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SOLUTIONS

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JP.106. Let $a, b, c > 0$. Prove that:

$$\frac{8(a+b+c)^3}{(a+b)(b+c)(c+a)}+5(ab+bc+ca)\geq 12+10(a+b+c)$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution by Rade Krenkov-Strumica-Macedonia

Inequality is equivalent with:

$$\frac{8(a+b+c)^2(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}+5(ab+bc+ca)^2\geq 12(ab+bc+ca)+$$

$+10(a+b+c)(ab+bc+ca)$. ***Using equality*** $(a+b+c)(ab+bc+ca)=$

$=(a+b)(b+c)(c+a)+abc$ ***have***

$$8(a+b+c)^2+\frac{8(a+b+c)^2abc}{(a+b)(b+c)(c+a)}+5(ab+bc+ca)^2\geq 12(ab+bc+ca)+$$

$+10(a+b+c)(ab+bc+ca)$. ***From AM-GM we get:***

$5(a+b+c)^2+5(ab+bc+ca)^2\geq 10(a+b+c)(ab+bc+ca)$. ***Now, enough to prove that:*** $3(a+b+c)^2+\frac{8(a+b+c)^2abc}{(a+b)(b+c)(c+a)}\geq 12(ab+bc+ca)$

$$3(a+b+c)^2+\frac{8(a+b+c)^3abc}{(a+b)(b+c)(c+a)\cdot(a+b+c)}\geq 12(ab+bc+ca)$$

From AM-GM we get:

$$8(a+b+c)^3=[(a+b)+(b+c)+(c+a)]^3\geq\left[3\sqrt[3]{(a+b)(b+c)(c+a)}\right]^3=$$

$=27(a+b)(b+c)(c+a)$. ***We, must prove that:***

$$3(a+b+c)^2+\frac{27abc}{a+b+c}\geq 12(ab+bc+ca)\Leftrightarrow$$

$$\Leftrightarrow (a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca)$$

From Schur's inequality we have:

$$(a+b+c)^3=a^2+b^3+c^3+3\sum a^2b+3\sum ab^2+6abc\geq$$

$\geq 4\sum a^2b+4\sum ab^2+3abc$. ***Now,*** $(a+b+c)^3+9abc\geq$

$$\geq 4\sum a^2b+4\sum ab^2+12abc=4(a+b+c)(ab+bc+ca)$$

JP.107. Prove that in any triangle ABC with incentre I the following relationship holds:

$$AI\cdot\frac{a^2}{w_a}+BI\cdot\frac{b^2}{w_b}+CI\cdot\frac{c^2}{w_c}\le 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)},$$

where R is the circumradius, r is the inradius of triangle ABC and w_a, w_b, w_c are the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Rajsekhar Azaad-India

$$\text{For any }\Delta:\frac{AI}{w_a}=\frac{b+c}{2s},\frac{BI}{w_b}=\frac{c+a}{2s},\frac{CI}{w_c}=\frac{a+b}{2s}$$

$$\Rightarrow LHS=\sum\frac{AI\cdot a^2}{w_a}=\sum\frac{a^2(b+c)}{2s}=\frac{(a+b+c)(ab+bc+ca)-3abc}{2s}$$

