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JP.121. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} \le \left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursarescu-Romania

We have $x^5 + y^5 \ge xy(x^3 + y^3), \forall x, y > 0 \Leftrightarrow x^5 - x^4y + y^5 - xy^4 \ge 0 \Leftrightarrow$

$$\Leftrightarrow x^4(x-y) - xy^4(x-y) \ge 0 \Leftrightarrow (x-y)(x^4 - y^4) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (x-y)^2(x+y)(x^2+y^2) \ge 0 \text{ true}$$

$$\Rightarrow \sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} \le \sum \frac{1}{\sin A \sin B} \quad (1)$$

$$\text{But } \sum \frac{1}{\sin A \sin B} = \frac{2R}{r} \quad (2)$$

$$\text{From (1) + (2)} \Rightarrow \sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} \le \frac{2R}{r}$$

$$\text{We must show: } \frac{2R}{r} \le \left(\frac{R}{r}\right)^2 \Leftrightarrow 2 \le \frac{R}{r} \Leftrightarrow 2r \le R \text{ true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} =$$

$$= \frac{(\sin A + \sin B)(\sin^2 A - \sin A \sin B + \sin^2 B)}{(\sin A + \sin B)(\sin^4 A - \sin^3 A \sin B + \sin^2 A \sin^2 B - \sin A \sin^3 B + \sin^4 B)}$$

$$= \frac{(\sin A + \sin B)(\sin^2 A - \sin A \sin B + \sin^2 B)}{\sin^3 A(\sin A - \sin B) - \sin^3 B(\sin A - \sin B) + \sin^2 A \sin^2 B}$$

$$= \frac{\sin^2 A - \sin A \sin B + \sin^2 B}{(\sin A - \sin B)(\sin A - \sin B)(\sin^2 A + \sin A \sin B + \sin^2 B) + \sin^2 A \sin^2 B}$$

$$\overset{A-G}{\le} \frac{\sin^2 A - \sin A \sin B + \sin^2 B}{3 \sin A \sin B(\sin A - \sin B)^2 + \sin^2 A \sin^2 B} =$$

$$= \frac{\sin^2 A - \sin A \sin B + \sin^2 B}{\sin A \sin B(3\sin^2 A + 3\sin^2 B - 5 \sin A \sin B)}$$

$$= \frac{(3\sin^2 A + 3\sin^2 B - 5\sin A \sin B) + 2\sin A \sin B}{3 \sin A \sin B(3\sin^2 A + 3\sin^2 B - 5\sin A \sin B)} =$$

$$= \frac{1}{3 \sin A \sin B} + \frac{2}{3}\left(\frac{1}{3\sin^2 A + 3\sin^2 B - 5\sin A \sin B}\right)$$

$$\overset{A-G}{\leq}\frac{1}{3\sin A\sin B}+\frac{2}{3}\left(\frac{1}{\sin A\sin B}\right)=\frac{1}{\sin A\sin B}\Rightarrow\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\leq\frac{1}{\sin A\sin B}$$

$$\Rightarrow\sum\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\leq\sum\frac{1}{\sin A\sin B}\leq\sum\frac{1}{\sin^2 A}=4R^2\sum\frac{1}{a^2}\overset{Goldstone}{\leq}\frac{4R^2}{4r^2}=\frac{R^2}{r^2}\quad \textit{(Done)}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\overset{CHEBYSHEV'S\ INEQUALITY}{\leq}\sum_{cyc}\frac{2}{\sin^2 A+\sin^2 B}$$

$$\overset{A.M\geq G.M}{\leq}\sum_{cyc}\frac{1}{\sin A\sin B}=\sum_{cyc}\frac{4R^2}{ab}=4R^2\cdot\frac{2p}{4Rrp}=\frac{2R}{r}\leq\left(\frac{R}{r}\right)^2\ [\because 2r\leq R]$$

Hence Proved

JP.122. Prove that in ΔABC the following relationship holds:

$$\min\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)\leq 2\left(\frac{R}{r}-1\right)\leq\max\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $\min\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)=\frac{a}{s-a}=m$ *(say) and*

$$\max\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)=\frac{c}{s-c}\therefore\frac{a}{s-a}\leq\frac{b}{s-b}\leq\frac{c}{s-c}\Rightarrow a\leq b\leq c.$$

Now, $\because\frac{a}{s-a}\leq\frac{b}{s-b}$ *and* $\frac{a}{s-a}\leq\frac{c}{s-c}\therefore 3\frac{a}{s-a}\leq\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\Rightarrow 3m\leq\sum\frac{a-s+s}{s-a}=$

$$=-3+s\sum\frac{1}{s-a}=-3+\left(\frac{s}{sr^2}\right)\left(\sum(s-b)(s-c)\right)=$$

$$=-3+\frac{3s^2-4s^2+s^2+4Rr+r^2}{r^2}=\frac{4R+r}{r}-3=\frac{2(2R-r)}{r}\Rightarrow m\leq\frac{2(2R-r)}{3r}\leq$$

$$\overset{?}{\leq}2\left(\frac{R}{r}-1\right)\Leftrightarrow\frac{2R-r}{3}\overset{?}{\leq}R-r\Leftrightarrow R\overset{?}{\geq}2r\rightarrow \textit{true (Euler)}\Rightarrow m\leq 2\left(\frac{R}{r}-1\right)\Rightarrow$$

$$\Rightarrow\min\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)\leq 2\left(\frac{R}{r}-1\right).\ \textit{Now, }\max\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)\geq 2\left(\frac{R}{r}-1\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{2c}{a+b-c}\geq 2\left(\frac{R}{r}-1\right)\Leftrightarrow\frac{c}{a+b-c}+1\geq\frac{R}{r}\Leftrightarrow\frac{a+b}{a+b-c}\geq\frac{abc}{4S}\cdot\frac{s}{S}=$$

$$=\frac{sabc}{4s(s-a)(s-b)(s-c)}\Leftrightarrow\frac{a+b}{a+b-c}\geq\frac{abc}{2(s-a)(s-b)(a+b-c)}\Leftrightarrow$$

$$\Leftrightarrow\frac{a+b}{a+b-c}\geq\frac{abc}{2(s-a)(s-b)(a+b-c)}\Leftrightarrow 2(a+b)(s-a)(s-b)\overset{(1)}{\geq} abc$$

Let $s-a=x, s-b=y, s-c=z\Rightarrow a=y+z, b=z+x, c=x+y (x,y,z>0)$

$\therefore$ ***(1)*** $\Leftrightarrow 2xy(2z+x+y)\geq(x+y)(y+z)(z+x)\Leftrightarrow x^2y+xy^2+2xyz\overset{(2)}{\geq} x^2z+xz^2+$
$+y^2z+yz^2$ $\because$ ***by our assumption,*** $a\leq b\leq c\therefore y+z\leq z+x\leq x+y\Rightarrow x\geq y\geq z$

$\therefore$ ***we can consider*** $y=z+m$ ***and*** $x=z+m+n$, ***where*** $m,n\geq 0$

$\therefore$ ***(2)*** $\Leftrightarrow (z+m)(z+m+n)^2+(z+m+n)(z+m)^2+2z(z+m)(z+m+n)\geq$
$\geq z(z+m+n)^2+(z+m+n)z^2+z(z+m)^2+(z+m)z^2\Leftrightarrow$

$\Leftrightarrow 2m^3+3m^2n+6m^2z+mn^2+6mnz+4mz^2+2nz^3\geq 0\to$ ***true*** $\because m,n\geq 0$ ***and***

$z>0\Rightarrow$ ***(2) is true*** $\Rightarrow\max\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)\geq 2\left(\frac{R}{r}-1\right)$ ***(Proved)***

Solution 2 by Myagmarsuren-Yadamsuren-Darkhan-Mongolia

$$\left.\begin{array}{c}a\geq b\geq c\\ \frac{1}{s-a}\geq\frac{1}{s-b}\geq\frac{1}{s-c}\end{array}\right\}\Rightarrow\frac{a}{s-a}\geq\frac{b}{s-b}\geq\frac{c}{s-c}$$

1) $LHS: 2\left(\frac{R}{r}-1\right)\geq\frac{c}{s-c}; a\geq b\geq c\Rightarrow a+b\geq 2c\Rightarrow a+b-c\geq c\Rightarrow 2(s-c)\geq c\Rightarrow$

$$\Rightarrow 2\geq\frac{c}{s-c};\frac{c}{s-c}\leq 2=2\cdot 1=2(2-1)\overset{Euler}{\leq}2\left(\frac{R}{r}-1\right) LHS$$

2) $\left.\begin{array}{l}x=s-a\\ y=s-b\\ z=s-c\end{array}\right\} a\geq b\geq c\Rightarrow z\geq y\geq x\Rightarrow 2z\geq y+x\Rightarrow(x+y-2z)xy\leq 0\Rightarrow$

$$\Rightarrow(x+y)xy+zx(z+x)-zx(z+x)\leq 2xyz\overset{zx(z+x)\leq yz(z+y)}{\Rightarrow}$$

$$\Rightarrow 2xyz\geq(x+y)xy+zx(z+x)-zx(z+x)\geq(x+y)xy+zx(z+x)-zy(z+y)$$

$$2xyz\geq(x+y)xy+zx(z+x)-zy(z+y)\Leftrightarrow\sum xy(x+y)-2xyz\leq 2zy(z+y)\Rightarrow$$

$$\Rightarrow\prod(x+y)-4xyz\leq 2zy(z+y)\Rightarrow\frac{\prod(x+y)-4xyz}{4xyz}\leq\frac{y+z}{2x}\Leftrightarrow$$

$$\Leftrightarrow 2\left(\underbrace{\frac{\prod(x+y)}{4xyz}}_{\substack{\Downarrow \\ \frac{R}{2}}}-1\right)\le \underbrace{\frac{y+z}{x}}_{\substack{\Downarrow \\ \frac{a}{s-a}}}$$

$$2\left(\frac{R}{r}-1\right)\le \frac{a}{s-a}\quad RHS$$

JP.123. Solve for real numbers:

$$\log_a(b^x+a-b)=\log_b(a^x+b-a),\ b>a>1$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Let $f(x)=\log_a(b^x+a-b)$ *and* $g(x)=\log_b(a^x+b-a)$. *Let*

$$y=f(x)=\log_a(b^x+a-b)\Rightarrow a^y=b^x+a-b\Rightarrow b^x=(a^y+b-a)\Rightarrow$$

$$\Rightarrow x\log b=\log(a^y+b-a)\ \{\text{Taking log}\}\Rightarrow x=\log_b(a^y+b-a)$$

$$f^{-1}(y)=\log_b(a^y+b-a)\Rightarrow f^{-1}(x)=\log_b(a^x+b-a)=g(x)$$

Therefore, $f(x)$ *and* $g(x)$ *are inverse of each other. Also, both* $f(x)$ *and* $g(x)$ *are increasing and continuous functions {since they are log functions}*

Considering the last two arguments/statements we can say, the only possible solution lies on the line $y=x$; *i.e.,* $f(x)=g(x)=x\Rightarrow \log_a(b^x+a-b)=\log_b(a^x+b-a)=x$

Taking, $\log_a(b^x+a-b)=x\Rightarrow b^x+a-b=a^x\Rightarrow b^x-a^x+a-b=0$

Let $h(x)=b^x-a^x+a-b;\ h'(x)=b^x\ln b-a^x\ln a;\ h'(x)=b^x\ln b-a^x\ln a>0;$

$h'(x)>0\Rightarrow h(x)$ *is an increasing function, so it can have atmost one real root*

$$\begin{cases} \because b>a>1 \\ b^x>a^x\quad (i) \\ \ln b>\ln a\quad (ii) \\ multiplying\ (i);(ii) \\ b^x\ln b>a^x\ln a \end{cases}$$

Clearly $h(x)=0$ *when* $x=1$, *which is the only solution.*

$\therefore x=1$ *is the unique solution.*

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$\log_a(b^x + a - b) = y \Rightarrow b^x + a - b = a^y \quad (1)$$

$$\log_b(a^x + b - a) = y \Rightarrow a^x + b - a = b^y \quad (2)$$

$$(1)+(2) \Rightarrow a^x + b^x = a^y + b^y \quad (3)$$

Since the function $f(x) = a^x + b^x$ increases, then (3) $\Rightarrow x = y$.

(1) $\Rightarrow b^x + a - b = a^x \Rightarrow b^x - a^x = b - a$ function $f(x) = b^x - a^x$ increases, because the $f(x) = b^x - a^3 = a^x\left(\left(\frac{b}{a}\right)^x - 1\right)$, as a product of two increasing functions, then

$f \uparrow \Rightarrow f(x) = b - a$ has a unique solution, $x = 1$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Give $k > 0, k \neq 1$, we have $\log_a(b^x + a - b) \leq \log_b(a^x + b - a) \leftrightarrow$

$$\leftrightarrow \frac{\log_k(b^x + a - b)}{\log_k a} = \frac{\log_k(a^x + b - a)}{\log_k b} \leftrightarrow \log_k(b^x - b) = \log_k(a^x - a) \leftrightarrow$$

$$\leftrightarrow b^x - b = a^x - a \leftrightarrow b^x - a^x = (b - a) \leftrightarrow x = 1; b > a > 1$$

Therefore the answer is $\{1\}$.

JP.124. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r. Prove that:

$$\frac{(2r)^{a+b+c}}{a^a \cdot b^b \cdot c^c} \leq \left(\tan\frac{A}{2}\right)^b \cdot \left(\tan\frac{B}{2}\right)^c \cdot \left(\tan\frac{C}{2}\right)^a \leq \left(\frac{\sqrt{3}}{36}\right)^{a+b+c} \cdot \frac{a^{2b} \cdot b^{2c} \cdot c^{2a}}{r^{2(a+b+c)}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Weighted GM} \geq \text{weighted HM} \Rightarrow \sqrt[a+b+c]{\left(b\tan\frac{A}{2}\right)^b \left(c\tan\frac{B}{2}\right)^c \left(a\tan\frac{C}{2}\right)^a} \geq$$

$$\geq \frac{a+b+c}{\frac{b}{b\tan\frac{A}{2}} + \frac{c}{c\tan\frac{B}{2}} + \frac{a}{a\tan\frac{C}{2}}} = \frac{2s}{\sum\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}} = \frac{2s}{\sum\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}} =$$

$$= \frac{2s}{\frac{s(3s-2s)}{rs}} = 2r$$

$$\therefore \left(b\tan\frac{A}{2}\right)^b\left(c\tan\frac{B}{2}\right)^c\left(a\tan\frac{C}{2}\right)^a \ge (2r)^{a+b+c} \Rightarrow \left(\tan\frac{A}{2}\right)^b\left(\tan\frac{B}{2}\right)^c\left(\tan\frac{C}{2}\right)^a \ge \frac{(2r)^{a+b+c}}{a^a b^b c^c}.$$

Now, $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{rs} \overset{G-A}{\le} \frac{\left(\frac{s-b+s-c}{2}\right)^2}{rs} = \frac{a^2}{4rs} \Rightarrow \left(\frac{\tan\frac{A}{2}}{a^2}\right)^b \overset{(1)}{\le} \left(\frac{1}{4rs}\right)^b$***. Similarly,***

$$\left(\frac{\tan\frac{B}{2}}{b^2}\right)^c \overset{(2)}{\le} \left(\frac{1}{4rs}\right)^c \ \& \ \left(\frac{\tan\frac{C}{2}}{c^2}\right)^a \overset{(3)}{\le} \left(\frac{1}{4rs}\right)^a$$

(1)· (2)·(3)$\Rightarrow \left(\frac{\tan\frac{A}{2}}{a^2}\right)^b\left(\frac{\tan\frac{B}{2}}{b^2}\right)^c\left(\frac{\tan\frac{C}{2}}{c^2}\right)^a \le \left(\frac{1}{4rs}\right)^{a+b+c} \overset{?}{\le} \left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\left(\frac{1}{r^2}\right)^{a+b+c} \Leftrightarrow$

$$\Leftrightarrow \frac{1}{4rs} \overset{?}{\le} \frac{\sqrt{3}}{36}\cdot\frac{1}{r^2} \Leftrightarrow 4rs \overset{?}{\ge} 12\sqrt{3}r^2 \Leftrightarrow s \overset{?}{\ge} 3\sqrt{3}r \to \textit{true}$$

$$\therefore \left(\frac{\tan\frac{A}{2}}{a^2}\right)^b\left(\frac{\tan\frac{B}{2}}{b^2}\right)^c\left(\frac{\tan\frac{C}{2}}{c^2}\right)^a \le \left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\cdot\frac{1}{r^{2(a+b+c)}} \Leftrightarrow$$

$$\Leftrightarrow \left(\tan\frac{A}{2}\right)^b\left(\tan\frac{B}{2}\right)^c\left(\tan\frac{C}{2}\right)^a \le \left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\cdot\frac{a^{2b}\cdot b^{2c}\cdot c^{2a}}{r^{2(a+b+c)}} \quad \textit{(Done)}$$

JP.125. Prove that in $\Delta\ ABC$ the following relationship holds:

$$\frac{4}{3}\left(r_a^2 + r_b^2 + r_c^2\right) \ge 4\sqrt{3}S + (a-b)^2 + (b-c)^2 + (c-a)^2$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Lahiru Samarakoon-Sri Lanka

Prove that $\frac{4}{3}\left(r_a^2 + r_b^2 + r_c^2\right) \ge 4\sqrt{3}\Delta + (a-b)^2 + (b-c)^2 + (c-a)^2$

we have to prove,

$$\frac{4}{3}\left[\left(\sum r_a\right)^2 - 2\left(\sum r_a r_b\right)\right] \ge 4\sqrt{3}\Delta + 2\left[\sum a^2 - \sum ab\right]$$

$$\frac{4}{3}\left[(4R+r)^2 - 2S^2\right] \ge 4\sqrt{3}\Delta + 2\left[2S^2 - 2r^2 - 8Rr - S^2 - r^2 - 4Rr\right]$$

$$4(4R+r)^2 - 8S^2 \ge 12\sqrt{3}\Delta + 6s^2 - 18r^2 - 72Rr$$

$$16 \times 4R^2 + 22r^2 + 104Rr \ge 12\sqrt{3}\Delta + 14S^2$$

But, $S^2 \le 4R^2 + 4Rr + 3r^2$***. So, we have to prove,***

$$16\ \times\ 4R^2+22r^2+104Rr\geq 12\sqrt{3}\Delta+14(4R^2+4Rr+3r^2)$$

$$4\ \times\ 2R^2-20R^2+48Rr\geq 12\sqrt{3}\Delta$$

We have to prove, $(2R^2-8r^2+12Rr)\geq 3\sqrt{3}Sr$

We have to prove, $(2R^2-5r^2+12Rr)^2\geq 27r^2(4R^2+4Rr+3r^2)$

$$\because S^2\leq 4R^2+4Rr+3r^2$$

$$4R^4+48R^3r+16R^2r^2-228Rr^3-56r^4\geq 0$$

$$R^4+12R^3+4R^2r^2-57Rr^3-14r^4\geq 0$$

L.H.S $=R^3(R-2r)+14R^2r(R-2r)+32Rr^2(R-2r)+7r^3(R-2r)$

$$=\underbrace{(R-2r)}_{(+)}\underbrace{(R^3+14R^2r+32Rr^2+7r^3)}_{(+)}$$

Euler. Its true. Proved

$$\frac{4}{3}\left(\sum r_a^2\right)\geq 4\sqrt{3}\Delta+(a-b)^2+(b-c)^2+(c-a)^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{4}{3}\sum r_a^2\overset{(1)}{\geq}4\sqrt{3}S+\sum(a-b)^2$$

$$\textbf{(1)}\Leftrightarrow \frac{4}{3}\sum r_a^2+2\sum ab-\sum a^2\overset{(2)}{\geq}4\sqrt{3}S+\sum a^2$$

Now, Hadwiger – Finsler $\Rightarrow 2\sum ab-\sum a^2\overset{(i)}{\geq}4\sqrt{3}S$ **&** $\sum a^2\underset{(ii)}{\overset{Leibnitz}{\leq}}9R^2$

(i), (ii), (2) $\Rightarrow$ ***it suffices to prove:***

$$4\{(4R+r)^2-2s^2\}\geq 27R^2\Leftrightarrow 8s^2\overset{(3)}{\leq}37R^2+32Rr+4r^2$$

LHS of (3) $\overset{Gerretsen}{\leq}32R^2+32Rr+24r^2\overset{?}{\leq}37R^2+32Rr+4r^2\Leftrightarrow 5R^2\overset{?}{\geq}20r^2\Leftrightarrow$

$\Leftrightarrow R\overset{?}{\geq}2r\rightarrow$ ***true (Euler) (proved)***

JP.126. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\cot A+\cot B+\cot C\geq\sqrt{\frac{3R}{2r}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

