ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0, ab + bc + ca > 0$. Prove that

$$\frac{15}{2} \cdot \frac{a^2 + b^2 + c^2}{(a + b + c)^2} + \frac{ab}{a^2 + b^2} + \frac{bc}{b^2 + c^2} + \frac{ca}{c^2 + a^2} \geq 4$$

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The desired inequality can be rewritten as follows

$$\frac{15(a^2 + b^2 + c^2)}{(a + b + c)^2} + \frac{(a + b)^2}{a^2 + b^2} + \frac{(b + c)^2}{b^2 + c^2} + \frac{(c + a)^2}{c^2 + a^2} \geq 11.$$

WLOG, assume that $a + b + c = 1$. Let $p := a + b + c = 1, q := ab + bc + ca \leq \frac{p^2}{3} = \frac{1}{3}, r := abc$.

By CBS inequality, we have

$$\sum_{cyc} \frac{(b + c)^2}{b^2 + c^2} \geq \frac{\left(\sum_{cyc}(b + c)(5a + 4)\right)^2}{\sum_{cyc}(b^2 + c^2)(5a + 4)^2} = \frac{(10q + 8p)^2}{50(q^2 - 2pr) + 40(pq - 3r) + 32(p^2 - 2q)}$$

$$= \frac{2(5q + 4)^2}{16 - 12q + 25q^2 - 110r}.$$

So it suffices to prove that

$$15(1 - 2q) + \frac{2(5q + 4)^2}{16 - 12q + 25q^2 - 110r} \geq 11 \Leftrightarrow \frac{(5q + 4)^2}{16 - 12q + 25q^2 - 110r} \geq 15q - 2 \quad (1)$$

If $15q - 2 \leq 0$, the inequality (1) is true. Assume now that $15q - 2 > 0$.

$$(1) \Leftrightarrow r \geq \frac{375q^3 - 255q^2 + 224q - 48}{110(15q - 2)}.$$

From the known identity

$$0 \leq (a - b)^2(b - c)^2(c - a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3$$

It follows

$$r \geq \frac{-2p^3 + 9pq - 2\sqrt{(p^2 - 3q)^3}}{27} = \frac{-2 + 9q - 2\sqrt{(1 - 3q)^3}}{27}.$$

So it suffices to prove that

$$\frac{-2+9q-2\sqrt{(1-3q)^3}}{27} \ge \frac{375q^3-255q^2+224q-48}{110(15q-2)}$$

$$\Leftrightarrow (1-3q)\left(1736-6120q+3375q^2\right) \ge 220(15q-2)\sqrt{(1-3q)^3}$$

$$\overset{squaring}{\Leftrightarrow} \quad 27(1-3q)^2(25q-8)^2\left(675q^2-80q+1632\right) \ge 0,$$

which is true and the proof is complete.

Equality holds iff $a = b = c$ and $a = b = 2c$ and its permutation.