$$=\frac{2s(s^2+r^2+4Rr)-3\cdot 4Rrs}{2s}=s^2+r^2-2Rr$$

$$\le 4R^2+4Rr+3r^2+r^2-2Rr\quad\text{(Gerretsen)}=4R^2+2Rr+4r^2$$

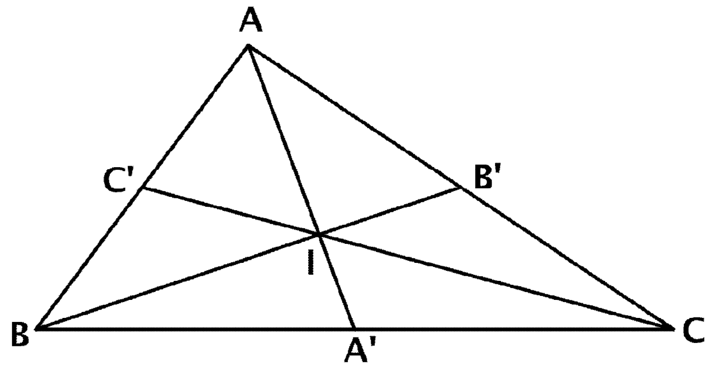
$$\le 4R^2+R^2+R^2=6R^2\quad (i)$$

$$RHS=3\sqrt{2}\frac{R^2}{r}\sqrt{r(R-r)}=3\sqrt{2}R^2\sqrt{\frac{R}{r}\left(\frac{R}{r}-1\right)}\ge 3\sqrt{2}R^2\sqrt{2(2-1)}\ (\because R\ge 2r)$$

$$=6R^2\quad (ii)$$

From (i) & (ii) inequality proved.

Solution 2 by Marian Ursarescu-Romania



From bisector theorem $\Rightarrow \frac{BA'}{A'C}=\frac{c}{b}\Rightarrow\frac{BA'}{a}=\frac{c}{b+c}\Rightarrow BA'=\frac{ac}{b+c}$

$$\frac{AI}{IA'}=\frac{c}{BA'}=\frac{c}{\frac{ac}{b+c}}=\frac{b+c}{a}\Rightarrow\frac{AI}{AA'}=\frac{b+c}{a+b+c}\Rightarrow\frac{AI}{w_a}=\frac{b+c}{2p}$$

Inequalities become: $\frac{1}{2p}\sum a^2(b+c)\le 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)}$ *(1)*

But $\sum a^2(b+c)=2p(p^2+r^2-2Rr)$ *(2)*

From (1)+(2) we show: $p^2+r^2-2Rr\le 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)}$ *(3)*

But $R\ge 2r\Rightarrow 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)}\ge 6R^2$ *(4)*

From (3)+(4) we must show: $p^2+r^2-2Rr\le 6R^2$ *(5)*

But (5) its true from Blundon – Gerretsen inequality: $p^2\le 4R^2+4Rr+3r^2$.

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$AI=r\csc\frac{A}{2},BI=r\csc\frac{B}{2},CI=r\csc\frac{C}{2},w_a=\frac{2bc\cos\frac{A}{2}}{b+c},$$

$$w_b=\frac{2ac\cos\frac{B}{2}}{c+a}\text{ and }w_c=\frac{2ab\cos\frac{C}{2}}{a+b}$$

$$\sum_{cyc}\frac{AI\cdot a^2}{w_a}\overset{\substack{Chebyshev's\\ Inequality}}{\le}\frac{a^2+b^2+c^2}{3}\cdot\left(\sum_{cyc}\frac{AI}{w_a}\right)$$

$$\left[let\ a^2\ge b^2\ge c^2\ then\ \frac{r\csc\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}}\le\frac{r\csc\frac{B}{2}}{\frac{2ca\cos\frac{B}{2}}{c+a}}\le\frac{r\csc\frac{C}{2}}{\frac{2ab\cos\frac{C}{2}}{a+b}}\right]$$

$$\le 3R^2\left(\sum_{cyc}\frac{r\csc\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}}\right)=3R^2r\left(\sum_{cyc}\frac{b+c}{bc\sin A}\right)=6R^3r\left(\sum_{cyc}\frac{b+c}{abc}\right)$$

$=6R^3r\cdot\frac{2\cdot 2p}{4Rrp}=6R^2$*, we need to prove,* $6R^2\le\frac{3\sqrt{2}R^2}{r}\sqrt{R(R-r)}$

$$\Leftrightarrow\sqrt{2}r\le\sqrt{R(R-r)}\Leftrightarrow R^2-Rr-2r^2\ge 0\Leftrightarrow(R+r)(R-2r)\ge 0,$$

which is also true

$$\therefore \sum_{cyc} \frac{AI \cdot a^2}{w_a} \le 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)}$$

(proved)