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Solution 1 by Marian Ursarescu-Romania

In any ΔABC ***we have:*** $\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4S}$ ***(1)***

From (1) inequality becomes: $\frac{a^2+b^2+c^2}{4S} \ge \sqrt{\frac{3R}{2r}} \Leftrightarrow \frac{a^2+b^2+c^2}{4\sqrt{s(s-a)(s-b)(s-c)}} \ge \sqrt{\frac{3R}{2r}} \Leftrightarrow$

$$\Leftrightarrow \frac{(a^2+b^2+c^2)^2}{16s(s-a)(s-b)(s-c)} \ge \frac{3}{2} \cdot \frac{R}{2} \quad (2)$$

But in any $\Delta\ ABC$ ***we have:*** $\frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$ ***(3)***

From (2)+(3) we must show: $\frac{(a^2+b^2+c^2)^2}{16s(s-a)(s-b)(s-c)} \ge \frac{3}{2} \cdot \frac{abc}{4(s-a)(s-b)(s-c)} \Leftrightarrow$

$$\Leftrightarrow \frac{(a^2+b^2+c^2)^2}{2s} \ge 3abc \Leftrightarrow (a^2+b^2+c^2)^2 \ge 3(a+b+c)abc \quad (4)$$

But from Cauchy $\Rightarrow 3(a^2+b^2+c^2) \ge (a+b+c)^2 \Rightarrow a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}$ ***(5)***

From (4)+(5) we must show: $\frac{(a+b+c)^4}{9} \ge 3(a+b+c)abc \Leftrightarrow (a+b+c)^3 \ge 27abc$***,***

which its true because $a+b+c \ge 3\sqrt[3]{abc}$***.***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left(\sum a^2\right)^2 \ge \left(\sum ab\right)^2 \ge 3abc(a+b+c) \quad \textit{(True)}$$

$$\left(\sum a^2\right)^2 \ge 3abc(a+b+c) = 24s^2Rr$$

$$\frac{1}{8s^2r} \cdot \left(\sum a^2\right)^2 \ge 3R;\ \frac{1}{8 \cdot s^2 \cdot r \cdot 2r}\left(\sum a^2\right)^2 \ge \frac{3R}{2r}$$

$$\frac{1}{16s^2r^2}\left(\sum a^2\right)^2 \ge \frac{3R}{2r};\ \frac{1}{4\Delta} \cdot \sum a^2 \ge \sqrt{\frac{3R}{2r}}$$

$$\frac{R}{abc} \cdot \sum a^2 \ge \sqrt{\frac{3R}{2r}} \quad (*)$$

$$\cot A = \frac{R}{abc}(b^2+c^2-a^2) \quad (**)$$

$$(*), (**) \to \sum \cot A = \frac{R}{abc}(a^2+b^2+c^2) \ge \sqrt{\frac{3R}{2r}}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum \cot A = \sum \frac{\cos A}{\sin A} = \sum \frac{b^2+c^2-a^2}{2bc} \cdot \frac{2R}{a} = \frac{R}{4Rrs}\sum (b^2+c^2-a^2) = \frac{\sum a^2}{4rs} =$$

$$= \frac{s^2-4Rr-r^2}{2rs} \geq \sqrt{\frac{3R}{2r}} \Leftrightarrow \frac{(s^2-4Rr-r^2)^2}{4r^2s^2} \geq \frac{3R}{2r} \Leftrightarrow s^4 + r^2(4R+r)^2 -$$

$-2s^2(4R+r^2) \overset{(1)}{\geq} 6Rrs^2$. ***Now, LHS of (1)*** $\overset{Gerretsen}{\geq} s^2(16Rr-5r^2) - 2s^2(4Rr+r^2) +$

$$+r^2(4R+r)^2 = s^2(8Rr-7r^2) + r^2(4R+r)^2 \overset{?}{\geq} 6Rrs^2 \Leftrightarrow s^2(2R-7r) +$$

$+r(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow s^2(2R-4r) + r(4R+r)^2 \underset{(2)}{\overset{?}{\geq}} 3rs^2$. ***Now LHS of (2)*** $\overset{Gerretsen}{\geq}$

$$\geq (16Rr-5r^2)(2R-4r) + r(4R+r)^2 \geq 3rs^2 \Leftrightarrow 3s^2 \overset{(3)}{\leq} (16R-5r)(2R-4r) +$$

$+(4R+r)^2$. ***Now, LHS of (3)*** $\overset{Gerretsen}{\leq} 3(4R^2+4Rr+3r^2) \overset{?}{\leq}$

$$\leq (16R-5r)(2R-4r)+(4R+r)^2 \Leftrightarrow 6R^2-13Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(6R-r) \overset{?}{\geq} 0$$

$\rightarrow$ ***true*** $\because R \overset{Euler}{\geq} 2r$ ***(proved)***

Solution 4 by Shyama P. Mandal-India

$$\sum \cot(A) = \sum \frac{b^2+c^2-a^2}{2bc\sin(A)} = \sum \frac{b^2+c^2-a^2}{4\Delta} = \frac{a^2+b^2+c^2}{4\Delta}$$

$$4\Delta = 4\sqrt{\Delta}\sqrt{\Delta} = 4\sqrt{rs}\sqrt{\frac{abc}{4R}} = \sqrt{\frac{2r}{R}}\sqrt{abc(a+b+c)}$$

$$abc \leq \left(\sqrt{\frac{a^2+b^2+c^2}{3}}\right)^3 \ GM-RMS \quad a+b+c \leq 3\sqrt{\frac{a^2+b^2+c^2}{3}} \ AM-RMS$$

$$4\Delta \leq \sqrt{\frac{2r}{3R}}(a^2+b^2+c^2)$$

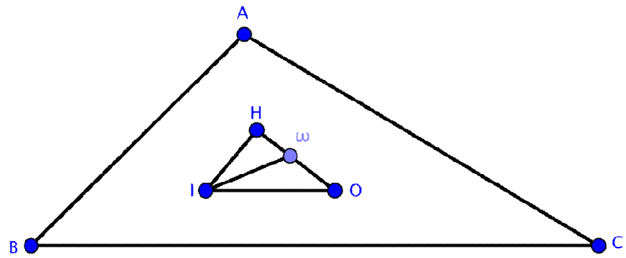
$$\sum \cot(A) \geq \sqrt{\frac{3R}{2r}}$$

JP.127. Prove that in any non-equilateral triangle the following inequality holds:

$$\tan\left(\frac{\widehat{OIH}}{2}\right) < 3\sqrt{\frac{R+2r}{R-2r}}$$

Proposed by Marian Ursărescu – Romania

Solution by Marian Ursărescu – Romania



$$m_a \le 2R\cos^2\frac{A}{2} \Rightarrow m_a \le \frac{a}{\sin A}\cos^2\frac{A}{2} \Rightarrow m_a \le \frac{a}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cdot\cos^2\frac{A}{2}$$

$$\Rightarrow m_a \le \frac{a}{2}\cot\frac{A}{2} \Rightarrow m_a \le \frac{a}{2\tan\frac{A}{2}} \Rightarrow \tan\frac{A}{2} \le \frac{a}{2m_a} \Rightarrow \tan\left(\frac{\widehat{OIH}}{2}\right) \le \frac{OH}{2I\omega} =$$

$$= \frac{\sqrt{9R^2-(a^2+b^2+c^2)}}{2\cdot\frac{R-2R}{2}} \Rightarrow \tan\left(\frac{\widehat{OIH}}{2}\right) \le \frac{\sqrt{9R^2-(a^2+b^2+c^2)}}{R-2r}$$

$$a^2+b^2+c^2 \ge 36r^2 \Rightarrow -(a^2+b^2+c^2) \le -36r^2$$

$$\tan\left(\frac{\widehat{OIH}}{2}\right) < \frac{3\sqrt{R^2-4r^2}}{R-2r} \Rightarrow \tan\left(\frac{\widehat{OIH}}{2}\right) < 3\sqrt{\frac{R+2r}{R-2r}}$$

JP.128. In ΔABC the following relationship holds:

$$\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a} \le \frac{1}{2r}$$

Proposed by Marian Ursărescu – Romania

Solution by Seyran Ibrahimov-Maasilli-Azerbaijan

$$LHS = \sum\frac{1}{m_a+m_b} \le \frac{1}{2r}$$

$$\frac{4}{m_a+m_b}\le\frac{1}{m_a}+\frac{1}{m_b}$$

$$LHS\le\frac{1}{4}\cdot 2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\le\frac{1}{2}\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{1}{2r}$$

JP.129. Let a, b and c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

a. $\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2}\le\frac{3\sqrt{3}}{2}\cdot\frac{R(R-r)}{r}$

b. $\frac{a}{\sqrt{b^2+c^2}}+\frac{b}{\sqrt{c^2+a^2}}+\frac{c}{\sqrt{a^2+b^2}}\le\sqrt{\frac{6R-3r}{2r}}$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursarescu-Romania

b) $b^2+c^2\ge 2bc\Rightarrow\frac{1}{\sqrt{b^2+c^2}}\le\frac{1}{\sqrt{2bc}}\Rightarrow\sum\frac{a}{\sqrt{b^2+c^2}}\le\frac{1}{\sqrt{2}}\sum\frac{a}{\sqrt{bc}}\Rightarrow$ *we must show:*

$$\frac{1}{\sqrt{2}}\sum\frac{9}{\sqrt{bc}}\le\sqrt{\frac{6R-3r}{2r}}\Leftrightarrow$$

$$\left(\sum\frac{a}{\sqrt{bc}}\right)^2\le\frac{6R-3r}{r}\quad (1)$$

But from Cauchy's inequality $\Rightarrow\left(\sum\frac{a}{\sqrt{bc}}\right)^2\le 3\sum\frac{a^2}{bc}$ *(2)*

From (1)+(2) we must show: $\sum\frac{a^2}{bc}\le\frac{2R-r}{r}\Leftrightarrow\frac{\sum a^3}{abc}\le\frac{2R-r}{r}$ *(3)*

But $\sum a^3=2s(s^2-3r^2-6Rr)$ *and* $abc=4sRr$ *(4)*

From (3)+(4) we must show: $\frac{s^2-3r^2-6Rr}{2Rr}\le\frac{2R-r}{r}\Leftrightarrow s^2-3r^2-6Rr\le 4R^2-2Rr\Leftrightarrow$

$\Leftrightarrow s^2\le 4R^2+4Rr+3r^2$ *with its Gerretsen's inequality.*

a) Again, $b^2+c^2\ge 2bc\Rightarrow\frac{a^3}{b^2+c^2}\le\frac{a^3}{2bc}\Rightarrow\sum\frac{a^3}{b^2+c^2}\le\frac{1}{2}\sum\frac{a^3}{bc}\Rightarrow$ *we must show this:*

$$\sum\frac{a^3}{bc}\le 3\sqrt{3}\frac{R(R-r)}{r}\quad (5)$$

Now, using sine law $\Rightarrow a=2R\sin A\Rightarrow$ *(5)* $\Leftrightarrow\sum\frac{2R\sin^3 A}{\sin B\sin C}\le 3\sqrt{3}\frac{R(R-r)}{r}\Leftrightarrow$

$$\sum\frac{\sin^3 A}{\sin B\sin C}\le\frac{3\sqrt{3}}{2}\left(\frac{R}{r}-1\right)\quad (6)$$

But (6) its true, proved in IneMath (2/2016) or (5) $\Leftrightarrow\frac{\sum a^4}{abc}\le 3\sqrt{3}\frac{R(R-r)}{r}$ ***(6)***

$$\sum a^4 = 2[s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2]\quad (7)$$

$$abc = 4sRr\quad (8)$$

From (6)+(7)+(8) $\Rightarrow\frac{s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2}{sR}\le 3\sqrt{3}R(R-r)$ ***and using again***

Gerretsen's $s^2\le 4R^2+4Rr+3r^2$ ***and*** $s\le\frac{3\sqrt{3}}{2}R$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$RHS\overset{Mitrinovic}{\ge}\frac{s(R-r)}{r}\therefore$ ***it suffices to prove:***

$$\sum_{cyc}\frac{a^3}{b^2+c^2}\le\frac{s(R-r)}{r}\Leftrightarrow\sum_{cyc}\left(\frac{a^3}{b^2+c^2}+a\right)-2s\le\frac{s(R-r)}{r}\Leftrightarrow$$

$$\Leftrightarrow\left(\sum a^2\right)\left(\sum\frac{a}{b^2+c^2}\right)\le\frac{s(R+r)}{r}\Leftrightarrow\left(\sum a^2\right)\frac{\sum a(c^2+a^2)(a^2+b^2)}{\prod(a^2+b^2)}\le\frac{s(R+r)}{r}\quad (1)$$

Now, $\sum a(c^2+a^2)(a^2+b^2)=\sum a\left(\sum a^2b^2+a^4\right)=\sum a^5+2s\left(\sum a^2b^2\right)$ ***(a)***

Now, $\left(\sum a^2\right)\left(\sum a^3\right)=\sum a^5+\sum a^2b^2(2s-c)=\sum a^5+2s\left(\sum a^2b^2\right)-abc\left(\sum ab\right)\Rightarrow$

$$\Rightarrow\sum a^5=\left(\sum a^2\right)\left(\sum a^3\right)+abc\left(\sum ab\right)-2s\left(\sum a^2b^2\right)\quad (b)$$

(a), (b) $\Rightarrow\sum a(c^2+a^2)(a^2+b^2)=\left(\sum a^2\right)\left(\sum a^3\right)+abc\left(\sum ab\right)=$

$$=\left(\sum a^2\right)\left(3abc+2s\left(\sum a^2-\sum ab\right)\right)+4Rrs(s^2+4Rr+r^2)=$$

$$=\left(\sum a^2\right)\left(12Rrs+2s(s^2-12Rr-3r^2)\right)+4Rrs(s^2+4Rr+r^2)=$$

$$=4s\{(s^2-4Rr-r^2)(s^2-6Rr-3r^2)+Rr(s^2+4Rr+r^2)\}=$$

$$=4s\{s^4-s^2(9Rr+4r^2)+r^2(4R+r)(7R+3r)\}\quad (c)$$

Again, $\prod(a^2+b^2)=2a^2b^2c^2+\sum a^2b^2\left(\sum a^2-c^2\right)=\left(\sum a^2\right)\left(\sum a^2b^2\right)-a^2b^2c^2=$

$$=\left(\sum a^2\right)\left(\left(\sum ab\right)^2-2abc(2s)\right)-16R^2r^2s^2=$$

$$=\left(\sum a^2\right)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-16R^2r^2s^2=$$

$$=\left(\sum a^2\right)\left(s^4-s^2(8Rr-2r^2)+r^2(4R+r)^2\right)-16R^2r^2s^2\quad (d)$$

(c), (d) ⇒ (1) becomes:

$$\left(\sum a^2\right)\begin{bmatrix}(R+r)\{s^4-s^2(8Rr-2r^2)+r^2(4R+r)^2\}-\\-4r\{s^4-s^2(9Rr+4r^2)+r^2(4R+r)(7R+3r)\}\end{bmatrix}\geq 16(R+r)R^2r^2s^2$$

$$\Leftrightarrow\left(\sum a^2\right)\{(R-2r)s^4-s^2(8R^2r-30Rr^2-18r^3)+r^2(4R+r)(4R^2-23Rr-11r^2)\}\geq$$

$$\geq 16(R+r)R^2r^2s^2+rs^4\left(\sum a^2\right)\quad \textbf{\textit{(2)}}$$

LHS of (2) $\overset{Gerretsen}{\geq}\left(\sum a^2\right)\begin{Bmatrix}(R-2r)(16Rr-5r^2)s^2-s^2(8R^2r-30Rr^2-18r^3)+\\+r^2(4R+r)(4R^2-23Rr-11r^2)\end{Bmatrix}\overset{?}{\geq}$

$$\geq 16(R+r)R^2r^2s^2+rs^4\left(\sum a^2\right)\quad \textbf{\textit{(3)}}$$

$$\Leftrightarrow\left(\sum a^2\right)\{s^2(8R^2-7Rr+28r^2)+r(4R+r)(4R^2-23Rr-11r^2)\}\geq$$

$$\overset{?}{\geq} 16(R+r)R^2rs^2+s^4\left(\sum a^2\right)\quad \textbf{\textit{(3)}}$$

RHS of (3) $\overset{Gerretsen}{\leq} 16(R+r)R^2rs^2+s^2\left(\sum a^2\right)(4R^2+4Rr+3r^2)\overset{?}{\leq}$

$$\leq\left(\sum a^2\right)\{s^2(8R^2-7Rr+28r^2)+r(4R+r)(4R^2-23Rr-11r^2)\}\Leftrightarrow$$

$$\Leftrightarrow\left(\sum a^2\right)\{s^2(4R^2-11Rr+25r^2)+r(4R+r)(4R^2-23Rr-11r^2)\}\overset{?}{\geq}$$

$$\overset{?}{\geq} 16R^2r(R+r)s^2\quad \textbf{\textit{(4)}}$$

LHS of (4) $\overset{Gerretsen}{\geq} 2(s^2-4Rr-r)\begin{Bmatrix}(16Rr-5r^2)(4R^2-11Rr+25r^2)+\\+r(4R+r)(4R^2-23Rr-11r^2)\end{Bmatrix}\overset{?}{\geq}$

$$\overset{?}{\geq} 16R^2r(R+r)s^2\Leftrightarrow$$

$$\Leftrightarrow s^2\{(16R-5r)(4R^2-11Rr+25r^2)+(4R+r)(4R^2-23Rr-11r^2)-8R^2(R+r)\}$$

$$\overset{?}{\geq} r(4R+r)\{(16R-5r)(4R^2-11Rr+25r^2)+(4R+r)(4R^2-23Rr-11r^2)\}\quad \textbf{\textit{(5)}}$$

LHS of (5) $\overset{Gerretsen}{\geq}(16Rr-5r^2)\begin{Bmatrix}(16R-5r)(4R^2-11Rr+25r^2)+\\+(4R+r)(4R^2-23Rr-11r^2)-8R^2(R+r)\end{Bmatrix}\overset{?}{\geq}$

$$\overset{?}{\geq} r(4R+r)\{(16R-5r)(4R^2-11Rr+25r^2)+(4R+r)(4R^2-23Rr-11r^2)\}\Leftrightarrow$$

$$\Leftrightarrow 104t^4-497t^3+800t^2-495t+102\geq 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$

$$\Leftrightarrow (t-2)[(t-2)\{(t-2)(104+127)+314\}+69]\geq 0\rightarrow \textbf{\textit{true}}$$

$\because R\geq 2r$ ***(proved)***

$$\frac{a}{\sqrt{b^2+c^2}}+\frac{b}{\sqrt{c^2+a^2}}+\frac{c}{\sqrt{a^2+b^2}}\le\sqrt{\frac{6R-3r}{2r}}$$

$$\because b^2+c^2\overset{A-G}{\ge}\therefore\frac{a}{\sqrt{b^2+c^2}}\le\frac{a\sqrt{a}}{\sqrt{2abc}}\quad \textit{(1)}$$

Similarly, $\frac{b}{\sqrt{c^2+a^2}}\overset{(2)}{\le}\frac{b\sqrt{b}}{\sqrt{2abc}}$ & $\frac{c}{\sqrt{a^2+b^2}}\overset{(3)}{\le}\frac{c\sqrt{c}}{\sqrt{2abc}}$

(1)+(2)+(3) $\Rightarrow LHS\le\frac{\sum a\sqrt{a}}{\sqrt{8Rrs}}\overset{C-B-S}{\le}\frac{\sqrt{\sum a^2}\sqrt{2s}}{\sqrt{8Rrs}}=\sqrt{\frac{\sum a^2}{4Rr}}\overset{?}{\le}\sqrt{\frac{6R-3r}{2r}}\Leftrightarrow 2R(6R-3r)\overset{?}{\ge}$

$\ge 2(s^2-4Rr-r^2)\Leftrightarrow 6R^2-3Rr\overset{?}{\ge}s^2-4Rr-r^2\Leftrightarrow s^2\overset{?}{\le}6R^2+Rr+r^2$ *(a)*

LHS of (a) $\overset{Gerretsen}{\le}4R^2+4Rr+3r^2\overset{?}{\le}6R^2+Rr+r^2\Leftrightarrow 2R^2-3Rr-2r^2\overset{?}{\ge}0\Leftrightarrow$

$\Leftrightarrow(R-2r)(2R+r)\overset{?}{\ge}0\to true\because R\ge 2r$ ***(Euler)*** $\Rightarrow$ ***(a) is true (proved)***

JP.130. Solve the equation in real numbers:

$$3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}=2(x^4-3x+4)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Omran Kouba-Damascus Syria

Step 1. Consider $f(x)=2(2-3x+2x^2)-1-x^8$ ***then*** $f(x)\ge 0$ ***with equality if and only if*** $x=1$. ***Indeed, with some algebra we see that*** $f(1+t)=t^4h(t)$ ***with***

$h(t)=31\left(t^2+\frac{28t}{31}\right)^2+\frac{1820}{31}\left(t+\frac{31}{65}\right)^2+\frac{952}{65}>0$. ***This proves the inequality:***

$$\sqrt[4]{\frac{1+x^8}{2}}\le 2x^2-3x+2\quad \textit{(1)}$$

With equality if and only if $x=1$.

