 1 \cdot 2 = 2$$

□

Solution 2 by Prithwijit De - India and Kee - Wai Lau - Hong Kong.
By the binomia theorem, we have

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

Integrating from 0 to $\frac{1}{2}$, we obtain

$$\int_0^{\frac{1}{2}} (1-x)^n dx = -\frac{(1-x)^{n+1}}{n+1}\Big|_0^{\frac{1}{2}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^{k+1}}{k+1}\Big|_0^{\frac{1}{2}}$$

or

$$\frac{-1}{n+1}\left(\left(\frac{1}{2}\right)^{n+1} - 1\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1}\left(\frac{1}{2}\right)^{n+1}$$

so

$$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1} 2^{n-k}$$

Hence

$$\lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot 2^{n-k} \cdot \binom{n}{k}} = \lim_{n\to\infty} \left(\frac{2^{n+1}}{n}\right)^{\frac{1}{2}} \left(\frac{1 - \frac{1}{2}^{n+1}}{1 + \frac{1}{n}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{2^{1+\frac{1}{n}}}{n^{\frac{1}{n}}}\right) \left(\frac{1 - \frac{1}{2}^{n+1}}{1 + \frac{1}{n}}\right)^{\frac{1}{n}}$$

$$= 2 \lim_{n\to\infty} \left(\frac{2}{n}\right)^{\frac{1}{n}} = 2$$

□

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