JP.108. If $a, b > 0$, then:

$$4\sqrt{ab} \cdot \frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a > 6\sqrt{ab}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $0 < x < \frac{\pi}{2}, a, b > 0$

$$4\sqrt{ab}\frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a \ge 4\sqrt{ab}\frac{\sin x}{x} + 2\sqrt{b\left(\frac{\tan x}{x}\right)^2 a}$$

$$= 2\sqrt{ab}\left[\frac{2\sin x}{x} + \frac{\tan x}{x}\right] \quad (1)$$

Let $g(x) = 2\sin x + \tan x - 3x, 0 \le x < \frac{\pi}{2}$

$$g'(x) = 2\cos x + \sec^2 x - 3, 0 < x < \frac{\pi}{2} :> 3[\cos^2 x \sec^2 x]^{\frac{1}{3}} - 3, 0 < x < \frac{\pi}{2}$$

$$\Rightarrow g'(x) > 0 \text{ for } 0 < x < \frac{\pi}{2} \Rightarrow g(x) > g(0) \text{ for } 0 < x < \frac{\pi}{2}$$

$$\Rightarrow 2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \text{ for } 0 < x < \frac{\pi}{2} \quad (2)$$

From (1), (2), we get: $4\sqrt{ab}\frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a > 6\sqrt{ab}$ *for* $0 < x < \frac{\pi}{2}$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know, $\sin x \ge x - \frac{x^3}{6}, \tan x \ge x + \frac{x^3}{3}$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$

$$4\sqrt{ab} \cdot \frac{\sin x}{x} + g\left(\frac{\tan x}{x}\right)^2 + a \ge 4\sqrt{ab} \cdot \frac{\sin x}{x} + 2\sqrt{ab} \cdot \frac{\tan x}{x}$$

$$= 2\sqrt{ab}\left(2\frac{\sin x}{x} + \frac{\tan x}{x}\right) > 2\sqrt{ab}\left(2 - \frac{x^2}{3} + 1 + \frac{x^2}{3}\right) \text{ for all } x \in \left(0, \frac{\pi}{2}\right)$$

$$= 6\sqrt{ab} \quad \textit{(proved)}$$

JP.109. If $a, b > 0$, then:

$$(a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} > \frac{6ab}{a+b}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}\overset{(1)}{>}\frac{6ab}{a+b}$$

$$\textit{(1)}\Leftrightarrow(a+b)^2\sin x+2ab\tan x\overset{(2)}{>}6abx$$

$$\because (a+b)^2\geq 4ab\ \therefore \textit{LHS of (2)}\geq 4ab\sin x+2ab\tan x\overset{?}{>}6abx\Leftrightarrow 2\sin x+\tan x-3x\underset{(3)}{\overset{?}{>}}0$$

$$\textit{Let } f(x)=2\sin x+\tan x-3x\ \forall x\in\left[0,\frac{\pi}{2}\right)$$

$$f'(x)=\sec^2 x+2\cos x-3\ \textit{and}\ f''(x)=2(\sec^2 x\tan x-\sin x)=$$

$$=2(\tan x(1+\tan^2 x)-\sin x)=2(\tan x-\sin x+\tan^3 x)\geq 2(\sin x-\sin x+\tan^3 x)$$

$$\left(\because \forall x\in\left[0,\frac{\pi}{2}\right),\tan x\geq x\geq\sin x\right)=2\tan^3 x\geq 0\ \therefore f''(x)\geq 0\ \forall x\in\left[0,\frac{\pi}{2}\right)$$

$$\Rightarrow f'(x)\uparrow \textit{on}\left[0,\frac{\pi}{2}\right)\Rightarrow f'(x)\geq f'(0)=0,\forall x\in\left[0,\frac{\pi}{2}\right)\Rightarrow f(x)\uparrow\textit{on}\left[0,\frac{\pi}{2}\right)$$

$$\Rightarrow f(x)\geq f(0)=0\Rightarrow\forall x\in\left[0,\frac{\pi}{2}\right),2\sin x+\tan x-3x\geq 0,\ \textit{equality at } x=0$$

$$\therefore \forall a,b>0,x\in\left(0,\frac{\pi}{2}\right),2\sin x+\tan x-3x>0\Rightarrow \textit{(3) is true (Proved)}$$