Step 2. By the AM-GM inequality, we have for all real x ***the following:***

$$3\sqrt[3]{x^2-x+1}\le x^2-x+1+1+1=x^2-x+3\quad \textit{(2)}$$

Step 3. For all real x ***we have:***

$$3x^2-4x+5\le 2(x^4-3x+4)\quad \textit{(3)}$$

with equality if and only if $x = 1$. Indeed:

$$2(x^4-3x+4)-(3x^2-4x+5)=2x^4-3x^2-2x+3=(x-1)^2(2(x+1)^2+1)\geq 0$$

Step 4. Combining (1), (2) and (3) we conclude that:

$$3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{1+x^8}{2}}\leq 2(x^4-3x+4)$$

with equality if and only if $x = 1$.

Solution 2 by Orlando Irahola Ortega-Bolivia

$$\to 6\sqrt[3]{x^2-x+1}+\sqrt[4]{8(x^8+1)}=4(x^4-3x)+16$$

$$\to 6\left(\sqrt[3]{x^2-x+1}-1\right)+\left(\sqrt[4]{8(x^8+1)}-2\right)=4(x^4-3x+2)$$

$$\to \frac{6(x^2-x)}{\underbrace{\left(\sqrt[3]{x^2-x+1}+\sqrt[3]{x^2-x+1}+1\right)}_{f(x)}}+\frac{\sqrt{8(x^8+1)}-4}{\underbrace{\left(\sqrt[4]{8(x^8+1)}+2\right)}_{g(x)}}=4[x^4-x-2(x-1)]$$

$$\Rightarrow \frac{6x(x-1)}{f(x)}+\frac{8(x^8-1)}{g(x)\underbrace{\left(\sqrt{8(x^8+1)}+4\right)}_{h(x)}}=4(x-1)(x^3+x^2+x-2)$$

$$\Rightarrow \frac{6x(x-1)}{f(x)}+\frac{8(x^4+1)(x^2+1)(x-1)(x+1)}{g(x)h(x)}=4(x-1)(x^3+x^2+x-2)\Rightarrow$$

$$\Rightarrow F(x), g(x)\wedge h(x)\neq 0$$

$$\Rightarrow (x-1)\left[\frac{6x}{f(x)}+\frac{8(x^4+1)(x^3+x^2+x+1)}{g(x)h(x)}-4(x^3+x^2+x-2)\right]=0$$

$$\Rightarrow x-1=0; x=1\wedge 6xg(x)h(x)+8f(x)(x^4+1)(x^3+x^2+x+1)=$$

$$=4f(x)g(x)h(x)(x^3+x^2+x-2)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

Case 1: $x\geq 1$. *Let* $f(x)=3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}-2(x^4-3x+4)$

$$f'(x)=\frac{2^{\frac{3}{4}}x^7}{(x^8+1)^{\frac{3}{4}}}+\frac{2x-1}{(x^2-x+1)^{\frac{2}{3}}}-2(4x^3-3)\leq\frac{16^{\frac{3}{4}}x^7}{(x^2+1)^3}+2x-1-2(4x^3-3)$$

$$\left(\because (x^8+1)^{\frac{3}{4}} \ge \left\{\frac{1}{8}(x^2+1)^4\right\}^{\frac{3}{4}} = \frac{(x^2+1)^3}{8^{\frac{3}{4}}} \,\&\, (x^2-x+1)^{\frac{2}{3}} \ge 1 \text{ as } x(x-1) \ge 0\right)$$

$$= \frac{8x^7}{(x^2+1)^3} + 2x - 1 - 2(4x^3-3) = \frac{8x^7 + (2x-1)(x^2+1)^3 - 2(4x^3-3)(x^2+1)^3}{(x^2+1)^3}$$

$$= \frac{-(x-1)(8x^8+8x^7+22x^6+17x^5+35x^4+20x^3+22x^2+7x+5)}{(x^2+1)^3} \le 0 \ (\because x \ge 1)$$

$\therefore$ ***on*** $[1,\infty)$, $f(x)$ ***is a decreasing*** f^n & $\because f(1) = 0$

$\therefore \forall x \ge 1, f(x) \le 0$, ***equality at*** $x = 1$, ***&*** $\because f(x)$ ***actually*** $= 0, \therefore x = 1$

$\therefore \forall x \ge 1$, ***given equation has only 1 root, which is*** $x = 1$.

Case 2: $-1 \le x < 1$. ***Now,*** $\sqrt[3]{(x^2-x+1)\cdot 1\cdot 1} \overset{GM\le AM}{\le} x^2-x+3$. ***Also,***

$$\sqrt[4]{\frac{x^8+1}{2}} = \sqrt[4]{\left(\frac{x^8+1}{16}\right)\cdot 2\cdot 2\cdot 2} \overset{GM<AM}{<} \frac{1}{8}(x^8+7)$$

$\therefore LHS < x^2 - x + 3 + \frac{x^8+7}{8}$ ***(adding the last 2 inequalities)*** $\underset{(1)}{\overset{?}{<}} 2x^4 - 6x + 8 \Leftrightarrow$

$$\Leftrightarrow x^8 - 16x^4 + 8x^2 + 40x - 33 \overset{?}{<} 0 \Leftrightarrow$$

$$\Leftrightarrow (x-1)^2(x^6+2x^5+3x^4+4x^3-11x^2-26x-33) \underset{(2)}{\overset{?}{<}} 0 \Leftrightarrow$$

$\Leftrightarrow x^6+2x^5+3x^4+4x^3-11x^2-26x-33 \overset{?}{<} 0$. ***Now,*** $4x^2(x-1) < 0 \ (\because x < 1) \Rightarrow$

$\Rightarrow 4x^3 < 4x^2$. ***Also,*** $3x^2(x^2-1) \le 0 \ (\because x^2 \le 1) \Rightarrow 3x^4 \le 3x^2$ ***&*** $2x^2(x^3-1) =$

$= 2x^2(x^2+x+1)(x-1) < 0 \Rightarrow 2x^5 < 2x^2$ ***& finally,*** $x^2(x^4-1) =$

$= x^2(x^2+1)(x+1)(x-1) \le 0 (\because x < 1 \,\&\, x+1 \ge 0) \Rightarrow x^6 \le x^2$

Adding the last 4 inequalities, we get: $4x^3 + 3x^4 + 2x^5 + x^6 < 10x^2$

$$\therefore (x^6+2x^5+3x^4+4x^3) - 11x^2 - 26x - 33 < -x^2 - 26x - 33 \le 26 - 33$$

$$(\because -x^2 \le 0 \,\&\, -26x \le 26)$$

$$= -7 \overset{(3)}{<} 0$$

(3) $\Rightarrow$ ***(2) is true*** $\Rightarrow$ ***(1) is true*** $\Rightarrow LHS < RHS \Rightarrow LHS = RHS$ ***never occurs when*** $x \in [-1, 1)$

$\Rightarrow$ ***now sollution in this case***

Case (3) $x < -1$

$$GM < AM \Rightarrow LHS < x^2 - x + 3 + \sqrt[4]{\frac{x^8+1}{2}} \overset{?}{\underset{(4)}{<}} 2x^4 - 6x + 8 \Leftrightarrow 2x^4 - x^2 - 5x + 5 \overset{?}{>}$$

$$\overset{?}{>} \sqrt[4]{\frac{x^8+1}{2}} \because 2x^4 - x^2 + 5 > 0\ (as\ \Delta < 0)\ \ -5x > 5 \therefore 2x^4 - x^2 - 5x + 5 > 0$$

$$\therefore 2x^4 - x^2 - 5x + 5 > \sqrt[4]{\frac{x^8+1}{2}} \Leftrightarrow 2(2x^4 - x^2 - 5x + 5)^4 - x^8 - 1 \overset{(5)}{>} 0$$

Let $x + 1 = t\ (t < 0) \therefore x = t - 1$

$\therefore$ ***(5) becomes*** $2\{2(t-1)^4 - (t-1)^2 - 5(t-1) + 5\}^4 - (t-1)^8 - 1 > 0 \Leftrightarrow$

$\Leftrightarrow (t-2)^2(32t^{14} - 384t^{13} + 2112t^{12} - 7360t^{11} + 19056t^{10} - 39712t^9 +$

$+68352t^8 - 98736t^7 + 121777t^6 - 127564t^5 + 112420t^4 - 83512t^3 +$

$+49408t^2 - 12960t + 7320) > 0 \Leftrightarrow p$ ***(say)*** > 0

$\because t < 0, -384t^{13} - 7360t^{11} - 39712t^9 - 98736t^7 - 127564t^5 - 83512t^3 -$

$-21960t > 0$ ***& of course,*** $32t^{14} + 2112t^{12} + 19056t^{10} + 68352t^8 + 121777t^6 +$

$+112420t^4 + 49408t^2 + 73220 > 0$

$\therefore$ ***adding the last 2 inequalities,*** $p > 0 \Rightarrow$ ***(5) is true*** $\Rightarrow$ ***(4) is true***

$\Rightarrow LHS < RHS\ \forall x < -1$

$\Rightarrow$ ***(4) is true*** $\Rightarrow LHS < RHS\ \forall x < -1 \Rightarrow$ ***no solution under this case. Combining all 3 cases, only solution is*** $x = 1$.

JP.131. Solve the system of equation in positive real numbers:

$$\begin{cases} 3\left(\sqrt[3]{x^2} + \sqrt[3]{y^2} + \sqrt[3]{z^2}\right) + 21 = 10(xy + yz + zx) \\ x + y + z = 3 \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt[3]{x} = a, \sqrt[3]{y} = b, \sqrt[3]{z} = c$. ***Then,*** $\sum a^3 = 3$ **&** $3\sum a^2 + 21 \overset{(1)}{=} 10\sum a^3b^3$

$$\because \sum a^3 = 3 \therefore \left(\sum a^3\right)^2 = 9 \Rightarrow \sum a^6 + 2\sum a^3b^3 = 9 \Rightarrow 5\sum a^6 + 10\sum a^3b^3 = 45$$

$$\Rightarrow 5\sum a^6 + 3\sum a^2 + 21 = 45 \Rightarrow 5\sum a^6 + 3\sum a^2 = 24 \overset{(2)}{=} 8\sum a^3 \left(\because \sum a^3 = 3\right)$$

Now, $a^6 + a^2 + a^2 + a^2 \overset{A-G}{\geq} 4\sqrt[4]{a^{12}} = 4a^3 \Rightarrow a^6 + 3a^2 \overset{(a)}{\geq} 4a^3$.

Similarly, $b^6 + 3b^2 \overset{(b)}{\geq} 4b^3$

& $c^6 + 3c^2 \overset{(c)}{\geq} 4c^3$. *Again,* $4\sum a^6 \overset{Chebyshev}{\geq} \frac{4}{3}(\sum a^3)^2 = \frac{4}{3}\cdot 3\cdot \sum a^3 \ (\because \sum a^3 = 3)$

$$\Rightarrow 4\sum a^6 \overset{(d)}{\geq} 4\sum a^3$$

(a)+(b)+(c)+(d)$\Rightarrow 5\sum a^6 + 3\sum a^2 \overset{(3)}{\geq} 8\sum a^3$, *with equality occuring when* $a = b = c$

$\therefore$ *(2), (3)* $\Rightarrow a = b = c$ *&* $\because \sum a^3 = 3 \therefore 3a^3 = 3 \Rightarrow a = 1 \Rightarrow a = b = c = 1 \Rightarrow$

$\Rightarrow x = y = z = 1 \therefore$ *only solution is:* $x = y = z = 1$ *(answer)*

JP.132. Let $x, y \in \left(0, \frac{\pi}{2}\right)$. Denote $k = 2 - min\{sin^2 x, cos^2 x\}$. Prove that:

$$\left(\frac{sin^2 x}{1 - cos\, y} + \frac{cos^2 x}{1 - sin\, y}\right)\left(\frac{sin^2 x}{1 + cos\, y} + \frac{cos^2 x}{1 + sin\, y}\right) \leq \left(\frac{\pi^2}{4} + \frac{cos^2 x}{cos^2 y}\right)^k$$

Proposed by Stefan Andrei Mihalcea-Romania

Solution by proposer

First, Milne Inequality is used:

Let $w_j(j = \overline{1,n}) > 0$, *with sum* 1; $P_j \in [0,1]$ $(j = \overline{1,n})$, *then*

$$\left(\sum_{j=1}^{n}\frac{w_j}{1 - p_j}\right)\left(\sum_{j=1}^{n}\frac{w_j}{1 + p_j}\right) \leq \left(\sum_{j=1}^{n}\frac{w_j}{1 - p_j^2}\right)^{2 - \min\limits_{1\leq j\leq n} w_j}$$

Let's take $\begin{cases} w_1 = \sin^2 x, where\ x, y \in \left(0, \frac{\pi}{2}\right) \\ w_2 = \cos^2 x \\ p_1 = \cos y \\ p_2 = \sin y \end{cases}$

$$\Rightarrow \left(\frac{\sin^2 x}{1 - \cos y} + \frac{\cos^2 x}{1 - \sin y}\right)\left(\frac{\sin^2 x}{1 + \cos y} + \frac{\cos^2 x}{1 + \sin y}\right) \leq \left(\frac{\sin^2 x}{\sin^2 y} + \frac{\cos^2 x}{\cos^2 y}\right)^{2 - \min\limits_{1\leq j\leq n} w_j}$$

Jordan

But $\forall\, t \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{2t}{\pi} \le \sin t \le t$

$$\Rightarrow \left(\frac{\sin^2 x}{1-\cos y} + \frac{\cos^2 x}{1-\sin y}\right)\left(\frac{\sin^2 x}{1+\cos y} + \frac{\cos^2 x}{1+\sin y}\right) \le \left(\frac{\pi^2}{4} + \frac{\cos^2 x}{\cos^2 y}\right)^k$$

where $k = 2 - \min\{\sin^2 x, \cos^2 x\}$

JP. 133. Let a, b, c be positive real numbers. Prove that:

$$\sqrt[8]{\frac{a^8+b^8}{2}} + \sqrt[8]{\frac{b^8+c^8}{2}} + \sqrt[8]{\frac{c^8+a^8}{2}} \le (a+b+c)^{10}\left(\frac{1}{9a} + \frac{1}{9b} + \frac{1}{9c}\right)^9$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

- We have: $a^8 + b^8 = (a^4+b^4)^2 - (a^2b^2\sqrt{2})^2 = (a^4 - a^2b^2\sqrt{2} + b^4)(a^4 + a^2b^2\sqrt{2} + b^4)$

$$\Leftrightarrow a^8 + b^8 = \left(a^2 - \sqrt{2-\sqrt{2}}ab + b^2\right)\left(a^2 + \sqrt{2-\sqrt{2}}ab + b^2\right)\left(a^2 - \sqrt{2+\sqrt{2}}ab + b^2\right)\left(a^2 + \sqrt{2+\sqrt{2}}ab + b^2\right)$$

- Therefore, by inequality AM – GM for 8 positive real numbers.

$$\frac{a^2 + \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2+\sqrt{2}}\right)} + \frac{a^2 - \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2+\sqrt{2}}\right)} + \frac{a^2 - \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2-\sqrt{2}}\right)} + \frac{a^2 + \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2-\sqrt{2}}\right)} + b + b + b + b \ge$$

$$\ge 8 \cdot \sqrt[8]{\frac{a^2 + \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2+\sqrt{2}}\right)} \cdot \frac{a^2 - \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2+\sqrt{2}}\right)} \cdot \frac{a^2 - \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2-\sqrt{2}}\right)} \cdot \frac{a^2 + \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2-\sqrt{2}}\right)} \cdot b \cdot b \cdot b \cdot b} \Leftrightarrow$$

$\Leftrightarrow \frac{8a^2}{b} - 12a + 12b \ge 8\sqrt[8]{\frac{a^8+b^8}{2}}$. **Similar:** $\frac{8b^2}{c} - 12b + 12c \ge 8 \cdot \sqrt[8]{\frac{b^8+c^8}{2}}$; $\frac{8c^2}{a} - 12c + 12a \ge 8 \cdot \sqrt[8]{\frac{c^8+a^8}{2}}$

$$\Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \sqrt[8]{\frac{a^8+b^8}{2}} + \sqrt[8]{\frac{b^8+c^8}{2}} + \sqrt[8]{\frac{c^8+a^8}{2}} \quad (1)$$

- By inequality AM – GM. We have:

$$(a+b+c)^{10}\left(\frac{1}{9a} + \frac{1}{9b} + \frac{1}{9c}\right)^9 \ge \frac{1}{9^9}(a+b+c)^{10}\left(3 \cdot \sqrt[3]{\frac{1}{abc}}\right)^9 = \frac{(a+b+c)^{10}}{3^9 \cdot (abc)^3} \quad (2)$$

- Other: $(a+b+c)^6 = [(a^2+b^2+c^2) + (ab+bc+ca) + (ab+bc+ca)]^3 \ge$

$\ge 27(a^2+b^2+c^2)(ab+bc+ca)^2 \ge 27(a^2+b^2+c^2) \cdot 3abc(a+b+c)$

$$\Rightarrow (a+b+c)^6 \ge 81abc(a+b+c)(a^2+b^2+c^2) \Leftrightarrow \frac{(a+b+c)^{10}}{3^9(abc)^3} \ge \frac{(a^2+b^2+c^2)^2}{3abc} \quad (3)$$

- Let (2), (3): $\Rightarrow (a+b+c)^{10}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9 \ge \frac{(a^2+b^2+c^2)^2}{3abc}$ ***(4)***

- Let (1), (4). We need to prove:

$$\frac{(a^2+b^2+c^2)^2}{3abc} \ge \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \Leftrightarrow \frac{(a^2+b^2+c^2)^2}{3abc} \ge \frac{ab^3+bc^3+ca^3}{abc} \Leftrightarrow$$

$$\Leftrightarrow (a^2+b^2+c^2)^2 \ge 3(ab^3+bc^3+ca^3) \Leftrightarrow \frac{1}{2}\sum(a^2-ac+2ab-b^2-bc)^2 \ge 0 \quad \textit{(True)}$$

- Therefore: $\sqrt[8]{\frac{a^8+b^8}{2}}+\sqrt[8]{\frac{b^8+c^8}{2}}+\sqrt[8]{\frac{c^8+a^8}{2}} \le (a+b+c)^{10}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9$ ***and we get the result.***

Solution 2 by Michael Sterghiou-Greece

$$a,b,c>0 \to \sum_{cyc}\sqrt[8]{\frac{a^8+b^8}{2}} \le (a+b+c)^{10}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9 \quad (1)$$

Let $(p,q,r)=\left(\sum_{cyc}a,\sum_{cyc}ab,abc\right)$***. WLOG we can assume*** $p=1$ ***[(1) homogeneous].***

The function $f(x)=x^{\frac{1}{8}}$ ***is concave so by Jensen's inequality*** $\sum_{cyc}\sqrt[8]{\frac{a^8+b^8}{2}} \le 3\sqrt[8]{\frac{a^8+b^8+c^8}{3}}$

which must be $\le \left(\frac{q}{9r}\right)^9$***. Expanding*** $\sum_{cyc}a^8$ ***we get*** $\sum_{cyc}a^8 = 2q^4-16q^3+24q^2r+$

$+20q^2-8qb^2-32qr-8q+12r^2+8r+1=f(q,r)$***. It suffices to prove that***

$$3^{79}f(q,r) \le \left(\frac{1}{r}\right)^{24} \text{ because } 3r^{\frac{2}{3}} \le q \text{ or } 3^{79}f(q,r)-\left(\frac{1}{r}\right)^{24} \le 0 \quad (2)$$

Let (2) be a function of r ***with*** q ***as parameter; call it*** $g(r)$; $g'(r)>0$ ***as***

$f'(r)=24q^2-16qr-32q+24r+8$ ***and*** $f''(r)=-16q+24>0$ ***(because***

$0<q\le\frac{1}{3}$***)*** $\to f'(r)\uparrow\to f'(r)>f'(0)=24q^2-32q+8=5(q)$

$s'(q)=48q-32<0\to s(q)\downarrow\to s(q)\ge s\left(\frac{1}{3}\right)=0\to f'(r)\ge 0$ ***and as*** $\left(-\frac{1}{r^{24}}\right)'>0$

$g(r)$ ***is increasing function of*** r***. We will use V. Cîrtoaje theorem that with*** p,q ***fixed*** r ***is maximal when*** $a=b$ ***assuming WLOG*** $a\le b\le c$***.*** $g(r)\le g(r_{max})$ ***which needs to be*** ≤ 0***. Therefore we have to prove that*** $h(a)=[2a^8+(1-2a)^8]\cdot a^{48}\cdot$

$\cdot(1-2a)^{24}-\frac{1}{3^{79}}\le 0$ ***with*** $0\le a\le\frac{1}{3}$ $\quad h'(a)=16a^{47}(2a-1)(3a-1)\cdot\omega(a)$,

$$\omega(a)=860a^8-3385a^7+5689a^6-5809a^5+3589a^4-1417a^3+349a^2-49a+3$$

We can observe that all even order derivations of $\omega(a)$ are ≥ 0 and all of odd order increasing with one root and a $min > 0$ for the $n-1$ derivative.

