4704

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If $a,b,c,d\in[0,1)$ then:

$$\frac{1}{a^6-1}+\frac{1}{b^6-1}+\frac{1}{c^6-1}+\frac{1}{d^2-1}\le\frac{2}{(abc)^2-1}+\frac{2}{abcd-1}$$

Solution 1 by proposer.
If $x,y,z,t\ge 0$ then:

$$x^6+y^6+z^6+t^2\overset{\text{AM-GM}}{\ge}3\sqrt[3]{x^6y^6z^6}+t^2=$$
$$=3x^2y^2z^2+t^2=2x^2y^2z^2+x^2y^2z^2+t^2\ge$$
$$\overset{\text{AM-GM}}{\ge}2x^2y^2z^2+2\sqrt{(x^2y^2z^2)\cdot t^2}=$$
$$=2x^2y^2z^2+2xyzt$$

$$x^6+y^6+z^6+t^2\ge 2x^2y^2z^2+2xyzt \tag{1}$$

In (1) we take $x=a^n;y=b^n;z=c^n;t=d^n$

$$a^{6n}+b^{6n}+c^{6n}+d^{2n}\ge 2(abc)^{2n}+2(abcd)^n$$

$$\sum_{n=0}^{\infty}a^{6n}+\sum_{n=0}^{\infty}b^{6n}+\sum_{n=0}^{\infty}c^{6n}+\sum_{n=0}^{\infty}d^{2n}\ge$$
$$\ge 2\sum_{n=0}^{\infty}(abc)^{2n}+2\sum_{n=0}^{\infty}(abcd)^n$$

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6}+\frac{1}{1-d^2}\ge\frac{2}{1-(abc)^2}+\frac{2}{1-abcd} \tag{2}$$

We use the fact: $a,b,c,d\in[0,1)\Rightarrow a^\infty=b^\infty=c^\infty=d^\infty=0$.
By multiplying (2) with "-1":

$$\frac{1}{a^6-1}+\frac{1}{b^6-1}+\frac{1}{c^6-1}+\frac{1}{d^2-1}\le\frac{2}{(abc)^2-1}+\frac{2}{abcd-1}$$

Equality holds for:

$$b=a;c=a;d=a^3$$

□

Solution 2 by Mohamed Amine Ben Ajiba - Tanger - Morocco.
By the AM-GM inequality, we have for all $x,y\in[0,1)$ that

$$\frac{1}{1-x^2}+\frac{1}{1-y^2}\ge\frac{2}{\sqrt{(1-x^2)(1+y^2)}}=\frac{2}{\sqrt{(1-xy)^2-(x-y)^2}}\ge\frac{2}{1-xy}$$

with equality when $x=y$. Using this inequality, we have

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}\ge\frac{2}{1-(ab)^3} \tag{1}$$

$$\frac{1}{1-c^6}+\frac{1}{1-(abc)^2}\ge\frac{2}{1-abc^4}\tag{2}$$

$$\frac{1}{1-d^2}+\frac{1}{1-(abc)^2}\ge\frac{2}{1-abcd}\tag{3}$$

$$2\Big(\frac{1}{1-(ab)^3}+\frac{1}{1-abc^4}\Big)\ge\frac{4}{1-(abc)^2}\tag{4}$$

Adding (1) – (4), we get:

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6}+\frac{1}{1-d^6}+\frac{2}{1-(abc)^2}+\frac{2}{1-(ab)^3}+\frac{2}{1-abc^4}$$
$$\ge\frac{2}{1-(ab)^3}+\frac{2}{1-abc^4}+\frac{2}{1-abcd}+\frac{4}{1-(abc)^2},$$

which simplifies to

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6}+\frac{1}{1-d^6}\ge\frac{2}{1-(abc)^2}+\frac{2}{1-abcd},$$

completing the proof. Equality holds if and only if $a=b=c=\lambda, d=\lambda^3$ for some $\lambda\in[0,1)$ □

Solution 3 by Marian Dincă.
Using the AM-GM inequality repeatedly, we obtain

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6}\ge\sqrt[3]{\frac{1}{1-a^6}\cdot\frac{1}{1-b^6}\cdot\frac{1}{1-c^6}}=\frac{3}{\sqrt[3]{(1-a^6)(1-b^6)(1-c^6)}}\tag{5}$$

and

$$\sqrt[3]{(1-a^6)(1-b^6)(1-c^6)}\le\frac{(1-a^6)+(1-b^6)+(1-c^6)}{3}=1-\frac{a^6+b^6+c^6}{3}$$
$$\le 1-\sqrt[3]{a^6b^6c^6}=1-(abc)^2\tag{6}$$

From (5) and (6), we have

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6}\ge\frac{3}{1-(abc)^2}=\frac{2}{1-(abc)^2}+\frac{1}{1-(abc)^2}\tag{7}$$

Next,

$$\frac{1}{1-(abc)^2}+\frac{1}{1-d^2}\ge 2\sqrt{\frac{1}{1-(abc)^2}\cdot\frac{1}{1-d^2}}$$
$$=\frac{2}{\sqrt{1-(abc)^2(1-d^2)}}\ge\frac{2}{\frac{1-(abc)^2+1-d^2}{2}}$$
$$=\frac{2}{1-\frac{(abc)^2+d^2}{2}}\ge\frac{2}{1-\sqrt{(abc)^2d^2}}=\frac{2}{1-abcd}\tag{8}$$

Finally, from (7) and (8), the conclusion follows. □

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