JP.110. If $x,y,z\in(0,1)$ and ABC is a triangle, then prove that:

$$\frac{\sin^2\frac{A}{2}}{x(1-x^3)}+\frac{\sin^2\frac{B}{2}}{y(1-y^3)}+\frac{\sin^2\frac{C}{2}}{z(1-z^3)}\geq\frac{2\sqrt[3]{4}}{3R}(2R-r)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\textit{Let } f(x)=x(1-x^3),0<x<1;f'(x)=1-4x^3;f'(x)=0\Rightarrow x=\left(\frac{1}{4}\right)^{\frac{1}{3}}$$

$$f''(x)=-12x^2\Rightarrow f''\left(\left(\frac{1}{4}\right)^{\frac{1}{3}}\right)<0\Rightarrow f(x)\ \textit{has a maximum for}\ x=\left(\frac{1}{4}\right)^3$$

$$\therefore \max_{0<x<1}f(x)=\frac{1}{4^{\frac{1}{3}}}\left(1-\frac{1}{4}\right)=\frac{3}{4^{\frac{4}{3}}}\Rightarrow\frac{1}{x(1-x^3)}\geq\frac{4^{\frac{4}{3}}}{3}\ \textit{for}\ 0<x<1$$

$$\therefore \frac{\sin^2\left(\frac{A}{2}\right)}{x(1-x^3)}+\frac{\sin^2\left(\frac{B}{2}\right)}{y(1-y^3)}+\frac{\sin^2\left(\frac{C}{2}\right)}{z(1-z^3)} \geq \frac{4^{\frac{4}{3}}}{3}\left[\sin^2\left(\frac{A}{2}\right)+\sin^2\left(\frac{B}{2}\right)+\sin^2\left(\frac{C}{2}\right)\right]$$

$$=\frac{4^{\frac{4}{3}}}{3}\left[1-\frac{r}{2R}\right]=\frac{2\left(3^{\frac{1}{4}}\right)(2R-r)}{3R}$$

$$\left[\begin{array}{c} 1-\sin^2\frac{A}{2}-\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=\cos^2\frac{A}{2}-\sin^2\frac{B}{2}-\sin^2\frac{C}{2} \\ =\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)-\sin^2\frac{C}{2}=\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\frac{A+B}{2}\right] \\ =2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=2\frac{(s-a)(s-b)(s-c)}{abc}=\frac{2}{3}\cdot\frac{\Delta^2}{abc} \\ =\frac{1}{2}\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right)=\frac{1}{2}\cdot\frac{r}{R} \end{array}\right]$$

Solution 2 by Marian Ursarescu-Romania

Let $f:(0,1)\to\mathbb{R}, f(x)=x(1-x^3)=x-x^4$

$$f(x)=1-4x^3=0\Rightarrow x=\frac{1}{\sqrt[3]{4}}$$

x	0		$\frac{1}{\sqrt[3]{4}}$		1
$f'(x)$		+ + + + + + + + + + + + + + +	0	− − − − − − − − − − − − − − −	
$f(x)$		$\nearrow$	$\frac{3}{4\sqrt[3]{4}}$ M	$\searrow$	

$$\Rightarrow f(x)\leq\frac{3}{4\sqrt[3]{4}}\Rightarrow$$

$$x(1-x^3)\leq\frac{3}{4\sqrt[3]{4}}\Rightarrow\frac{1}{x(1-x^3)}\geq\frac{4\sqrt[3]{4}}{3}\quad \textbf{\textit{(1)}}$$

From (1) inequalities become: $\frac{\sin^2\frac{A}{2}}{x(1-x^3)}+\frac{\sin^2\frac{B}{2}}{y(1-y^3)}+\frac{\sin^2\frac{C}{2}}{z(1-z^3)}\geq\frac{4\sqrt[3]{4}}{3}\left(\sum\sin^2\frac{A}{2}\right)$ ***(2)***

But $\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}$ ***(3)***

From (2)+(3)$\Rightarrow\frac{\sin^2\frac{A}{2}}{x(1-x^3)}+\frac{\sin^2\frac{B}{2}}{y(1-y^3)}+\frac{\sin^2\frac{C}{2}}{z(1-z^3)}\geq\frac{2\sqrt[3]{4}}{3R}(2R-r)$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = x - x^4$ for all $x \in (0,1)$ then $f'(x) = 1 - 4x^3$