(i.e. $h^{(6)}(a) > 0$, $h^5(a) \uparrow$, $h^4(a)$ has min and is ≥ 0). Hence $h'(a) \geq 0$ because

$$2a-1 \leq 0, 3a-1 \leq 0, \omega(a) \geq 0 \to h(a) \uparrow \to h(a) < h\left(\frac{1}{3}\right) = 0$$

The proof is complete.

JP.134. Let $a, b, c > 0$ such that: $a^2+b^2+c^2 = 3abc$. Find the maximum value of:

$$P = \frac{ab}{2a^6-a^5+b^4+a^2+1} + \frac{bc}{2b^6-b^5+c^4+b^2+1} + \frac{ca}{2c^6-c^5+a^4+c^2+1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

*** We have** $2a^6-a^5-3a^3+a^2+1 = 2a^5(a-1)+a^4(a-1)+a^3(a-1)-$

$$-2a^2(a-1)-a(a-1)-(a-1) = (a-1)(2a^5+a^4+a^3-2a^2-a-1)$$

$$= (a-1)\Big(2a^4(a-1)+3a^3(a-1)+4a^2(a-1)+2a(a-1)+(a-1)\Big)$$

$$= (a-1)^2(2a^4+3a^3+4a^2+2a+1) \geq 0 \text{ (because } a > 0 \text{ and } (a-1)^2 \geq 0)$$

$$\Rightarrow 2a^6-a^5-3a^3+a^2+1 \geq 0 \Leftrightarrow 2a^6-a^5+a^2+1 \geq 3a^3 \Leftrightarrow$$

$$\Leftrightarrow 2a^6-a^5+b^4+a^2+1 \geq 3a^3+b^4$$

$$\Leftrightarrow \frac{1}{2a^6-a^5+b^4+a^2+1} \leq \frac{1}{3a^3+b^2} \Leftrightarrow \frac{ab}{2a^6-a^5+b^4+a^2+1} \leq \frac{ab}{3a^3+b^4} \quad (1)$$

- By AM-GM inequality we have:

$$3a^3+b^4 = a^3+a^3+a^3+b^4 \geq 4\sqrt[4]{a^3\cdot a^3\cdot a^3\cdot b^4} = 4\sqrt[4]{a^9\cdot b^4} = 4a^2b\sqrt[4]{a} \Leftrightarrow$$

$$\Leftrightarrow \frac{ab}{3a^3+b^4} \leq \frac{ab}{4a^2b\sqrt[4]{a}} = \frac{1}{4a\sqrt[4]{a}}$$

- Hence (1) and AM-GM inequality:

$$\Rightarrow \frac{ab}{2a^6-a^5+b^4+a^2+1} \leq \frac{1}{4a\sqrt[4]{a}} \leq \frac{1}{4a}\cdot\frac{1}{4}\left(\frac{1}{a}+1+1+1\right) = \frac{1}{16a}\left(\frac{1}{a}+3\right)$$

$$+ \textit{Similar: } \frac{bc}{2b^6-b^5+c^4+b^2+1} \leq \frac{1}{16b}\left(\frac{1}{b}+3\right); \frac{ca}{2c^6-c^5+a^4+c^2+1} \leq \frac{1}{16c}\left(\frac{1}{c}+3\right)$$

$$- \textit{Hence: } \Rightarrow P = \frac{ab}{2a^6-a^5+b^4+a^2+1} + \frac{bc}{2b^6-b^5+c^4+b^2+1} + \frac{ca}{2c^6-c^5+a^4+c^2+1} \leq$$

$$\le \frac{1}{16a}\left(\frac{1}{a}+3\right)+\frac{1}{16b}\left(\frac{1}{b}+3\right)+\frac{1}{16c}\left(\frac{1}{c}+3\right)$$

$$\Leftrightarrow P \le \frac{1}{16}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad \textit{(2)}$$

– Other $a^2+b^2+c^2=3abc$ *and inequality:* $(x+y+z) \ge \sqrt{3(xy+yz+zx)}$ *that:*

$$x=\frac{a}{bc}, y=\frac{b}{ca}, z=\frac{c}{ab}$$

We have:

$$3=\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab} \ge \sqrt{3\left(\frac{a}{bc}\cdot\frac{b}{ca}+\frac{b}{ca}\cdot\frac{c}{ab}+\frac{c}{ab}\cdot\frac{a}{bc}\right)}=\sqrt{3\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)} \Leftrightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le 3 \quad \textit{(3)}$$

- Let (3) and AM-GM inequality:

$$3 \ge \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\left(\frac{1}{a^2}+1\right)+\left(\frac{1}{b^2}+1\right)+\left(\frac{1}{c^2}+1\right)-3 \ge \frac{2}{a}+\frac{2}{b}+\frac{2}{c}-3 \Leftrightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le 3 \quad \textit{(4)}$$

- Let (2), (3), (4): $\Rightarrow P \le \frac{1}{16}\cdot 3+\frac{3}{16}\cdot 3=\frac{12}{16}=\frac{3}{4} \Rightarrow P \le \frac{3}{4} \Rightarrow P_{\max}=\frac{3}{4}$

+ Equality occurs if: $\begin{cases} a,b,c>0; a^2+b^2+c^2=3abc \\ a-1=b-1=c-1=0 \\ a^3=b^4; b^3=c^4; c^3=a^4 \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1; \frac{a}{bc}=\frac{b}{ca}=\frac{c}{ab} \end{cases} \Leftrightarrow a=b=c=1.$

Maximum value of P *is* $\frac{3}{4}$ *when* $a=b=c=1$.

JP.135. Let x, y, z be positive real numbers such that: $x+y+z=3$. Prove that:

$$\frac{x^3y^3}{x^4+y^3-x+2}+\frac{y^3z^3}{y^4+z^3+y+2}+\frac{z^3x^3}{z^4+x^3-z+2} \le \frac{x^4+y^4+z^4+3xyz}{6}.$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Marian Ursărescu – Romania

First, we show this: $x^4-x+2 \ge x^3+1, \forall x \in \mathbb{R}$ *(1)*

(1) $\Leftrightarrow x^4-x^3-x+1 \ge 0 \Leftrightarrow x^3(x-1)-(x-1) \ge 0 \Leftrightarrow (x-1)(x^3-1) \ge 0 \Leftrightarrow$

$\Leftrightarrow (x-1)^2(x^2+x+1) \ge 0$ *(true). From (1)* $\Rightarrow \frac{1}{x^4+y^3-x+2} \le \frac{1}{x^3+y^3+1} \Rightarrow$ *inequality*

becomes: $\sum \frac{x^3y^3}{x^3+y^3+1} \le \frac{x^4+y^4+z^4+3xyz}{6}$ *(2)*

From AM-GM $\Rightarrow x^3+y^3+1 \ge 3xy$ *(3)*

From (2)+(3) we must show this: $\sum\frac{x^2y^2}{3}\le\frac{x^4+y^4+z^4+3xyz}{6}\Leftrightarrow\sum x^2y^2\le\frac{x^4+y^4+z^4+3xyz}{2}\Leftrightarrow$

$\Leftrightarrow x^4+y^4+z^4+3xyz-2(x^2y^2+x^2z^2+y^2z^2)\ge 0$ ***(4). Now, let*** $x=\frac{3a}{a+b+c}$,

$y=\frac{3b}{a+b+c}, z=\frac{3c}{a+b+c}$ ***with*** $a,b,c>0$.

$$\textbf{(4)}\Leftrightarrow\frac{81(a^4+b^4+c^4)}{(a+b+c)^4}+\frac{81abc}{(a+b+c)^3}-\frac{2\cdot 81(a^2b^2+b^2c^2+c^2a^2)}{(a+b+c)^4}\ge 0$$

$$\Leftrightarrow\frac{a^4+b^4+c^4}{(a+b+c)^4}+\frac{abc}{(a+b+c)^3}-\frac{2(a^2b^2+a^2c^2+b^2c^2)}{(a+b+c)^4}\ge 0\Leftrightarrow$$

$$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c)-2(a^2b^2+b^2c^2+c^2a^2)\ge 0\ \textbf{(5)}$$

Now, use Cârtoaje theorem: let $f_4(a,b,c)$ ***be a symmetric polynomial of degree four. Then:***

$f_4(a,b,c)\ge 0,\forall a,b,c\ge 0\Leftrightarrow f_4(a,1,1)\ge 0,\forall a\ge 0$. ***Let*** $f_4(a,b,c)=a^4+b^4+c^4+$
$+abc(a+b+c)-2(a^2b^2+b^2c^2+c^2a^2)$

$$f_4(a,1,1)=a^4+2+a(a+2)-2(2a^2+c)=a^4+2+a^2+2a-4a^2-2=$$
$$=a^4-3a^2+2a=a(a^3-3a+2)=a(a-1)^2(a+2)\ge 0,\forall a\ge 0\Rightarrow f_4(a,b,c)\ge 0$$

$\Rightarrow$ ***(5) its true.***

SP.121. Let x,y,z be positive real numbers such that: $x+y+z=3$. Prove that:

$$\frac{x^4}{5-3\sqrt[3]{y}}+\frac{y^4}{5-3\sqrt[3]{z}}+\frac{z^4}{5-3\sqrt[3]{x}}+\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}\ge 3$$

Proposed by Hoang Le Nhat Tung – Hanoi – VietNam

Solution by proposer

- Because $\begin{cases}x,y,z>0\\x+y+z=3\end{cases}\Rightarrow 0<x,y,z<3\Rightarrow 5-3\sqrt[3]{x}>0; 5-2\sqrt[3]{y}>0; 5-3\sqrt[3]{z}>0$

- Be Cauchy – Schwarz inequality we have:

$$\frac{x^4}{5-3\sqrt[3]{y}}+\frac{y^4}{5-3\sqrt[3]{z}}+\frac{z^4}{5-3\sqrt[3]{x}}+\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}\ge\frac{(x^2+y^2+z^2)^2}{15-3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}+\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}\quad\textbf{(1)}$$

+ ***Other, by AM-GM inequality:*** $\begin{cases} \sqrt{x}+\sqrt{x}+x^2 \geq 3\sqrt[3]{\sqrt{x}\cdot\sqrt{x}\cdot x^2} = 3x \\ \sqrt{y}+\sqrt{y}+y^2 \geq 3\sqrt[3]{\sqrt{y}\cdot\sqrt{y}\cdot y^2} = 3y \\ \sqrt{z}+\sqrt{z}+z^2 \geq 3\sqrt[3]{\sqrt{z}\cdot\sqrt{z}\cdot z^2} = 3z \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 2\sqrt{x} \geq 3x - x^2 \\ 2\sqrt{y} \geq 3y - y^2 \\ 2\sqrt{z} \geq 3z - z^2 \end{cases} \Rightarrow$$

$$\Rightarrow 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) \geq 3(x+y+z) - (x^2+y^2+z^2) = (x+y+z)^2 - (x^2+y^2+z^2)$$

$\Leftrightarrow 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) \geq 2(xy+yz+zx) \Leftrightarrow \sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx$ ***(because*** $x+y+z+3$***)*** ***(2)***

$$\begin{cases} \sqrt[3]{x}+\sqrt[3]{x}+\sqrt[3]{x}+x^2+x^2 \geq 5\sqrt[5]{x^5} = 5x \\ \sqrt[3]{y}+\sqrt[3]{y}+\sqrt[3]{y}+y^2+y^2 \geq 5\sqrt[5]{y^5} = 5y \\ \sqrt[3]{z}+\sqrt[3]{z}+\sqrt[3]{z}+z^2+z^2 \geq 5\sqrt[5]{z^5} = 5z \end{cases} \Leftrightarrow \begin{cases} 3\sqrt[3]{x} \geq 5x - 2x^2 \\ 3\sqrt[3]{y} \geq 5y - 2y^2 \\ 3\sqrt[3]{z} \geq 5z - 2z^2 \end{cases}$$

$$\Rightarrow 3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right) \geq 5(x+y+z) - 2(x^2+y^2+z^2) = 15 - 2(x^2+y^2+z^2)$$

$$\Leftrightarrow 15 - 3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right) \leq 2(x^2+y^2+z^2) \Leftrightarrow \frac{(x^2+y^2+z^2)^2}{15-3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)} \geq \frac{x^2+y^2+z^2}{2} \quad \textbf{(3)}$$

- Let (1), (2), (3):

$$\Rightarrow \frac{x^4}{5-3\sqrt[3]{y}} + \frac{y^4}{5-3\sqrt[3]{z}} + \frac{z^4}{5-3\sqrt[3]{x}} + \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2} \geq \frac{x^2+y^2+z^2}{2} + \frac{xy+yz+zx}{2} \quad \textbf{(4)}$$

+ ***We have:*** $\frac{x^2+y^2+z^2}{2} + \frac{xy+yz+zx}{2} = \frac{(x+y+z)^2-(xy+yz+zx)}{2} \geq \frac{(x+y+z)^2 - \frac{(x+y+z)^2}{3}}{2} = \frac{(x+y+z)^2}{3} = \frac{3^2}{3} = 3$ ***(5)***

- Let (4), (5): $\Rightarrow \frac{x^4}{5-3\sqrt[3]{y}} + \frac{y^4}{5-3\sqrt[3]{z}} + \frac{z^4}{5-3\sqrt[3]{x}} + \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2} \geq 3$ ***and we get the result.***

+ ***Equality occurs if*** $x = y = z = 1$.

SP.122. If $z_1, z_2, z_3 \in \mathbb{C}$ are different in pairs and $|z_1| = |z_2| = |z_3| = 1$ then:

$$|z_1 - z_3| + |z_2 - z_3| \leq 3 + |z_1 + z_2|$$

Proposed by Marian Ursărescu – Romania

Solution by Omran Kouba-Damascus-Syria

We will prove the stronger inequality: $|z_1 - z_3| + |z_2 - z_3| \leq 3 + \frac{1}{2}|z_1 + z_2|$ ***(1)***

Consider x and y from $\left[0,\frac{\pi}{2}\right]$ such that $\frac{z_1}{z_3}=e^{4ix}$ and $\frac{z_2}{z_3}=e^{4iy}$. With this notation (1) is equivalent to $2\sin(2x)+2\sin(2y)\le 3+|\cos(2x+2y)|$ ***(2)***

Now, with $z=x+y\in[0,\pi]$, clearly we have:

$$2\sin(2x)+2\sin(2y)=4\sin(z)\cos(x-y)\le 4\sin z \quad (3)$$

and $(3+|\cos 2z|)^2-(4\sin z)^2=9+\cos^2 2z+6|\cos 2z|-8(1-\cos 2z)=$ $=(1+\cos 2z)^2+6(|\cos 2z|+\cos 2z)\ge 0$. *But, $\sin z\ge 0$, so the previous inequality implies: $4\sin z\le 3+|\cos 2z|$, thus (3) implies (2), and this is equivalent to (1). The proof of the stronger inequality (1) is completed.*

SP.123. Let $A\in M_n(\mathbb{R})$ be a symmetric and invertible matrix.

Prove that: $det(A^2+A^{-2}+2I_n)\ge 4^n$

Proposed by Marian Ursărescu – Romania

Solution by proposer

A symmetric and invertible ⇒ eigen values $\lambda_1,\lambda_2,\dots\lambda_n\in\mathbb{R}^*$. Let be the polynomial $p(x)=x^2+1\Rightarrow \det p(A)=p(\lambda_1)p(\lambda_2)-p(\lambda_n)=(\lambda_1^2+1)(\lambda_2^2+1)\dots.(\lambda_n^2+1)$ ***(1)***

A^{-1} has the eigen values

$$\lambda_1^{-1},\lambda_2^{-1},\dots,\lambda_n^{-1}\in\mathbb{R}^*\Rightarrow \det p(A^{-1})=(\lambda_1^{-2}+1)(\lambda_2^{-2}+1)\dots(\lambda_n^{-1}+1)$$

$$\Rightarrow \det p(A^{-1})=(\lambda_1^{-2}+1)(\lambda_2^{-2}+1)\dots(\lambda_n^{-1}+1)=$$

$$=\frac{(\lambda_1^2+1)(\lambda_2^2+1)\dots(\lambda_n^2+1)}{\lambda_1^2\lambda_2^2\dots\lambda_n^2} \quad (2)$$

But $A^2+A^{-2}+2I_n=(A^2+I_n)(A^{-2}+I_n)\Rightarrow \det(A^2+A^{-2}+2I_n)\Rightarrow$

$$\det(A^2+A^{-2}+2I_n)=\det(A^2+I_n)\cdot\det(A^{-2}+I_n) \quad (3)$$

From (1)+(2)+(3) $\Rightarrow \det(A^2+A^{-2}+2I_n)=$

$$=\left[\frac{(\lambda_1^2+1)(\lambda_2^2+1)\dots(\lambda_n^2+1)}{\lambda_1\lambda_2\dots\lambda_n}\right]^2 \quad (4)$$

But $\lambda_k^2+1\ge 2\lambda_k$ ***(5)***

But (4) + (5) $\Rightarrow \det(A^2+A^{-2}+2I_n)\ge\left(\frac{2^n\lambda_1\lambda_2\dots\lambda_n}{\lambda_1\lambda_2\dots\lambda_n}\right)^2=4^n$

SP.124. Let a, b, c be the side lenghts of a triangle ABC with inradius r and circumradius R. Prove:

$$\frac{3}{2} \le \frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \le \frac{2R-r}{2r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{a^2}{b^2+c^2} \overset{A-G}{\le} \sum \frac{a^2}{2bc} = \frac{\sum a^3}{2abc} = \frac{3abc + 2s(\sum a^2 - \sum ab)}{2 \cdot 4Rrs} =$$

$$= \frac{2s(s^2-12Rr-3r^2)+12Rrs}{8Rrs} = \frac{2s(s^2-6Rr-3r^2)}{8Rrs} = \frac{s^2-6Rr-3r^2}{4Rr} \le \frac{2R-r}{2r} \Leftrightarrow$$

$$\Leftrightarrow s^2 - 6Rr - 3r^2 \le 2R(2R-r) = 4R^2 - 2Rr \Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2 \to \textit{true}$$

(Gerretsen) $\Rightarrow \sum \frac{a^2}{b^2+c^2} \le \frac{2R-r}{2r}$. *Also,* $\sum \frac{a^2}{b^2+c^2} \overset{Nesbitt}{\ge} \frac{3}{2}$ *(Done).*

SP.125. Let triangle ABC have exradii r_a, r_b, r_c, altitudes h_a, h_b, h_c and a, b, c be the lengths of the sides. Prove that:

$$\left(\frac{h_a}{r_a}\right)^2 + \left(\frac{h_b}{r_b}\right)^2 + \left(\frac{h_c}{r_c}\right)^2 \le \frac{1}{2}\left(\frac{a^4+b^4}{c^4} + \frac{b^4+c^4}{a^4} + \frac{c^4+a^4}{b^4}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{h_a}{r_a}\right)^2 = \left(\frac{2\Delta}{a} \times \frac{s-a}{\Delta}\right)^2 = 4\frac{(s-a)^2}{a^2} \textit{ etc} \therefore \textit{given inequality becomes:}$$

$$\sum a^4b^4(a^4+b^4) \overset{(1)}{\ge} 8a^2b^2c^2\{b^2c^2(s-a)^2 + c^2a^2(s-b)^2 + a^2b^2(s-c)^2\}.$$

Let $s-a = x, s-b = y,$

$s - c = z \Rightarrow s = \sum x \therefore a = y+z, b = z+x, c = x+y \ (x,y,z > 0).$

