ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0, ab + bc + ca > 0$, then prove that :

$$\frac{\sqrt{b+c}}{a+\sqrt{bc}} + \frac{\sqrt{c+a}}{b+\sqrt{ca}} + \frac{\sqrt{a+b}}{c+\sqrt{ab}} \geq 3\sqrt{\frac{a+b+c}{2(ab+bc+ca)}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\left(\sum_{cyc} \frac{\sqrt{b+c}}{a+\sqrt{bc}}\right)^2 \sum_{cyc} (b+c)^2\left(a+\sqrt{bc}\right)^2 \geq 8(a+b+c)^3,$$

so it suffices to prove that

$$9\sum_{cyc} (b+c)^2\left(a+\sqrt{bc}\right)^2 \leq 16(a+b+c)^2(ab+bc+ca).$$

By AM − GM inequality, we have

$$18\sum_{cyc} (b+c)^2\left(a+\sqrt{bc}\right)^2 = 9\sum_{cyc} (b+c)^2\left(2a^2+2bc+4a\sqrt{bc}\right) \leq 9\sum_{cyc} (b+c)^2\left[2a^2+2bc+a\left(b+c+\frac{4bc}{b+c}\right)\right]$$

$$= 9\sum_{cyc} (b+c)\left[2\left(a^2+bc\right)(b+c)+a\left(b^2+c^2+6bc\right)\right] =$$

$$= 27\sum_{cyc} a^3(b+c) + 72\sum_{cyc} b^2c^2 + 162abc\sum_{cyc} a \overset{?}{\leq} 32\left(\sum_{cyc} a\right)^2 \sum_{cyc} bc$$

$$\Leftrightarrow 8\sum_{cyc} b^2c^2 + 2abc\sum_{cyc} a \leq 5\sum_{cyc} a^3(b+c) \Leftrightarrow 5\sum_{cyc} bc(b-c)^2 + \sum_{cyc} a^2(b-c)^2 \geq 0,$$

which is true and the proof is complete. Equality holds if $a = b = c$.