$f''(x) = -12x^2$ now let $f'(c) = 0 \Rightarrow c = \frac{1}{\sqrt[3]{4}}$ then $f''(x) < 0$ at $x = \frac{1}{\sqrt[3]{4}}$

hence at $x = \frac{1}{\sqrt[3]{4}}$ the function attains maximum then $f\left(\frac{1}{\sqrt[3]{4}}\right) \geq f(x)$

$$\sum_{cyc} \frac{\sin^2 \frac{A}{2}}{x(1-x^3)} \geq \frac{4\sqrt[3]{4}}{3} \sum_{cyc} \sin^2 \frac{A}{2} = \frac{4\sqrt[3]{4}}{3} \sum_{cyc} \frac{(p-b)(p-c)}{bc}$$

$$= \frac{4\sqrt[3]{4}}{3} \cdot \frac{a(p-b)(p-c) + b(p-a)(p-c) + c(p-a)(p-b)}{abc}$$

$$= \frac{4\sqrt[3]{4}}{3} \cdot \frac{p^2(a+b+c) - 2p(ab+bc+ca) + 3abc}{abc} = \frac{4\sqrt[3]{4}}{3} \cdot \frac{4Rrp - 2pr^2}{4Rrp} = \frac{2\sqrt[3]{4}}{3R}(2R - r) \quad \textbf{(Proved)}$$

JP.111. Prove that:

(i) If $a, b, c, d, x, y, z, t \in \mathbb{R}_+^*$, then:

$$4(a^2 + b^2 + c^2 + d^2 + x^2 + y^2 + z^2 + t^2) +$$
$$+8\sqrt{(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + t^2)} \geq$$
$$\geq (a + b + c + d + x + y + z + t)^2;$$

(ii) If $a, b, c, m, n, p, x, y, z \in \mathbb{R}_+^*$ then:

$$5(a^3 + b^3 + c^3 + m^3 + n^3 + p^3 + x^3 + y^3 + z^3) +$$
$$+3\sqrt[3]{(a^3 + b^3 + c^3)(m^3 + n^3 + p^3)(x^3 + y^3 + z^3)} \geq$$
$$\geq 2(a + b + c)(m + n + p)(x + y + z)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$4(a^2 + b^2 + c^2 + d^2 + x^2 + y^2 + z^2 + t^2) + 8\sqrt{(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + t^2)} \geq$$

$\geq (a + b + c + d + x + y + z + t)^2$; Let $a + b + c + d = \sum a$, $x + y + z + t = \sum x$,

$$a^2 + b^2 + c^2 + d^2 = \sum a^2, x^2 + y^2 + z^2 + t^2 = \sum x^2$$

$$LHS \overset{Chebyshev}{\geq} 4\left(\sum a^2+\sum x^2\right)+8\sqrt{\frac{(\sum a)^2}{4}\cdot\frac{(\sum x)^2}{4}}=$$

$$=4\sum a^2+4\sum x^2+2\left(\sum a\right)\left(\sum x\right)$$

$$\overset{Chebyshev}{\geq}(\sum a)^2+(\sum x)^2+2(\sum a)(\sum x)=(\sum a+\sum x)^2=RHS \quad \textit{(Proved)}$$

$$5(a^3+b^3+c^3+m^3+n^3+p^3+x^3+y^3+z^3)+$$

$$+3\sqrt[3]{(a^3+b^3+c^3)(m^3+n^3+p^3)(x^3+y^3+z^3)}\geq$$

$$\geq 2(a+b+c)(m+n+p)(x+y+z)$$

Let $a+b+c=\sum a\,, m+n+p=\sum m\,, x+y+z=\sum x,$

$$a^3+b^3+c^3=\sum a^3\,, m^3+n^3+p^3=\sum m^3\,, x^3+y^3+z^3=\sum x^3$$

$$LHS \overset{Chebyshev}{\geq} \frac{5}{9}\left[\left(\sum a\right)^3+\left(\sum m\right)^3+\left(\sum x\right)^3\right]+3\sqrt[3]{\frac{(\sum a)^3}{9}+\frac{(\sum m)^3}{9}+\frac{(\sum x)^3}{9}}$$