Then (1) becomes:

$$\sum_{cyc}[\{(y+z)(z+x)\}^4\{(y+z)^4 + (z+x)^4\}] \ge 8(x+y)^2(y+z)^2(z+x)^2 \cdot$$

$$\cdot\left[\sum_{cyc}\{x^2(z+x)^2(x+y)^2\}\right] \Leftrightarrow 2\sum x^{12} + 12\left(\sum x''y + \sum xy''\right) +$$

$$+26\left(\sum x^{10}y^2 + \sum x^2y^{10}\right) + 48xyz\left(\sum x^9\right) + 28\left(\sum x^9y^3 + \sum x^3y^9\right) +$$

$$+64xyz\left(\sum x^8y + \sum xy^8\right) + 25\left(\sum x^8y^4 + \sum x^4y^8\right) + 40xyz\left(\sum x^7y^2 + \sum x^2y^7\right)$$

$$+72xyz\left(\sum x^6y^3 + \sum x^3y^6\right) + 32\left(\sum x^7y^5 + \sum x^5y^7\right) + 40\sum x^6y^6 +$$

$$+144x^2y^2z^2\left(\sum x^3y^3\right) + 160xyz\left(\sum x^5y^4 + \sum x^4y^5\right) +$$

$$+36x^2y^2z^2\left(\sum x^4y^2 + \sum x^2y^4\right) \overset{(2)}{\geq} 4x^2y^2z^2\left(\sum x^6\right) + 80x^2y^2z^2\left(\sum x^5y + \sum xy^5\right)$$

$$+224x^3y^3z^3\left(\sum x^3\right) + 312x^3y^3z^3\left(\sum x^2y + \sum xy^2\right) + 636x^4y^4z^4$$

$$2\sum x^9 = \sum(x^9+y^9) \overset{Chebyshev}{\geq} \sum\frac{1}{2}(x^2+y^2)(x^8+y^7) \overset{A-G}{\geq} \sum xy(x^7+y^7) =$$

$$= \sum y(x^8+z^8) \overset{Chebyshev}{\geq} \sum\frac{1}{2}y(x^2+z^2)(x^6+z^6) \overset{A-G}{\geq} \sum xyz(x^6+z^6) =$$

$$= 2xyz(\textstyle\sum x^6) \Rightarrow 4xyz(\sum x^9) \geq 4x^2y^2z^2(\sum x^6) \quad \textbf{(a)}$$

Again, $2\sum x^6y^6 = \sum(x^6y^6 + y^6z^6) \underset{(i)}{\overset{A-G}{\geq}} 2\sum x^3z^3y^6 = 2\sum x^3y^3z^3(\sum x^3) \Rightarrow$

$$\Rightarrow 40\textstyle\sum x^6y^6 \geq 40x^3y^3z^3(\sum x^3) \quad \textbf{(b)}$$

Also, $32(\sum x^7y^5 + \sum x^5y^7) \overset{A-G}{\geq} 64\sum x^6y^6 \underset{(c)}{\overset{by\ (i)}{\geq}} 64x^3y^3z^3(\sum x^3)$

Also, $25(\sum x^8y^4 + \sum x^4y^8) \overset{A-G}{\geq} 50\sum x^6y^6 \underset{(d)}{\overset{by\ (i)}{\geq}} 50x^3y^3z^3(\sum x^3)$

Also, $28(\sum x^9y^3 + \sum x^3y^9) \overset{A-G}{\geq} 56\sum x^6y^6 \underset{(c)}{\overset{by\ (i)}{\geq}} 56x^3y^3z^3(\sum x^3)$

Lastly, $7(\sum x^{10}y^2 + \sum x^2y^{10}) \overset{A-G}{\geq} 14\sum x^6y^6 \underset{(f)}{\overset{by\ (i)}{\geq}} 14x^3y^3z^3(\sum x^3)$

Now, $144x^2y^2z^2(\sum x^3y^3) = 72x^2y^2z^2(2\sum x^3y^3) \overset{(g)}{\geq} 72x^2y^2z^2\cdot xyz(\sum x^2y + \sum xy^2)$

$$\left(\because 2\sum u^3 \geq \sum u^2v + \sum uv^2\right)$$

$$= 72x^3y^3z^3\left(\sum x^2y + \sum xy^2\right)$$

Also, $160xyz(\sum x^5y^4 + \sum x^4y^5) = 160xyz\sum\{x^5(y^4 + z^4)\} \overset{A-G}{\geq} 320xyz(\sum x^5y^2z^2) =$

$$= 320x^3y^3z^2\left(\sum x^3\right) = 160x^3y^3z^3\left(2\sum x^3\right) \overset{(h)}{\geq} 160x^3y^3z^3\left(\sum x^2y + \sum xy^2\right)$$

Lastly,

$$44xyz\left(\sum x^6y^3 + \sum x^3y^6\right) = 44xyz\left\{\sum y^3\left(x^6 + z^6\right)\right\} \overset{(i)}{\geq} 44xyz\sum\{y^3x^2z^2(x^2 + z^2)\}$$

$$= 44x^3y^3z^3\left(\sum x^2y + \sum xy^2\right)$$

SP.126. Let m_a, m_b, m_c the lenghts of the medians of a triangle ABC with circumradius R. Prove that:

$$\frac{1}{m_am_b} + \frac{1}{m_bm_c} + \frac{1}{m_cm_a} + \frac{3}{m_am_b + m_bm_c + m_cm_a} \geq 4 \cdot \frac{\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}}{m_a + m_b + m_c}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Bogdan Fustei-Romania

the medians m_a, m_b, m_c of $\Delta\, ABC$ can be also the sides of a triangle of medians

denoted m_{a1}, m_{b1}, m_{c1}. But $\left.\begin{array}{l} m_{a1} = \frac{3}{4}a \\ m_{b1} = \frac{3}{4}b \\ m_{c1} = \frac{3}{4}c \end{array}\right\}$ ***we will write the inequality from enunciation***

for m_{a1}, m_{b1}, m_{c1}: $\frac{1}{\frac{9}{16}ab} + \frac{1}{\frac{9}{16}bc} + \frac{1}{\frac{9}{16}ac} + \frac{3}{\frac{9}{16}(ab+bc+ac)} \geq \frac{4\cdot\frac{4}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{\frac{3}{4}(a+b+c)}$

$$\frac{16}{9}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}\right) + \frac{16}{3}\cdot\frac{1}{ab + bc + ac} \geq \frac{\frac{16}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}{\frac{3}{4}(a + b + c)}$$

$$\frac{16}{9}\cdot\sum\frac{1}{ab} + \frac{16}{3}\cdot\frac{1}{ab + bc + ac} \geq \frac{16}{3}\cdot\frac{4}{3}\cdot\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{a + b + c}\Bigg| : \frac{16}{3}$$

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$$\frac{1}{3}\cdot\sum\frac{1}{ab}+\frac{1}{ab+bc+ac}\geq\frac{4}{3}\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{a+b+c}$$

But $\sum\frac{1}{ab}=\frac{1}{2Rr};\Rightarrow\frac{1}{6Rr}+\frac{1}{ab+bc+ac}\geq\frac{4}{3}\cdot\frac{\frac{ab+bc+ac}{abc}}{2s}$

$$\frac{1}{6Rr}+\frac{1}{ab+bc+ac}\geq\frac{4}{3}\cdot\frac{ab+bc+ac}{4RS\cdot 2s}[abc=4RS]$$

$$\frac{1}{6Rr}+\frac{1}{ab+bc+ac}\geq\frac{ab+bc+ac}{6RSs}$$

$$\frac{6Rr+ab+bc+ac}{6Rr(ab+bc+ac)}\geq\frac{ab+bc+ac}{6Rrs^2}\Leftrightarrow\frac{6Rr+ab+bc+ac}{ab+bc+ac}\geq\frac{ab+bc+ac}{s^2}$$

$s^2(6Rr+ab+bc+ac)\geq(ab+bc+ac)^2$. ***But*** $ab+bc+ac=2R(h_a+h_b+h_c)$

$$2Rs^2(h_a+h_b+h_c+3r)\geq 4R^2(h_a+h_b+h_c)^2$$

$\frac{s^2}{2R}\geq\frac{(h_a+h_b+h_c)^2}{h_a+h_b+h_c+3r}$; ***But*** $h_a+h_b+h_c=r\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)$

$$(h_a+h_b+h_c)^2=r^2\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2$$

$$3r+h_a+h_b+h_c=r\left(9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)$$

$$\frac{s^2}{2R}\geq\frac{r^2\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{r\left(9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)}\Rightarrow\frac{s^2}{2Rr}\geq\frac{\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}$$

$$\sum\frac{h_a}{r_a}=\sum\frac{bc(s-a)}{2Rrs}=\frac{s(s^2+4Rr+r^2)-12Rrs}{2Rrs}=\frac{s^2-8Rr+r^2}{2Rr}\overset{(1)}{=}$$

$$\frac{\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}=\frac{(s^2+4Rr+r^2)^2}{4R^2r^2}\cdot\frac{2Rr}{(s^2+10Rr+r^2)}=\frac{(s^2+4Rr+r^2)^2}{2Rr(s^2+16Rr+r^2)}\leq$$

$$\leq\frac{s^2}{2Rr}\Leftrightarrow s^4+s^2(10Rr+r^2)\geq s^4+r^2(4R+r)^2+s^2(8Rr+2r^2)\Leftrightarrow$$

$\Leftrightarrow s^2(2R-r)\geq r(4R+r)^2\overset{(2)}{=}s^2\geq 16Rr-5r^2$ ***– Gerretsen's inequality*** $\Rightarrow$

$\Rightarrow s^2(2R-r)\geq(16Rr-5r^2)(2R-r)$ ***– true. We will prove that:***

$$(16Rr-5r^2)(2R-r)\geq r(4R+r)^2$$

$$r(16R-5r)(2R-r)\ge r(4R+r)^2 \Leftrightarrow 32R^2-10Rr-16Rr+5r^2\ge 16R^2+r^2+8Rr$$

$$16R^2+4r^2\ge 34Rr \Rightarrow 8R^2+2r^2\ge 17Rr \Rightarrow 8R^2-17Rr+2r^2\ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R-r)\ge 0\ R-2r\ge 0 \Rightarrow R\ge 2r$ – ***Euler's inequality***

$8R>r$ – ***true. So, (2) true;*** $\Rightarrow \frac{s^2}{2Rr}\ge \frac{\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}$ – ***true*** $\Rightarrow$ ***inequality from enunciation***

is true, namely: $\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_am_c}+\frac{3}{m_am_b+m_bm_c+m_cm_a}\ge 4\cdot\frac{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}{m_a+m_b+m_c}$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{(\sum m_a)(\sum m_am_b)+3m_am_cm_c}{m_am_bm_c(\sum m_am_b)}\ge\frac{4\sum m_am_b}{m_am_bm_c(\sum m_a)}\Leftrightarrow$$

$$\Leftrightarrow\left(\sum m_am_b\right)\left(\sum m_a\right)^2+3m_am_bm_c\left(\sum m_a\right)\ge 4\left(\sum m_am_b\right)^2\Leftrightarrow$$

$$\Leftrightarrow\left(\sum m_am_b\right)\left(\sum m_a^2+2\sum m_am_b\right)+3m_am_bm_c\left(\sum m_a\right)\ge 4\left(\sum m_am_b\right)^2\Leftrightarrow$$

$\Leftrightarrow (\sum m_am_b)(\sum m_a^2)+3m_am_bm_c(\sum a)^{m_a}\ge 2(\sum m_am_b)^2$ ***(1) . We shall now prove:***

$$(\sum ab)(\sum a^2)+3abc(\sum a)\ge 2(\sum ab)^2 \quad \textit{(2)}$$

$$\Leftrightarrow 3\cdot 4Rrs(2s)\ge\left(\sum ab\right)\left(2\sum ab-\sum a^2\right)\Leftrightarrow 24Rrs^2\ge$$

$$\ge(s^2+4Rr+r^2)\{2(s^2+4Rr+r^2)-(2s^2-8Rr-2r^2)\}\Leftrightarrow 24Rrs^2\ge$$

$$\ge(s^2+4Rr+r^2)(16Rr+4r^2)\Leftrightarrow 6Rs^2\ge(s^2+4Rr+r^2)(4R+r)\Leftrightarrow$$

$$\Leftrightarrow(2R-r)s^2\ge r(4R+r)^2 \quad \textit{(3)}$$

LHS of (3) $\overset{Gerretsen}{\ge}(2R-r)(16Rr-5r^2)\overset{?}{\ge}r(4R+r)^2\Leftrightarrow 8R^2-17Rr+2r^2\overset{?}{\ge}0\Leftrightarrow$

$\Leftrightarrow(8R-r)(R-2r)\overset{?}{\ge}0\rightarrow$ ***true*** $\Rightarrow$ ***(2) is true. Let us consider a triangle with sides***

$\frac{2}{3}m_a,\frac{2}{3}m_b,\frac{2}{3}m_c$ ***and apply (2) on it. We shall have:***

$$\frac{4}{9}\sum m_am_b\cdot\frac{4}{9}\sum m_a^2+3\cdot\frac{8}{27}m_am_bm_c\cdot\frac{2}{3}\sum m_a\ge 2\cdot\frac{16}{81}\left(\sum m_am_b\right)^2\Leftrightarrow$$

$\Leftrightarrow(\sum m_am_b)(\sum m_a^2)+3m_am_bm_c(\sum m_a)\ge 2(\sum m_am_b)^2\Rightarrow$ ***(2) is true (proved)***

SP.127. Let be $A, B \in M_n(\mathbb{R})$ such that: $A^2 + B^2 = 2\sin\frac{\pi}{x}AB$.

If $AB - BA$ is invertible, then nx is an even integer.

Proposed by Marian Ursărescu – Romania

Solution by proposer

Let be the equation $x^2 - 2\sin\frac{\pi}{x}x + 1 = 0$ *which has the roots* $z_1 = \sin\frac{\pi}{x} - i\cos\frac{\pi}{x}$

$$z_2 = \sin\frac{\pi}{x} + i\cos\frac{\pi}{x}$$

$$(B - z_1A)(B - z_2A) = B^2 + z_1z_2A^2 - z_1AB - z_2BA =$$

$$= B^2 + A^2 - \left(2\sin\frac{\pi}{x} - z_2\right)AB - z_2BA =$$

$$B^2 + A^2 - 2\sin\frac{\pi}{x}AB + z_2AB - z_2BA = z_2(AB - BA)$$

$$\det\underbrace{(B - z_1A)}_{\geq 0}(B - z_2A) = \det\big(z_2(AB - BA)\big) \Rightarrow$$

$$z_2^n\det(AB - BA) \geq 0; \det(AB - BA) \neq 0 \Rightarrow$$

$$z_2^n\det(AB - BA) > 0 \Rightarrow z_2^n \in \mathbb{R} \Rightarrow$$

$$\left(\sin\frac{\pi}{x} + i\cos\frac{\pi}{x}\right)^n \in \mathbb{R} \Rightarrow \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{x}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{x}\right)\right)^n \in \mathbb{R}$$

$$\Rightarrow \cos n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) + i\sin n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) \in \mathbb{R} \Rightarrow \sin n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) = 0 \Rightarrow n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) = k\pi \Rightarrow$$

$$\Rightarrow n\left(\frac{x-2}{2}\right) = k \Rightarrow nx - 2n = 2k \Rightarrow nx = 2(n + k) \Rightarrow nx \text{ is even.}$$

SP.128. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum of the expression:

$$P = \frac{x^3}{y\left(\sqrt{2(y^4 + z^4)} + yz\right)^2} + \frac{y^3}{z\left(\sqrt{2(z^4 + x^4)} + zx\right)^2} + \frac{z^3}{x\left(\sqrt{2(x^4 + y^4)} + xy\right)^2} + \frac{\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}}{27}$$

Proposed by Hoang Le Nhat Tung – Hanoi – VietNam

Solution by proposer

- By Cauchy – Schwarz inequality we have:

$$\left(\sqrt{2(y^4 + z^4)} + 2yz\right)^2 \leq 2\left(2(y^4 + z^4) + 4y^2z^2\right) = 4(y^4 + 2y^2z^2 + z^4) = 4(y^2 + z^2)^2$$

$$\Rightarrow \sqrt{2(y^4+z^4)}+yz \le 2y^2-yz+2z^2 \Leftrightarrow \frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2} \ge \frac{x^3}{y(2y^2-yz+2z^2)^2}$$

+ Similar: $\frac{y^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2} \ge \frac{y^3}{z(2z^2-zx+2x^2)^2}$; $\frac{z^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2} \ge \frac{z^3}{x(2x^2-xy+2y^2)^2}$

- Therefore: $\frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2}+\frac{y^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2}+\frac{z^3}{x\left(\sqrt{2(x^4+y^4)}+xy\right)^2} \ge$

$$\ge \frac{x^3}{y(2y^2-yz+2z^2)^2}+\frac{y^3}{z(2z^2-zx+2x^2)^2}+\frac{z^3}{x(2x^2-xy+2y^2)^2} \quad (1)$$

- By Cauchy – Schwarz inequality:

$$\sum \frac{x^3}{y(2y^2-yz+2z^2)^2} = \sum \frac{\left(\frac{x^2}{2y^2-yz+2z^2}\right)^2}{xy} \ge \frac{\left(\sum \frac{x^2}{2y^2-yz+2z^2}\right)^2}{\sum xy} \quad (2)$$

+ Other, $\sum \frac{x^2}{2y^2-yz+2z^2} = \sum \frac{x^4}{2x^2y^2-x^2yz+2x^2z^2} \ge \frac{(\sum x^2)^2}{\sum(2x^2y^2-x^2yz+2x^2z^2)} \ge 1$

$$\Leftrightarrow (\sum x^2)^2 \ge 4\sum x^2y^2 - xyz\sum x \Leftrightarrow \sum x^4 + xyz\sum x \ge 2\sum x^2y^2 \quad (3)$$

+ By Schur and AM-GM inequality: $\sum x^2(x-y)(x-z) \ge 0 \Rightarrow \sum x^4 + xyz\sum x \ge \sum xy(x^2+y^2)$

$\sum xy(x^2+y^2) \ge \sum xy \cdot 2xy = 2\sum x^2y^2 \Rightarrow \sum x^4 + xyz\sum x \ge 2\sum x^2y^2 \Rightarrow$ ***(3) True.***

+ Let (2), (3): $\Rightarrow \sum \frac{x^3}{y(2y^2-yz+2z^2)^2} \ge \frac{1}{\sum xy}$. *Let (1):* $\Rightarrow \sum \frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2} \ge \frac{1}{\sum xy}$ ***(4)***

- By AM-GM inequality:

$$\begin{cases} \sqrt[4]{x}+\sqrt[4]{x}+\sqrt[4]{x}+\sqrt[4]{x}+x^3+x^2 \ge 6\sqrt[6]{x\cdot x^3\cdot x^2} = 6x \\ \sqrt[4]{y}+\sqrt[4]{y}+\sqrt[4]{y}+\sqrt[4]{y}+y^3+y^2 \ge 6\sqrt[6]{y\cdot y^3\cdot y^2} = 6y \\ \sqrt[4]{z}+\sqrt[4]{z}+\sqrt[4]{z}+\sqrt[4]{z}+z^3+z^2 \ge 6\sqrt[6]{z\cdot z^3\cdot z^2} = 6z \end{cases} \Leftrightarrow \begin{cases} 4\cdot\sqrt[4]{x} \ge 6x-x^2-x^3 \\ 4\cdot\sqrt[4]{y} \ge 6y-y^2-y^3 \\ 4\cdot\sqrt[4]{z} \ge 6z-z^2-z^3 \end{cases}$$

$$\Rightarrow 4\left(\sum\sqrt[4]{x}\right) \ge 6\sum x - \sum x^2 - \sum x^3 = 6\cdot 3 - (\sum x)^2 + 2\sum xy - \sum x^3 = 2\sum xy + 9 - \sum x^3 \quad (5)$$

+ Other, because $x+y+z=3; x,y,z>0 \Rightarrow \sum(x-3)(x-1)^2 \le 0 \Leftrightarrow \sum(x-3)(x^2-2x+1) \le 0$

$$\Leftrightarrow \sum x^3 - 5\sum x^2 + 7\sum x - 9 \le 0 \Leftrightarrow \sum x^3 \le 5\sum x^2 - 7\sum x + 9 = 5\cdot 3^2 - 10\sum xy - 7\cdot 3 + 9$$

$\Leftrightarrow \sum x^3 \le 33 - 10\sum xy$. *Let (5):* $\Rightarrow 4\left(\sum\sqrt[4]{x}\right) \ge 2\sum xy + 9 - (33 - 10\sum xy) \Leftrightarrow \sum\sqrt[4]{x} \ge 3\sum xy - 6$ ***(6)***

