ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0$,

$a+b+c=ab+bc+ca>0$ then find the minimum value of :

$$P=\frac{3a-b-c}{b^2+c^2}+\frac{3b-c-a}{c^2+a^2}+\frac{3c-a-b}{a^2+b^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$. We have $3p=3q\leq p^2 \Rightarrow p\geq 3$. Also,

$$P=\sum_{cyc}\frac{4a-p}{b^2+c^2}=\frac{\sum_{cyc}(4a-p)\left[a^2\left(a^2+b^2+c^2\right)+b^2c^2\right]}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}=$$

$$=\frac{4\sum a^3.\sum a^2+4abc\sum bc-p\left(\sum a^2\right)^2-p\sum b^2c^2}{\sum a^2.\sum b^2c^2-a^2b^2c^2}=$$

$$=\frac{4\left(p^3-3pq+3r\right)\left(p^2-2q\right)+4qr-p\left(p^2-2q\right)^2-p\left(q^2-2pr\right)}{(p^2-2q)(q^2-2pr)-r^2}=$$

$$\overset{q=p}{=}\frac{3p^5-16p^4+19p^3+p(14p-20)r}{p^4-2p^3-2p^2(p-2)r-r^2}=f(r).$$

f is clearly incresing, then we have : If $p\geq 4$: $P=f(r)\geq f(0)=\frac{3p^2-16p+19}{p-2}=$

$=\frac{3}{2}+\frac{(p-4)(6p-11)}{2(p-2)}\geq\frac{3}{2}$. If $3\leq p\leq 4$:
By the fourth degree Schur's inequality, we have

$$r\geq\frac{(p^2-q)(4q-p^2)}{6p}, \text{then}$$

$$P=f(r)\geq f\left(\frac{p(p-1)(4-p)}{6}\right)=\frac{24p^3-36p^2-252p+480}{11p^4-74p^3+171p^2-128p-16}=$$

$$=\frac{3}{2}+\frac{(4-p)(p-3)(33p^2-39p-84)}{2[(p-3)(p(p-3)(11p-8)+24p+16)+32]}\geq\frac{3}{2}.$$

So the maximum value of P is $\frac{3}{2}$ achieved at

$a=b=c=1$ & $a=b=2, c=0$ and permutations.