- Let (4), (6): $\Rightarrow P \ge \frac{1}{\sum xy} + \frac{3\sum xy - 6}{27} = \frac{1}{\sum xy} + \sum xy - \frac{2}{9} \ge 2\sqrt{\sum xy \cdot \frac{\sum xy}{9}} - \frac{2}{9} = \frac{2}{3} - \frac{2}{9} = \frac{4}{9}$

$\Rightarrow P \geq \frac{4}{9} \Rightarrow P_{\min} = \frac{4}{9}$. *Equality occurs if:* $\begin{cases} x = y = z > 0 \\ x + y + z = 3 \end{cases} \Leftrightarrow x = y = z = 1.$

SP.129. Let a, b, c be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{a^3}{b^2\left(\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\right)} + \frac{b^3}{c^2\left(\sqrt[4]{\frac{c^8+a^8}{2}}+5ca\right)} + \frac{c^3}{a^2\left(\sqrt[4]{\frac{a^8+b^8}{2}}+5ab\right)} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \geq 1 \quad (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

– By Cauchy-Schwarz inequality we have:

$$\left(\sqrt{2(b^8+c^8)} + 2b^2c^2\right)^2 \leq 2(2(b^8+c^8) + 4b^4c^4) = 4(b^8 + 2b^4c^4 + c^8) = 4(b^4+c^4)^2$$

$$\Rightarrow \sqrt{2(b^8+c^8)} + 2b^2c^2 \leq 2(b^4+c^4) \Leftrightarrow \sqrt[4]{\frac{b^8+c^8}{2}} \leq \sqrt{b^4 - b^2c^2 + c^4} = \sqrt{(b^2+c^2)^2 - \left(bc\sqrt{3}\right)^2}$$

$$\Leftrightarrow \sqrt[4]{\frac{b^8+c^8}{2}} = \sqrt{\left(2+\sqrt{3}\right)\left(b^2 - bc\sqrt{3} + c^2\right)\left(2-\sqrt{3}\right)\left(b^2 + bc\sqrt{3} + c^2\right)} \leq$$

$$\leq \frac{\left(2+\sqrt{3}\right)\left(b^2 - bc\sqrt{3} + c^2\right) + \left(2-\sqrt{3}\right)\left(b^2 + bc\sqrt{3} + c^2\right)}{2} = \frac{4b^2 - 6bc + 4c^2}{2} = 2b^2 - 3bc + 2c^2$$

$$\Rightarrow \sqrt[4]{\frac{b^8+c^8}{2}} + 5bc \leq 2(b^2 + bc + c^2) \Leftrightarrow \frac{a^3}{b^2\left(\sqrt[4]{\frac{b^8+c^8}{2}} + 5bc\right)} \geq \frac{a^3}{2b^2(b^2+bc+c^2)}$$

SP.130. Let a, b, c be positive real numers such that: $a + b + c = 3$. Prove that:

$$\frac{a^3}{b^2(b^2+bc+c^2)} + \frac{b^3}{c^2(c^2+ca+a^2)} + \frac{c^3}{a^2(a^2+ab+b^2)} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq 2 \quad (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

- By Cauchy-Schwarz inequality and $a + b + c = 3$, we have:

$$\frac{a^3}{b^2(b^2+bc+c^2)} + \frac{b^3}{c^2(c^2+ca+a^2)} + \frac{c^3}{a^2(a^2+ab+b^2)}$$

$$=\frac{\left(\frac{a^2}{b}\right)^2}{a(b^2+bc+c^2)}+\frac{\left(\frac{b^2}{c}\right)^2}{b(c^2+ca+a^2)}+\frac{\left(\frac{c^2}{a}\right)^2}{c(a^2+ab+b^2)}\ge\frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2}{a(b^2+bc+c^2)+b(c^2+ca+a^2)+c(a^2+ab+b^2)}$$

$$\Rightarrow\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}\ge\frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2}{(a+b+c)(ab+bc+ca)}=\frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2}{3(ab+bc+ca)}\quad (2)$$

- Using Cauchy-Schwarz inequality: $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^4}{a^2b}+\frac{b^4}{b^2c}+\frac{c^4}{c^2a}\ge\frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}$ ***(3)***

- By Bunhiacopxki we have:

$$(a\cdot ab+b\cdot bc+c\cdot ca)^2\le(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)\le(a^2+b^2+c^2)\cdot\frac{(a^2+b^2+c^2)^2}{3}$$

$$\Rightarrow(a^2b+b^2c+c^2a)^2\le\frac{(a^2+b^2+c^2)^3}{3}\Leftrightarrow a^2b+b^2c+c^2a\le\sqrt{\frac{(a^2+b^2+c^2)^3}{3}}$$

+ Let (3):

$$\Rightarrow\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge\frac{(a^2+b^2+c^2)^2}{\sqrt{\frac{(a^2+b^2+c^2)^3}{3}}}=\sqrt{3(a^2+b^2+c^2)}\Leftrightarrow\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2\ge3(a^2+b^2+c^2)\quad (4)$$

- Let (2), (4): $\Rightarrow\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}\ge\frac{3(a^2+b^2+c^2)}{3(ab+bc+ca)}$

$$\Rightarrow\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}+\frac{ab+bc+ca}{a^2+b^2+c^2}\ge$$

$$\ge\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{ab+bc+ca}{a^2+b^2+c^2}\ge2\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}\cdot\frac{ab+bc+ca}{a^2+b^2+c^2}}=2$$

⇒ (1) True and we get the result.

+ Equality occurs if: $\begin{cases}a=b=c\\a+b+c=3\end{cases}\Leftrightarrow a=b=c=1.$

SP.131. Let $a,b,c>0$, with sum 1. Prove that:

$$\sum ab^2\ge\left(\sum ab\right)^2-10\sum ab+12abc.$$

Proposed by Mihalcea Andrei Stefan-Romania

Solution by proposer

In Milne's Inequality we take the pairs:

$$(a,1-b),(b,1-c),(c,1-a)\to 3\sum\frac{a-ab}{1+a-b}\le 2.$$

But:

$$\sum\frac{a-ab}{1+a-b}=\frac{\sum(a-ab)(1+b-c)(1+c-a)}{\prod(2a+c)}=\frac{(\sum ab)^2-2\sum a^2b-\sum ab^2}{9abc+4\sum ab^2+2\sum a^2b}.$$

So, after calculations :

$$3\left(\sum ab\right)^2\le 18abc+10\sum ab(a+b)+\sum ab^2$$

Using that $\sum ab(a+b)=\sum ab-3abc$*, it results q.e.d.*

SP.132. Let a,b be two positive numbers.

Prove that:

$$\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)}+\frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2}\le 1.$$

Proposed by Mihalcea Andrei Ștefan – Romania

Solution by proposer

First, observe that $1-\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)}=\frac{a+1}{a^2+1}+\frac{b+1}{b^2+1}$.

Now

$$,\frac{a+1}{a^2+1}+\frac{b+1}{b^2+1}=\frac{1}{a+1-\frac{2a}{a+1}}+\frac{1}{b+1-\frac{2b}{b+1}}\ge\frac{4}{a+b+2-2\left(\frac{a^2}{a^2+a}+\frac{b^2}{b^2+b}\right)}\ge\frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2},$$

by C-B-S

SP.133. Let $a,b,c>0$, with sum 1. Prove that:

$$2\sqrt{abc}\sum\frac{a}{1+a^2}\le\frac{1+\sum a^2}{3+\sum a^2}.$$

Proposed by Mihalcea Andrei Ștefan – Romania

Solution 1 by proposer

In Milne's Inequality, we take the pairs:

$$(1-a,a+b^2),(1-b,b+c^2),(1-c,c+a^2)\to$$

$$\left(3+\sum a^2\right)\sum\frac{(b+c)(a+b^2)}{1+b^2}\le 2\left(1+\sum a^2\right).$$

But $(b+c)(a+b^2)\ge 4b\sqrt{abc}$.

So, $2\sqrt{abc}\sum\frac{a}{1+a^2}\le\frac{1+\sum a^2}{3+\sum a^2}$.

Solution 2 by Michael Sterghiou-Greece

$$2\sqrt{abc}\sum_{cyc}\frac{a}{1+a^2}\le\frac{1+\sum_{cyc}a^2}{3+\sum_{cyc}a^2}\quad (1)$$

Let $\left(\sum_{cyc}a,\sum_{cyc}ab,abc\right)=(p,q,r): p=1, q\le\frac{1}{3}, \sum_{cyc}a^2=1-2q$

$f(a)=\frac{a}{1+a^2}$, has $f''(a)=\frac{2a(a^2-3)}{(a^2+1)^3}<0$ for $0<\alpha<1$ hence

$\sum_{cyc}\frac{a}{1+a^2}\overset{Jensen}{\le}3\cdot\frac{\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}=\frac{9}{10}$. Also $r\le\left(\frac{q}{3}\right)^{\frac{3}{2}}$ so it is enough to show the stronger than

(1) inequality $2r^{\frac{1}{2}}\sum_{cyc}\frac{a}{a^2+1}\le 2\left(\frac{q}{3}\right)^{\frac{3}{4}}\cdot\frac{9}{10}\le\frac{1-q}{2-q}$ or

$$f(q)=-3^{\frac{5}{4}}q^{\frac{7}{4}}+2\cdot 3^{\frac{5}{4}}q^{\frac{3}{4}}+5q-5\le 0$$

$f'(q)=-\frac{21}{4}3^{\frac{1}{4}}q^{\frac{3}{4}}+\frac{9\cdot 3^{\frac{1}{4}}}{2q^{\frac{1}{4}}}+5>0$ because $-\frac{21}{4}\cdot 3^{\frac{1}{4}}\cdot\left(\frac{1}{3}\right)^{\frac{3}{4}}<5$ hence $f(q)\uparrow$ and

$f(q)\le f\left(\frac{1}{3}\right)$ for $\le\frac{1}{3}$ or $f(q)<\frac{5}{3}\left(\sqrt{3}-2\right)<0$

We are done!

SP.134. Let $ABCD$ be a cyclic quadrilateral with perimeter 2.

Denote $AB=a, BC=b, CD=c, DA=d$.

Prove that:

$$4\le\sum\tan\frac{A}{2}<\frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}.$$

Proposed by Mihalcea Andrei Ștefan – Romania

Solution by proposer

The function $f:(0,1)\to(0,\infty), f(x)=\sqrt{\frac{1-x}{(1-a)(1-b)(1-c)(1-d)}}$ is concave.

We know that $\tan\frac{A}{2}=\sqrt{\frac{(1-a)(1-d)}{(1-c)(1-b)}}=f(1-[(1-a)(1-d)]^2)$. ***Applying Jensen*** →

$$\sum\tan\frac{A}{2}\le 4f\left(\frac{4-\sum[(1-a)(1-d)]^2}{4}\right)=2\sqrt{\frac{\sum[(1-a)(1-d)]^2}{(1-a)(1-b)(1-c)(1-d)}}$$

But $\sum[(1-a)(1-d)]^2<[\sum(1-a)(1-d)]^2=[(a+c)(b+d)]^2$.

So, $\sum\tan\frac{A}{2}<\frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}$. ***Now,***

$$\sum\tan\frac{A}{2}=\sqrt{\frac{(1-a)(1-d)}{(1-c)(1-b)}}+\sqrt{\frac{(1-b)(1-c)}{(1-a)(1-d)}}+\sqrt{\frac{(1-c)(1-d)}{(1-a)(1-b)}}+\sqrt{\frac{(1-a)(1-b)}{(1-c)(1-d)}},$$

Which is bigger or equal than 4. ***Using this → q.e.d.***

SP.135. If $a,b,c>1$ then:

$$\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}\ge\frac{16}{3(a+b+c+d)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Tran Hong-Vietnam

Using Cauchy's inequality:

$$LHS\ge 4\sqrt[4]{\frac{\log_a b\cdot\log_b c\cdot\log_c d\cdot\log_d a}{(a+b+c)(b+c+d)(c+d+a)(d+a+b)}}$$

$$=\frac{4}{\sqrt[4]{(a+b+c)(b+c+d)(c+d+a)(d+a+b)}}\overset{Cauchy}{\ge}\frac{4}{\frac{3a+3b+3c+3d}{4}}=\frac{16}{3(a+b+c+d)}.$$ ***Proved***

Equality $\Leftrightarrow a=b=c=d$.

Solution 2 by Amit Dutta-Jamshedpur-India

Using Cauchy – Schwarz's Inequality:

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)^2\left(b_1^2+b_2^2+\cdots+b_n^2\right)\ge(a_1b_1+a_2b_2+\cdots+a_nb_n)^2$$

Putting $a_i=\frac{x_i}{\sqrt{yi}}$ ***and*** $b_i=\sqrt{y_i}$, ***we have:***

$$\left(\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\cdots+\frac{x_n^2}{y_n}\right)(y_1+y_2+\cdots+y_n)\ge(x_1+x_2+\cdots+x_n)^2$$

$$\Rightarrow\left(\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\cdots+\frac{x_n^2}{y_n}\right)\ge\frac{(x_1+x_2+\cdots+x_n)^2}{(y_1+y_2+\cdots+y_n)}\to \textit{Titu's Lemma}$$

Using this inequality, putting $x_1=\sqrt{\log_a b},\ y_1=(a+b+c)$

$$x_2=\sqrt{\log_b c},\ y_2=(b+c+d)$$

$$x_3=\sqrt{\log_c d},\ y_3=c+d+a$$

$$x_4=\sqrt{\log_d a},\ y_4=d+a+b$$

We have,

$$\left\{\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}\right\}\ge$$

$$\ge\frac{\left(\sqrt{\log_a b}+\sqrt{\log_b c}+\sqrt{\log_c d}+\sqrt{\log_d a}\right)^2}{3(a+b+c+d)}$$

AM-GM

$$\sqrt{\log_a b}+\sqrt{\log_b c}+\sqrt{\log_c d}+\sqrt{\log_d a}\ge 4$$

$$\therefore\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}\ge\frac{16}{3(a+b+c+d)}$$

(proved)

Solution 3 by Serban George Florin-Romania

$$\sum\frac{\log_a b}{a+b+c}\ge 4\sqrt[4]{\prod\frac{\log_a b}{a+b+c}}=\frac{4}{\prod(a+b+c)}\ge\frac{4}{\frac{\sum(a+b+c)}{4}}=\frac{16}{3(a+b+c+d)},\ (M_a\ge M_g)$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

for $a,b,c>1$ *we have:* $\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}=$

$$=\frac{\log_x b}{\frac{\log_x a}{a+b+c}}+\frac{\log_x c}{\frac{\log_x b}{b+c+d}}+\frac{\log_x d}{\frac{\log_x c}{c+d+a}}+\frac{\log_x a}{\frac{\log_x d}{d+a+b}},\ x>1$$

$$\ge\frac{\left(\sqrt{\frac{\log_x b}{\log_x a}}+\sqrt{\frac{\log_x c}{\log_x b}}+\sqrt{\frac{\log_x d}{\log_x c}}+\sqrt{\frac{\log_x a}{\log_x d}}\right)^2}{3(a+b+c+d)}\ge\frac{16}{3(a+b+c+d)}$$

Therefore it is to be true.

UP.121. Prove that:

$$\int_0^\infty x^{p-a}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx=\left(\frac{n}{n-p}\right)B(n-2p,p)$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Khalef Ruhemi-Jarash-Jordan

$$I=\int_0^\infty x^{p-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx$$

$$\textit{Let } \frac{2}{\sqrt{1+4x}+1}=y\Rightarrow x=\frac{1}{y^2}-\frac{1}{y}=\frac{1-y}{y^2}$$

$$dx=-\frac{2}{y^3}+\frac{1}{y^2}$$

$$\therefore I=-\int_0^1 (1-y)^{p-1}y^{2-2p}y^n(y-2)^{-3}y\cdot dy$$

$$\therefore I=-\int_0^1 y^{n-2p-1}(1-y)^{p-1}(y-2)dy=\int_0^1 x^{n-2p-1}(1-x)^{p-1}\big(1+(1-x)\big)dx=$$

$$=\int_0^1 x^{n-2p-1}(1-x)^{p-1}dx+\int_0^1 x^{n-2p-1}(1-x)^p dx=\int_0^1 x^{n-2p-1}(1-x)^{p-1}dx+$$

$$+\left.\frac{x^{n-2p}(1-x)^p}{n-2p}\right|_0^1+\frac{p}{n-2p}\int_0^1 x^{n-2p}(1-x)^{p-1}=\frac{\Gamma(n-2p)\Gamma(p)}{\Gamma(n-p)}+$$

$$+\left(\frac{p}{n-2p}\right)\frac{\Gamma(n-2p+1)\Gamma(p)}{\Gamma(n-p+1)}=$$

$$\beta(n-2p,p)+\left(\frac{p}{n-2p}\right)\left(\frac{(n-2p)\Gamma(n-2p)\Gamma(p)}{(n-p)\Gamma(n-p)}\right)=$$

$$=\mathrm{B}(n-2p,p)+\left(\frac{p}{n-p}\right)\mathrm{B}(n-2p,p)=\left(\frac{n}{n-p}\right)\mathrm{B}(n-2p,p)$$

$$\Rightarrow I=\left(\frac{n}{n-p}\right)\mathrm{B}(n-2p,p)$$

Solution 2 by Mohammed Alqottby-Hajjah-Yemen

$$\textit{Show that: } L = \int_0^\infty x^{p-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx = \frac{n}{n-p}, n \neq p$$

$$\textit{We have } \int_0^1 \frac{x^{b-a}(1-x)^{c-b-1}dx}{(1-zx)^a} = B(c-b,b)_2F_1(a,b;c;z)$$

$$\textit{So put } 1+4x = \frac{1}{y^2} \Rightarrow x = \frac{1}{4}\cdot\frac{1-y^2}{y^2}, x\in[0,\infty) \Rightarrow y\in[0,1] \Rightarrow L = \frac{2^{n+1}}{4^p}\int_0^1 \frac{x^{n-2p-1}(1-x)^{p-1}dx}{(1+x)^{n-p+1}}$$

$$= \frac{2^{n+1}}{4^p}B(p,n-2p)_2f_1(n-p+1,n-2p,n-p;-1) \to \ (1)$$

$$\textit{But } {}_2f_1(n-p+1,n-2p;n-p;\ -1)$$

$$= \sum_{k=0}^{\infty}(-1)\frac{\Gamma(k+n-p+1)\Gamma(k+n-2p)\Gamma(n-p)}{\Gamma(n-p+1)\Gamma(n-2p)\Gamma(k+n-p)k!} =$$

$$= \frac{1}{2}\cdot\frac{1}{(n-p)\Gamma(n-2p)}\sum_{k=0}^{\infty}(-1)^k\left[\frac{(k+n)\Gamma(k+n-2p)+\Gamma(k+n-2p+1)}{k!}\right]$$

$$= \frac{n}{2(n-p)\Gamma(n-2p)}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(k+n-2p)}{k!} +$$

$$+ \frac{1}{2(n-2p)\Gamma(n-2p)}\left(\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(k+n-2p)}{(k-1)!} - \sum_{k=1}^{\infty}(-1)^k\cdot\frac{\Gamma(k+n-2p)}{(k-1)!}\right) =$$

$$= \frac{n2^{2p-n-1}\Gamma(n-2p)}{(n-p)\Gamma(n-2p)} + 0 = \frac{2n^{2p-n-1}}{n-p} \to \ (2)$$

$$(2) \to (1)$$

$$\Rightarrow \int_0^\infty x^{p-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx = \frac{n}{n-p}B(n-2p,p) = \frac{n}{n-p}B(p,n-2p)$$

UP.122. Let be $a_n = \sum_{k=1}^{n} arctan\frac{1}{k^2+k+1}$. Find:

$$\lim_{n\to\infty} n\left(a_n^{\frac{\pi}{4}} - \frac{\pi^{a_n}}{4}\right)$$

Proposed by Marian Ursarescu-Romania

Solution by proposer

$$a_n=\sum\arctan\frac{1}{k^2+k+1}=\sum\arctan\frac{\frac{1}{k}-\frac{1}{k+1}}{1+\frac{1}{k(k+1)}}=$$

$$=\sum\left(\arctan\frac{1}{k}-\arctan\frac{1}{k+1}\right)=\frac{\pi}{4}-\arctan\frac{1}{n+1}\to\frac{\pi}{4}$$

$$\lim_{n\to\infty}\frac{a_n^{\frac{\pi}{4}}-\frac{\pi}{4}^{a_n}}{\frac{\pi}{4}-a_n}\cdot 4\left(\frac{\pi}{4}-a_n\right)\quad \textbf{(1)}$$

$$a_n=x\to\frac{\pi}{4}\Rightarrow\lim_{x\to\frac{\pi}{4}}\frac{x^{\frac{\pi}{4}}-\frac{\pi}{4}^{x}}{\frac{\pi}{4}-x}=\left(\frac{\pi}{4}\right)^{\frac{\pi}{4}}\left(\ln\frac{\pi}{4}-1\right)\quad \textbf{(2)}$$

$$\lim_{x\to a}\frac{x^a-a^x}{a-x}=a^a(\ln a-1)$$

$$\lim_{n\to\infty}n\left(\frac{\pi}{4}-a_n\right)=\lim_{n\to\infty}\frac{\frac{\pi}{4}-a_n}{\frac{1}{n}}\overset{C.S.}{=}\lim_{n\to\infty}\frac{\frac{\pi}{4}-a_{n+1}-\frac{\pi}{4}+a_n}{\frac{1}{n+1}-\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\frac{-\arctan\frac{1}{(n+1)^2+n+1+1}}{-\frac{1}{n(n+1)}}$$

$$\lim_{n\to\infty}\frac{\arctan\frac{1}{(n+1)^2+n+2}}{\frac{1}{(n+1)^2+(n+2)}\cdot\frac{(n+1)^2+n+2}{n(n+1)}}=1\quad \textbf{(3)}$$

From (1)+(2)+(3)

$$\Rightarrow\lim_{n\to\infty}n\left(a_n^{\frac{\pi}{4}}-\frac{\pi^{a_n}}{4}\right)=\left(\frac{\pi}{4}\right)^{\frac{\pi}{4}}\left(\ln\frac{\pi}{4}-1\right)$$

UP.123. Find:

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{\sqrt{n^4+k^2}-k}\right)$$

*Proposed by Marian Ursărescu – **Romania***

Solution 1 by Sagar Kumar-Patna Bihar-India

$$L = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{\sqrt{n^4+k^2}-k}\right)$$

$$L = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{n^2}\right)^2 = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\left(1+\frac{k^2}{n^4}\right)^{\frac{1}{2}}+\frac{k}{n^2}\right)^n =$$

$$= \lim_{n\to\infty}\prod_{k}^{n}\left(1+\frac{k^2}{2n^4}+\frac{k}{n^2}+0\left(n^{\frac{1}{8}}\right)\right)^2$$

$$\log L = 2\lim_{n\to\infty}\sum_{k=1}^{n}\log\left(1+\frac{k}{n^2}\right)$$

$$\log L = 2\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^2}{2n^4}\dots\right) = 2\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right) = 2\int_0^1 x\,dx$$

$$\log L = 1 \Rightarrow L = e \textit{ Answer}$$

Solution 2 by Shivam Sharma-New Delhi-India

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k}+k}{n^2}\right)^2 \Rightarrow \lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt{1+\frac{k^2}{n^4}}+\frac{k}{n^2}\right)^2 \Rightarrow \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k^2}{2n^4}+\frac{k}{n^2}+\cdots\right)^2$$

$$\ln(\Omega) = 2\lim_{n\to\infty}\ln\left(\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)\right)$$

$$\ln(\Omega) = 2\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)$$

As we know, $x-\frac{x^2}{2}\le\ln(1+x)\le x$***. Using this, we get,***

$$\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^2}{n^4}\right)\le\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\le\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)-\lim_{n\to\infty}\frac{1}{2n^2}\left(\sum_{k=1}^{n}\frac{k^2}{n^2}\right)\le\lim_{n\to\infty}\sum_{k}^{n}\ln\left(1+\frac{k}{n^2}\right)\le\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)$$

$$2\left[\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)-\lim_{n\to\infty}\frac{1}{2n^2}\left(\sum_{k=1}^{n}\frac{k^2}{n^2}\right)\right]\le 2\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\le 2\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)$$

$$2\left[\int_0^1 x\,dx-\lim_{n\to\infty}\left(\frac{1}{2n^2}\right)\int_0^1 x^2\,dx\right]\le 2\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\le 2\int_0^1 x\,dx$$

$$2\left[\frac{x^2}{2}\right]_0^1-0\le 2\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\le 2\left[\frac{x^2}{2}\right]_0^1$$

$$1\le 2\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\le 1$$

$$1\le\ln(\Omega)\le 1$$

By squeeze theorem, we get $\Omega = e$ (Answer)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{\sqrt{n^4+k^2}-k}\right)=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{n^2}\right)^2\Rightarrow$$

$$\Rightarrow\sqrt{\Omega}=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt{1+\frac{k^2}{n^4}}+\frac{k}{n^2}\right)\Rightarrow\frac{\ln\Omega}{2}=\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\sqrt{1+\frac{k^2}{n^4}}+\frac{k}{n^2}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{1}{2}\cdot\frac{\left(\frac{k}{n^2}\right)^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\left(\frac{k}{n^2}\right)^5}{5}-\cdots\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n^2}+o\left(\frac{1}{n^2}\right)\right)$$

$$\left[\because\ln\left(x+\sqrt{1+x^2}\right)=x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}-\cdots\right]$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}=\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=\frac{1}{2}\Rightarrow\Omega=e\ \ \textit{(Answer)}$$

UP.124. Find all continuous functions such that:

$$\int_{e^x}^{e^{2x}} f(t)\,dt=\int_1^{e^x} f(t)\,dt$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$\int_{e^x}^{e^{2x}} f(t)\,dt = \int_1^{e^x} f(t)\,dt$$

Differentiating w.r.t. x we get: $2e^{2x}f(e^{2x}) - e^x f(e^x) = e^x f(e^x) \Rightarrow 2e^{2x}f(e^{2x}) =$

$= 2e^x f(e^x) \Rightarrow e^x f(e^{2x}) = f(e^x)$. *Put* $x = \ln t\,, t > 0$ *to obtain:*

$$tf(t^2) = f(t)\ \ \forall t > 0 \quad \textit{(1)}$$

Taking $\lim t \to 0+$, *we get:* $0 = f(0)$. *For* $0 < t < 1$, $tf(t) = t^2 f(t^2) = t^{2^2} f\left(t^{2^2}\right) =$

$= t^{2^3} f\left(t^{2^3}\right) = \cdots t^{2^n} f\left(t^{2^n}\right)\ \forall n \in \mathbb{N}$. *Taking limit as* $n \to \infty$, *we get*

$$tf(t) = \lim_{n\to\infty} t^{2^n} f\left(t^{2^n}\right), 0 < t < 1$$

$= 0f(0) = 0 \Rightarrow f(t) = 0$ *for* $0 < t < 1$. *As* f *is continuous* $f(1) = \lim_{t\to 1-} f(t) = 0$.

$\therefore$ *for* $0 \le t \le 1$, $f(t) = 0$. *Let* $t > 1$, $t^2 f(t^2) = tf(t) = t^{\frac{1}{2}} f\left(t^{\frac{1}{2}}\right) = t^{\frac{1}{4}} f\left(t^{\frac{1}{4}}\right) = \cdots =$

$= t^{\frac{1}{2^n}} f\left(t^{\frac{1}{2^n}}\right)\ \forall n \in \mathbb{N}$. *Taking limit as* $n \to \infty$, *we get, for* $t > 1$

$tf(t) = \lim_{n\to\infty} t^{\frac{1}{2^n}} f\left(t^{\frac{1}{2^n}}\right) = (1)f(1)$ *[$\because$ f is continuous]* $\Rightarrow tf(t) = 0\ \forall t > 1$

$\Rightarrow f(t) = 0\ \forall t > 1$. *Thus,* $f(x) = 0, \forall x \ge 0$. *Let* g *be any continuous function on*

$(-\infty, 0]$ *such that* $g(0) = 0$, *then:* $f(x) = \begin{cases} g(x), \forall\ x < 0 \\ 0, \forall x \ge 0 \end{cases}$

UP.125. Prove that:

$$2\sum_{k=1}^{n} \ln^2\left(1 + \frac{1}{k}\right) < \frac{2n}{n+1} < \sum_{k}^{n} \ln\left(1 + \frac{2}{k^2 + k - 1}\right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\textit{If } x > 0 \textit{ then: } \frac{2}{x+1} \le \ln\left(1 + \frac{1}{x}\right) \le \frac{1}{\sqrt{x^2+x}} \quad \textit{(1)}$$

Let be $f(x) = \frac{1}{\sqrt{x^2+x}} - \ln\left(1 + \frac{1}{x}\right)$ *then:* $f'(x) = \frac{2\sqrt{x^2+1} - (2x+1)}{2x(x+1)\sqrt{x^2+x}} < 0$

therefore $x < \infty \Rightarrow f(x) > f(\infty) = 0$. *Let be* $g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{2}{2x+1}$ *then:*

$$g'(x)=\frac{-1}{x(x+1)}-\frac{2}{2x+1}<0 \text{ therefore } x<\infty \Rightarrow g(x)>g(\infty)=0$$

Using (1) we have: $\sum_{k=1}^{n}\ln^2\left(1+\frac{1}{k}\right)\le\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$ *and if* $x=\frac{k^2+k-1}{2}$

then $\ln\left(1+\frac{2}{k^2+k-1}\right)\ge\frac{2}{k(k+1)}\Rightarrow\sum_{k=1}^{n}\left(1+\frac{2}{k^2+k-1}\right)\ge 2\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{2n}{n+1}$

UP.126. Prove that:

$$\int_0^1\int_0^1\frac{\sqrt{x+y}+\sqrt{xy}}{\left(x\sqrt{y}+y\sqrt{x}\right)}dx\,dy=\ln\left(17-12\sqrt{2}\right)+\sqrt{2}\ln\left(17+12\sqrt{2}\right)-$$

$$-\frac{4}{3}\left(1-3\sqrt{2}+2\ln(2)\right)$$

Proposed by K. Srinivasa Raghava – AIRMC – India

Solution by Khalef Ruhemi-Jarash-Jordan

$$I:=\int_0^1\int_0^1\frac{\sqrt{x+y}+\sqrt{xy}}{x\sqrt{y}+y\sqrt{x}}dx\,dy\rightarrow \text{Let } x=t^2\Rightarrow dx=2t\,dt;\ y=v^2\Rightarrow dy=2vdv$$

$$\therefore I=4\int_0^1\int_0^1\frac{\left(\sqrt{t^2+v^2}+tv\right)tv\cdot dt\,dv}{t^2v+v^2t}=4\int_0^1\left(\frac{\sqrt{x^2+y^2}+xy}{x+y}\cdot dx\right)dy$$

Let $x=yt\Rightarrow dx=y\,dt$

$$=4\int_0^1\left(\int_0^{\frac{1}{y}}\frac{y\sqrt{1+x^2}+y^2x}{1+x}dx\right)dy=4\int_0^1\int_0^1\frac{y\sqrt{1+x^2}+y^2x}{1+x}dy\,dx+$$

$$+4\int_0^\infty\int_0^{\frac{1}{x}}\frac{y\sqrt{1+x^2}+y^2x}{1+x}dy\,dx=4\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx+4\int_0^\infty\frac{\frac{\sqrt{1+x^2}}{2x^2}+\frac{1}{3x^2}}{1+x}dx$$

Let $\frac{1}{x}=y\Rightarrow\frac{1}{y};dx=-\frac{dy}{y^2}$

$$=4\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx+4\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx$$

$$= 8\int_0^1 \frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx = 4\int_0^1 \frac{\sqrt{1+x^2}}{1+x}dx + \frac{8}{3}\int_0^1 \frac{1+x-1}{1+x}dx$$

$$= \left(\frac{8}{3}\right)\int_0^1 \left(1-\frac{1}{1+x}\right)dx + 4\int_0^1 \frac{x}{\sqrt{1+x^2}}dx - \underbrace{4\int_0^1 \frac{dx}{\sqrt{1+x^2}} + 8\int_0^1 \frac{dx}{(1+x)\sqrt{1+x^2}}}_{let\ x=\tan(\theta);dx=\sec^2(\theta)d\theta}$$

$$= \left(\frac{8}{3}\right)(x-\ln(1+x))\Big|_0^1 + \left(4\sqrt{1+x^2}\right)\Big|_0^1 - 4\int_0^{\frac{\pi}{4}} \sec(\theta)\,d\theta + \frac{8}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec\left(\frac{\pi}{4}-\theta\right)d\theta$$

$$= \left(\frac{8}{3}\right)(1-\ln(2)) + 4\sqrt{2} - 4 - \left(4\ln(\tan(\theta)+\sec(\theta))\Big|_0^{\frac{\pi}{4}}\right) -$$

$$-\frac{8}{\sqrt{2}}\left(\ln\left(\tan\left(\frac{\pi}{4}-\theta\right)+\sec\left(\frac{\pi}{4}-\theta\right)\right)\Big|_0^{\frac{\pi}{4}}\right) = \frac{8}{3}-\frac{8}{3}\ln(2) + 4\sqrt{2} - 4 - 4\ln(1+\sqrt{2}) +$$

$$+\frac{8}{\sqrt{2}}\ln(1+\sqrt{2}) = -\frac{4}{3}-\frac{8}{3}\ln(2) + 4\sqrt{2} + 4\sqrt{2}\ln(1+\sqrt{2}) - 4\ln(1+\sqrt{2}) =$$

$$= -\frac{4}{3}(1-3\sqrt{2}+2\ln(2)) + 4\ln(\sqrt{2}-1) + 4\sqrt{2}\ln(\sqrt{2}+1) =$$

$$= -\frac{4}{3}(1-3\sqrt{2}+2\ln(2)) + \ln\left((\sqrt{2}-1)^4\right) + \sqrt{2}\ln\left((\sqrt{2}+1)^4\right) =$$

$$= -\frac{4}{3}(1-3\sqrt{2}+2\ln(2)) + \ln(17-12\sqrt{2}) + \sqrt{2}\ln(17+12\sqrt{2}) = I$$

$$\therefore \int_0^1\int_0^1 \frac{\sqrt{x+y}+\sqrt{xy}}{x\sqrt{y}+y\sqrt{x}}dx\,dy = -\frac{4}{3}(1-3\sqrt{2}+2\ln(2)) + \ln(17-12\sqrt{2}) +$$

$$+\sqrt{2}\ln(17+12\sqrt{2})$$

UP.127. If $x,y,z,t \in \mathbb{R}$ then:

$$|\sin x| + |\sin y| + |\sin z| + |\sin t| + |\cos x| + |\cos y| + |\cos z| +$$

$$+|\cos t| + 2|\cos(x+y+z+t)| \geq 2$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\text{If } x_1+x_2+x_3+x_4+x_5=0 \text{ then } = \left|\cos\left(\sum_{k=1}^{5} x_k\right)\right| \le |\cos x_1| + \left|\sin\left(\sum_{k=2}^{5} x_k\right)\right| \le$$

$$\le |\cos x_1| + |\cos x_2| + |\cos(x_3+x_4+x_5)| \le |\cos x_1| + |\cos x_2| +$$

$$+|\cos x_3| + |\cos x_4| + |\cos x_5|$$

$$\text{If } x_1=x, x_2=y, x_3=z, x_4=t, x_5=-x-y-z-t \text{ then:}$$

$$|\cos x| + |\cos y| + |\cos z| + |\cos t| + |\cos(x+y+z+t)| \ge 1 \quad (1)$$

$$\text{In (1) we take } x=\frac{\pi}{2}-y; y=\frac{\pi}{2}-y_2, z=\frac{\pi}{2}-y_3, t=\frac{\pi}{2}-y_4 \Rightarrow$$

$$\Rightarrow |\sin y_1| + |\sin y_2| + |\sin y_3| + |\sin y_4| + |\cos(y_1+y_2+y_3+y_4)| \ge 1 \text{ or}$$

$$|\sin x| + |\sin y| + |\sin z| + |\sin t| + |\sin(x+y+z+t)| \ge 1 \quad (2)$$

Adding (1) and (2) we obtain the result.

UP.128. If $\alpha,\beta > 1$ and $2\alpha-\beta > 1$ then:

$\zeta(2\alpha-\beta)\zeta(\beta) \ge \zeta^2(\alpha)$ where ζ is the Riemann zeta function.

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\text{In inequality } \sum_{k=1}^{\infty}\frac{x_k^2}{a_k} \ge \frac{\left(\sum_{k=1}^{\infty} x_k\right)^2}{\sum_{k=1}^{\infty} a_k} \text{ we take } x_k=\frac{1}{k^\alpha}, a_k=\frac{1}{k^\wedge\beta} \text{ and we obtain}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^{2\alpha-\beta}} \ge \frac{\left(\sum_{k=1}^{\infty}\frac{1}{k^\alpha}\right)^2}{\sum_{k=1}^{\infty}\frac{1}{k^\beta}} \text{ or } \zeta(2\alpha-\beta) \ge \frac{\zeta^2(\alpha)}{\zeta(\beta)} \text{ or } \zeta(2\alpha-\beta)\zeta(\beta) \ge \zeta^2(\alpha)$$

UP.129. Let R_+^* be the set of real positive numbers, let $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ be two sequences of real positive numbers with

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in R_+^*, \lim_{n\to\infty}(b_{n+1}-b_n)=b\in R_+^*,$$

Let $P_n=\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$ and we denote $P_n! = P_1P_2\ldots P_n$, for any positive integer n. If $u,v\in R$ with $u+v=1$. Evaluate:

$$\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v} - b_n^u\sqrt[n]{(P_n!)^v}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

We use Cauchy – D'Alembert, Cesaró – Stolz and $f^9 = e^{9\ln 7}$ and $\frac{a^{x_n}-1}{x_n} \to \ln a$, with $x_n \to 0$.

$$L = \lim_{n\to\infty} b_{n+1}^{u} \sqrt[n+1]{(P_{n+1})!^{\gamma}} - b_n^u \sqrt[n]{(P_{n!})^{\gamma}} = \lim_{n\to\infty} b_n^u \sqrt[n]{(P_n)!^n}\left(\frac{b_{n+1}^u \sqrt[n+1]{(P_{n+1})!^v}}{b_n^u \cdot \sqrt[n]{(P_n)!^v}} - 1\right) =$$

$$= \lim_{n\to\infty} \frac{b_n^n \cdot \sqrt[n]{(P_n)^v}}{n} \cdot n\left(\frac{b_{n+1}^u \cdot \sqrt[n+1]{(P_{n+1})!^v}}{b_n^u \cdot \sqrt[n]{(P_n)!^v}} - 1\right) \quad (1)$$

$$\lim_{n\to\infty} \frac{b_n^u \cdot \sqrt[n+1]{(P_n)!^v}}{n} = \lim_{n\to\infty} \left(\frac{b_n}{n}\right)^u \cdot \left(\frac{\sqrt[n]{(P_n)!}}{n}\right)^v \quad (2)$$

$$\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} \frac{b_{n+1}-b_n}{n+1-n} = b \quad (3)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(P_n)!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(P_n)!}{n^n}} = \lim_{n\to\infty} \frac{(P_{n+1})!}{(n+1)^{n+1}} \cdot \frac{n^n}{(P_n)!} = \lim_{n\to\infty} \frac{P_{n+1}}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \lim_{n\to\infty} \frac{P_n}{n} =$$

$$= \frac{1}{e} \lim_{n\to\infty} \frac{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}{n} = \frac{1}{e} \lim_{n\to\infty} \sqrt{\frac{a_1^2 + \dots + a_n^2}{n^3}} = \frac{1}{e} \sqrt{\lim_{n\to\infty} \frac{a_1^2 + \dots + a_n^2}{n^3}} =$$

$$= \frac{1}{e} \sqrt{\lim_{n\to\infty} \frac{a_{n+1}^2}{3n^2 + 3n + 1}} = \frac{1}{e} \sqrt{\lim_{n\to\infty} \frac{a_{n+1}^2}{(n+1)^2} \cdot \frac{(n+1)^2}{3n^2 + 3n}} = \frac{1}{e\sqrt{3}} \lim_{n\to\infty} \frac{a_n}{n} =$$

$$= \frac{1}{e\sqrt{3}} \lim_{n\to\infty} (a_{n+1} - a_n) = \frac{a}{e\sqrt{3}} \quad (4)$$

From (2) + (3) + (4) $\Rightarrow \lim_{n\to\infty} \frac{b_n^u \cdot \sqrt[n]{(P_n)!^v}}{n} = \frac{b^n \cdot a^v}{e^v \cdot \sqrt{e}^v}$ ***(5)***

Now let

$$\lim_{n\to\infty} n\left(\frac{b_{n+1}^u \sqrt[n+1]{P_{n+1}!^v}}{b_n^n \cdot \sqrt[n]{P_{n!}^v}} - 1\right) = \lim_{n\to\infty} n\left(\frac{\ln \frac{b_{n+1}^u \cdot \sqrt[n+1]{P_{n+1}!^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}}}{\ln\left(\frac{b_{n+1}^u \cdot \sqrt[n+1]{P_{n+1}!^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}}\right)} - 1\right) \ln\left(\frac{b_{n+1}^u \sqrt[n+1]{P_{n+1}^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}}\right)$$

$$= \lim_{n\to\infty} n \ln\left(\frac{b_{n+1}^u \sqrt[n+1]{P_{n+1}^v!}}{b_n^n \cdot \sqrt[n]{P_{n!}^v}}\right) = \lim_{n\to\infty} \ln\left(\frac{b_{n+1}^u \cdot \sqrt[n+1]{P_{n+1}!^v}}{b_n^u \cdot \sqrt[n]{P_n^n}}\right)^n \quad (6)$$

$$\lim_{n\to\infty}\frac{b_{n+1}^n}{b_n^n}\cdot\lim_{n\to\infty}\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{P_{n+1}!}}{\sqrt[n]{P_n!}}\right)^v=$$

$$=\lim_{n\to\infty}\left(\frac{b_{n+1}}{n+1}\cdot\frac{n}{b_n}\cdot\frac{n+1}{n}\right)^n\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{P_{n+1}!}}{n+1}\cdot\frac{n}{\sqrt[n]{P_n}}\cdot\frac{n+1}{n}\right)$$

$$s\underset{(4)}{\overset{(3)}{=}}1\cdot\frac{a\sqrt{3}}{3e}\cdot\frac{3e}{a\sqrt{3}}=1\ \ (7)$$

From (6)+(7) we have :

$$\ln\lim_{n\to\infty}\left(\frac{b_{n+1}^n}{b_n^n}\cdot\frac{\sqrt[n+1]{P_{n+1}!^u}}{\sqrt[n]{P_n^u}}\right)^n=\ln\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)^{nu}\cdot\lim_{n\to\infty}\left(\frac{P_{n+1}!}{P_n!}\cdot\frac{1}{\sqrt[n]{P_{n+1}!}}\right)^v\right)$$

$$=\ln\left(\lim_{n\to\infty}\left[\left(1+\frac{b_{n+1}-b_n}{b_n}\right)^{\frac{b_n}{b_{n+1}-b_n}}\right]^{\frac{b_n}{b_{n+1}-b_n}nu}\cdot\left(\frac{a\sqrt{3}}{3}\cdot\frac{3e}{a\sqrt{3}}\right)\right)=$$

$$=\ln(e^u\cdot e^v)=\ln(e^{u+v})=\ln e=1\ \ (8)$$

From (1) + (5) + (8) $\Rightarrow L=\frac{a^v b^n}{e^v\cdot\sqrt{3}^v}$

UP.130. If $a,b,c,d\in[1,\infty);x\in\mathbb{R}$ then:

$$(ac)^{\sin x}\cdot(bd)^{\cos x}\le 2^{\sqrt{2(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d)}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu – Romania

We have: $a^{\sin x}\cdot b^{\cos x}\cdot c^{\sin x}\cdot d^{\cos x}\le 2^{\sqrt{2(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d)}}$ ***(1)***

But $a^{\sin x}\cdot b^{\cos x}=\left(2^{\log_2 a}\right)^{\sin x}\cdot\left(b^{\log_2 b}\right)^{\cos x}=$

$$=2^{\log_2 a\cdot\sin x}\cdot 2^{\log_2 b\cdot\cos x}=2^{\log_2 a\cdot\sin x+\log_2 b\cos x}\ \ (2)$$

From Cauchy's inequality $\Rightarrow$

$$(\log_2 a\cdot\sin x+\log_2 b\cos x)^2\le\left(\log_2^2 a+\log_2^2 b\right)(\sin^2 x+\cos^2 x)\ \ (3)$$

From (2) + (3) $\Rightarrow a^{\sin x}\cdot b^{\cos x}\le 2^{\sqrt{\log_2^2 a+\log_2^2 b}}$ ***(4)***

Similarly $\Rightarrow c^{\sin x}d^{\cos x} \le 2^{\sqrt{\log_2^2 c+\log_2^2 d}}$ *(5)*

From (4) + (5) $\Rightarrow a^{\sin x}\cdot b^{\cos x}\cdot c^{\sin x}\cdot d^{\cos x} \le 2^{\sqrt{\log_2^2 a+\log_2^2 b}+\sqrt{\log_2^2 c+\log_2^2 d}}$ *(6)*

From Cauchy's inequality $\Rightarrow 2(\alpha^2+\beta^2) \ge (\alpha+\beta)^2 \Rightarrow$

$$\Rightarrow \sqrt{\log_2^2 a+\log_2^2 b}+\sqrt{\log_2^2 c+\log_2^2 d} \le \sqrt{2\left(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d\right)} \quad (7)$$

From (6) + (7) $\Rightarrow (ac)^{\sin x}\cdot(bd)^{\cos x} \le 2^{\sqrt{2\left(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d\right)}}$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

1) $\sum \frac{1}{x^2+y^2+2z^2} \ge \frac{9}{4(x^2+y^2+z^2)}$ *(1)*

2) $\frac{2\sqrt{3\cdot x^2y^2z^2\cdot(x^2+y^2+z^2)}}{\prod(x^2+y^2)} \le \frac{2(x^2y^2+y^2z^2+z^2x^2)}{\prod(x^2+y^2)}$ *(2)*

(1); (2) $\Rightarrow \frac{9}{4(x^2+y^2+z^2)} \ge \frac{2(x^2y^2+y^2z^2+z^2x^2)}{\prod(x^2+y^2)}$ *(*) Assure*

$$\prod(x^2+y^2) = \sum x^2\cdot\sum x^2y^2 - x^2y^2z^2 \quad (3)$$

(*); (3) $\Rightarrow \frac{9}{4(x^2+y^2+z^2)} \ge \frac{2(x^2y^2+y^2z^2+z^2x^2)}{\sum x^2\cdot\sum x^2y^2-x^2y^2z^2} \Leftrightarrow 9\cdot\left(\sum x^2\cdot\sum x^2y^2 - x^2y^2z^2\right) \ge$

$$\ge 8\cdot\sum x^2\cdot\sum x^2y^2$$

$$\sum x^2\cdot\sum x^2y^2 \ge 9x^2y^2z^2 \quad (x=y=z)$$

$$(AM \ge GM)$$

UP.131. Prove that in any triangle ABC the following relationship holds:

$$2^{\sin A}+2^{\sin B}+2^{\sin C}+2^{\cos A}+2^{\cos B}+2^{\cos C} > \frac{6}{\left(\sqrt{2}\right)^{\sqrt{2}}}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

By AM-GM:

$$2^{\sin A}+2^{\cos A} \ge 2\sqrt{2^{\sin A}\cdot 2^{\cos A}} = 2\sqrt{2^{\sin A+\cos A}} = 2\sqrt{2^{\sqrt{2}\cos\left(\frac{\pi}{4}-A\right)}} =$$

$$= 2\cdot 2^{\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}-A\right)} > 2\cdot 2^{-\frac{\sqrt{2}}{2}} = \frac{2}{2^{\frac{\sqrt{2}}{2}}} = \frac{2}{\left(\sqrt{2}\right)^{\sqrt{2}}}$$

Analogous: $2^{\sin B} + 2^{\cos B} > \frac{2}{(\sqrt{2})^{\sqrt{2}}}$; $2^{\sin C} + 2^{\cos C} > \frac{2}{(\sqrt{2})^{\sqrt{2}}}$

By adding: $2^{\sin A} + 2^{\sin B} + 2^{\sin C} + 2^{\cos A} + 2^{\cos B} + 2^{\cos C} > \frac{6}{(\sqrt{2})^{\sqrt{2}}}$

UP.132. If $a, b > 0; m, n \geq 1$ then:

$$\int_0^b \left(\int_0^b (\sin x \sin y)^{2n} (\cos x \cos y)^{2m} dx\right) dy \leq \frac{ab}{4^{m+n}}\left(\frac{m}{n}\right)^{m-n}$$

Proposed by Daniel Sitaru – Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian

If $x, y > 0; x + y = 1, n, m \in \mathbb{N}$ ***then*** $x^n y^m \leq \frac{n^n\cdot m^m}{(n+m)^{n+m}}$***, if*** $x = \sin^2 x, y = \cos^2 x$ ***then:***

$$(\sin^2 x)^n \cdot (\cos^2 x)^m \leq \frac{n^n\cdot m^m}{(n+m)^{m+n}} \quad (1)$$

Using (1) we have: $LHS \leq \frac{n^n m^m}{(n+m)^{m+n}}\cdot a\cdot \frac{n^n\cdot m^m}{(n+m)^{m+n}} = \frac{n^{2n}\cdot m^{2m}\cdot ab}{(n+m)^{2(m+n)}}$ ***(2)***

$$n + m \geq 2\sqrt{nm} \Rightarrow \frac{1}{n+m} \leq \frac{1}{2\sqrt{nm}},$$

$$(2) \Rightarrow \frac{n^{2n}\cdot m^{2m}\cdot ab}{(n+m)^{2(m+n)}} \leq \left(\frac{1}{2\sqrt{nm}}\right)^{2(m+n)}\cdot n^{2n}\cdot m^{2m}\cdot ab = \frac{ab}{4^{m+n}}\cdot\left(\frac{m}{n}\right)^{m-n}$$

Q.E.D.

UP.133. Prove that if: $0 \leq b \leq a \leq \frac{\pi}{4}$ then:

$$\int_a^b \left(\int_a^b \left(\frac{\sin^2(x+y) + \sin^2(x-y) - 1}{1 + 2\sin x \sin y}\right) dx\right) dy \geq (a-b)(\sin^2 a - \sin^2 b + b - a)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\int_a^b\int_a^b\left(\frac{\sin^2(x+y)+\sin^2(x-y)-1}{1+2\sin x\cos y}\right)dx\,dy=2\int_a^b(a+b-x)\frac{\sin^2(2x)-1}{1+2\sin x\cos x}dx=$$

$$=2\int_a^b(a+b-x)(\sin 2x-1)dx=2(a+b)\int_a^b(\sin 2x-1)\,dx-2\int_a^b x(\sin 2x-1)dx$$

$$=2(a+b)\left(\frac{\cos 2b-\cos 2a}{2}+a-b\right)-2\int_b^a x(\sin 2x-1)dx$$

$$=2(a+b)(\sin^2 a-\sin^2 b+a-b)-2\int_b^a(a+b-z)(\sin(2a+2b-2z)-1)dz$$

$$\left[\begin{array}{c}\therefore\ let\ z=a+b-x,dx=-dz;when\ x=b,z=a\\ when\ x=a,z=b\end{array}\right]$$

$$\geq 2(a+b)(\sin^2 a-\sin^2 b+a-b)-2\int_b^a(a+b-z)(\sin(\pi-2z)-1)dz$$

$$\Rightarrow 2\int_b^a(a+b-x)(\sin 2x-1)dx\geq 2(a+b)(\sin^2 a-\sin^2 b+a-b)$$

$$\Rightarrow\int_a^a(a+b-x)(\sin 2x-1)dx\geq(a+b)(\sin^2 a-\sin^2 b+a-b)$$

We need need to prove, $(a+b)(\sin^2 a-\sin^2 b+a-b)\geq$

$$\geq(a-b)(\sin^2 a-\sin^2 b+a-b)$$

$\Leftrightarrow b(\sin^2 a-\sin^2 b+a-b)\geq 0$, ***which is true (Hence Proved)***

UP.134. In $\Delta\,ABC$ the following relationship holds:

$$\frac{a^2(m_a+m_b)}{h_c}+\frac{b^2(m_b+m_c)}{h_a}+\frac{c^2(m_c+m_a)}{h_b}\geq 8\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursarescu-Romania

$$\frac{a^2(m_a+m_b)}{h_c}+\frac{b^2(m_b+m_c)}{h_a}+\frac{c^2(m_c+m_a)}{h_b}\geq 8\sqrt{3}S$$

We must show this: $\frac{a^2m_a}{h_c}+\frac{b^2m_b}{h_a}+\frac{c^2m_c}{h_b}+\frac{a^2m_b}{h_c}+\frac{b^2m_c}{h_a}+\frac{c^2m_a}{h_b}\geq 8\sqrt{3}S$***. First we show this:*** $\frac{a^2m_a}{h_c}+\frac{b^2m_b}{h_a}+\frac{c^2m_c}{h_b}\geq 4\sqrt{3}S$ ***(1)***

We know: $m_a\geq\frac{b^2+c^2}{4R}\Rightarrow m_a\geq\frac{bc}{2R}\Rightarrow\frac{a^2m_a}{h_c}+\frac{b^2m_b}{h_a}+\frac{c^2m_c}{h_b}\geq\frac{a^2bc}{2Rh_c}+\frac{ab^2c}{2Rh_a}+\frac{c^2ab}{2Rh_b}=$

$$=\frac{abc}{2R}\left(\frac{a}{h_c}+\frac{b}{h_a}+\frac{c}{h_b}\right)=2S\left(\frac{ac}{ch_c}+\frac{ab}{ah_a}+\frac{h_c}{bh_b}\right)=(ab+bc+ac)\quad (2)$$

From (1)+(2) we must show: $ab+bc+ac\geq 4\sqrt{3}S$ ***(3)***

Now $ab+bc+ac=s^2+r^2+4Rr\Rightarrow$ ***(3)*** $\Leftrightarrow s^2+r^2+4Rr\geq 4\sqrt{3}sr$ ***(4)***

Now use Doucet inequality: $9r(4R+r)\leq 3s^2\leq(4R+r)^2$

From left side of Doucet and (4) we must show this:

$$16Rr+4r^2\geq 4\sqrt{3}sr\Leftrightarrow 4r(4R+r)\geq 4\sqrt{3}sr\Leftrightarrow 4R+r\geq\sqrt{3}s\Leftrightarrow(4R+r)^2\geq 3s^2$$

true $\Rightarrow$ ***then (3) its true. Now, we show this:***

$$\left.\begin{array}{l}\frac{a^2m_b}{h_c}+\frac{b^2m_c}{h_a}+\frac{c^2m_a}{h_b}\geq 4\sqrt{3}S\ (5)\\ \text{but } m_a\geq h_a, m_b\geq h_b, m_c\geq h_c\end{array}\right\}\Rightarrow(5)\Rightarrow \text{we must show: } \frac{a^2h_b}{h_c}+\frac{b^2h_c}{h_a}+\frac{c^2h_a}{h_b}\geq 4\sqrt{3}S\ \ (6)$$

But $\frac{a^2h_b}{h_c}+\frac{b^2h_c}{h_a}+\frac{c^2h_a}{h_b}=\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}=\frac{a^3c^2+b^3a^2+c^3b^2}{abc}$ ***(7)***

Now, we show this: $\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}\geq ab+bc+ac\Leftrightarrow$

$$a^3c^2+b^3a^2+c^3b^2\geq abc(ab+bc+ac)\quad (8)$$

Now, use inequality of generalized enviroments $\Rightarrow$

$$\left.\begin{array}{l}\frac{4}{7}a^3c^2+\frac{2}{7}b^3a^2+\frac{1}{7}c^3b^2\geq a^2b^2c\\ \frac{4}{7}b^3a^2+\frac{2}{7}c^3b^2+\frac{1}{7}a^3c^2\geq a^2bc^2\\ \frac{4}{7}c^3b^2+\frac{2}{7}a^3c^2+\frac{1}{7}b^2a^2\geq ab^2c^2\end{array}\right\}\Rightarrow a^3c^2+b^3a^2+c^2b^2\geq abc(ab+ac+bc)\Rightarrow$$

$\Rightarrow$ ***(8) its true. From (7) + (8) we must show:*** $ab+bc+ac\geq 4\sqrt{3}S$***, but this inequality its true from (3). From (1) + (5)*** $\Rightarrow$ ***inequality its true.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{a^2(m_a + m_b)}{h_c} \overset{Tereshin}{\geq} \sum \frac{a^2(\sum a^2 + c^2)}{4R} \cdot \frac{2R}{ab} =$$

$$= \frac{1}{2}\left(\sum a^2\right)\left(\sum \frac{a}{b}\right) + \frac{1}{2}\sum c^2\left(\frac{a}{b}\right) \overset{A-G}{\geq} \frac{\sum a^2}{2} \cdot 3 + \frac{1}{2}\sum c^2\left(\frac{a}{b}\right) \overset{Ionescu-Weitzenbock}{\geq}$$

$$\geq \frac{4\sqrt{3}S \cdot 3}{2} + \frac{1}{2}\sum c^2\left(\frac{a}{b}\right) \geq 6\sqrt{3}S + \frac{1}{2}4S\sqrt{\frac{a}{b}\cdot\frac{b}{c} + \frac{b}{c}\cdot\frac{c}{a} + \frac{c}{a}\cdot\frac{a}{b}}$$

$$\left(\because ma^2 + nb^2 + pc^2 \geq 4S\sqrt{\sum mn}, \forall m, n, p \geq 0\right)$$

$$= 6\sqrt{3}S + 2S\sqrt{\sum \frac{b}{a}} \overset{A-G}{\geq} 6\sqrt{3}S + 2S\sqrt{3} = 8\sqrt{3}S \text{ (Proved)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{matrix} m_a + m_b \geq h_a + h_b \\ m_b + m_c \geq h_b + h_c \\ m_c + m_a \geq h_c + h_a \end{matrix}\right\} \Leftrightarrow \sum a^2\left(\frac{m_a + m_b}{h_c}\right) \geq \sum a^2\left(\frac{h_a + h_b}{h_c}\right) = \left(a^2 \cdot \frac{h_a}{h_c} + b^2 \cdot \frac{h_c}{h_a}\right) +$$

$$+\left(b^2 \cdot \frac{h_b}{h_a} + c^2 \cdot \frac{h_a}{h_b}\right) + \left(c^2 \cdot \frac{h_c}{h_b} + a^2 \cdot \frac{h_b}{h_c}\right) \overset{M_a \geq M_g}{\geq} 2ab + 2bc + 2ca = 2(ab + bc + ca)$$

$$= 2\left(\frac{ab \cdot \sin C}{2} \cdot \frac{2}{\sin C} + \frac{bc \cdot \sin A}{2} \cdot \frac{2}{\sin A} + \frac{ca \cdot \sin B}{2} \cdot \frac{2}{\sin B}\right) =$$

$$= 4S\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right) \geq \left(\begin{matrix} f(x) = \frac{1}{\sin x} \\ f''(x) \geq 0 \end{matrix}\right)$$

$$= 4S \cdot \sum \frac{1}{\sin A} \geq 12S \cdot \frac{1}{\sin\frac{\pi}{3}} = \frac{24S}{\sqrt{3}} = 8\sqrt{3}S$$

UP.135. If $a, b, c > 0$ then:

$$\frac{a + c^2}{xb + yc} + \frac{b + a^2}{zc + ya} + \frac{c + b^2}{xa + yb} \geq \frac{3}{x + y} + \frac{a + b + c}{x + y}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursarescu-Romania

From Bergström inequality $\Rightarrow$

$$\frac{c^2}{xb+yc}+\frac{a^2}{xc+ya}+\frac{b^2}{xa+yb}\geq\frac{(a+b+c)^2}{(a+b+c)(x+y)}=\frac{a+b+c}{x+y}\quad (1)$$

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}=\frac{a^2}{xab+yac}+\frac{b^2}{xbc+yab}+\frac{c^2}{xac+ybc}\geq\frac{(a+b+c)^2}{(ab+ac+bc)(x+y)}\quad (2)$$

From (2)+(3)⇒ But $(a+b+c)^2\geq 3(ab+ac+bc)$ ***(3)***

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}\geq\frac{3}{x+y}\quad (4)$$

From (1)+(4)⇒ $\frac{a+c^2}{xb+yc}+\frac{b+a^2}{xc+ya}+\frac{c+b^2}{xa+yb}\geq\frac{3}{x+y}+\frac{a+b+c}{x+y}$

Solution 2 by Tran Hong-Vietnam

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}=\frac{a^2}{abx+acy}+\frac{b^2}{bcx+aby}+\frac{c^2}{acx+bcy}\overset{Schwarz}{\geq}$$

$$\frac{(a+b+c)^2}{(ab+ac+bc)(x+y)}\geq\frac{(a+b+c)^2}{\frac{(a+b+c)^2}{3}(x+y)}=\frac{3}{x+y}\quad (1)$$

$$\frac{c^2}{xb+yc}+\frac{a^2}{xc+ya}+\frac{b^2}{xa+yb}\overset{Schwarz}{\geq}\frac{(a+b+c)^2}{(a+b+c)(x+y)}=\frac{a+b+c}{x+y}\quad (2)$$

From (1) and (2) ⇒ proved. Equality ⇔ $a=b=c$.