$$=\frac{5}{9}\left[\left(\sum a\right)^3+\left(\sum m\right)^3+\left(\sum x\right)^3\right]+\frac{(\sum a)(\sum m)(\sum x)}{3}$$

$$\overset{A-G}{\geq}\frac{5}{9}\cdot 3\left(\sum a\right)\left(\sum m\right)\left(\sum x\right)+\frac{(\sum a)(\sum m)(\sum x)}{3}=2\left(\sum a\right)\left(\sum m\right)\left(\sum x\right)=RHS$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$4\left(\sum_{cyc}a^2+\sum_{cyc}x^2\right)+8\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}x^2\right)}$$

$$=4\left(\sqrt{\sum_{cyc}a^2}+\sqrt{\sum_{cyc}x^2}\right)^2 \overset{\substack{ROOT\ MEAN\\ SQUARE}}{\geq} 4\left(\frac{\sum_{cyc}a}{\sqrt{4}}+\frac{\sum_{cyc}x}{\sqrt{4}}\right)^2$$

$$=(a+b+c+d+x+y+z+t)^2 \quad \textit{(proved)}$$

$$5\left(\sum_{cyc}a^3+\sum_{cyc}m^3+\sum_{cyc}x^3\right)+3\sqrt[3]{\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}m^3\right)\left(\sum_{cyc}x^3\right)}$$

$$\overset{AM\geq GM}{\geq} 18\sqrt[3]{\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} m^3\right)\left(\sum_{cyc} x^3\right)}$$

$$\geq 18\sqrt[3]{\frac{(a+b+c)^3}{9}\cdot\frac{(m+n+p)^3}{9}\cdot\frac{(x+y+z)^3}{9}}$$

$$= 2(a+b+c)(m+n+p)(x+y+z)$$

JP.112. Prove that if $a, b \in R_+, x, y, u, v \in R_+^*$ then:

$$\left(a\cdot\frac{x}{y}+b\cdot\frac{u}{v}\right)^2+\left(a\cdot\frac{y}{x}+b\cdot\frac{v}{u}\right)^2\geq 2(a+b)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

By Cauchy-Schwarz we have: $LHS \geq \frac{\left(a\cdot\frac{x}{y}+b\cdot\frac{u}{v}+a\cdot\frac{y}{x}+b\cdot\frac{v}{u}\right)^2}{2} = \frac{\left(a\cdot\frac{x}{y}+a\cdot\frac{y}{x}+b\cdot\frac{u}{v}+b\cdot\frac{v}{u}\right)^2}{2}$

$\geq \frac{(2a+2b)^2}{2} = 2(a+b)^2 \Rightarrow$ **Q.E.D.**

Solution 2 by Henry Ricardo-New York-USA

Recalling that $\alpha + \frac{1}{\alpha} \geq 2$ ***for*** $\alpha > 0$ ***by the AM-GM inequality, we see that***

$$\left(a\cdot\frac{x}{y}+b\cdot\frac{u}{v}\right)^2+\left(a\cdot\frac{y}{x}+b\cdot\frac{v}{u}\right)^2=$$

$$=\left(a\cdot\frac{x}{y}\right)^2+2ab\frac{xu}{yv}+\left(b\cdot\frac{u}{v}\right)^2+\left(a\cdot\frac{y}{x}\right)^2+2ab\frac{yv}{xu}+\left(b\cdot\frac{v}{u}\right)^2$$

$$=a^2\left[\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2\right]+2ab\left(\frac{xu}{yv}+\frac{yv}{xu}\right)+b^2\left[\left(\frac{u}{v}\right)^2+\left(\frac{v}{u}\right)^2\right]$$

$$\geq 2a^2+2b^2+4ab = 2(a+b)^2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\left(a\frac{x}{y}+b\frac{u}{v}\right)^2+\left(a\frac{y}{x}+b\frac{v}{u}\right)^2=$$

$$=a^2\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+b^2\left(\frac{u^2}{v^2}+\frac{v^2}{u^2}\right)+2ab\left(\frac{xu}{yv}+\frac{yv}{xu}\right)\geq 2a^2+2b^2+2ab(2)=2(a+b)^2$$

JP.113. Let x, y, z be real numbers such that:

$$x^2y + y^2z + z^2x + 4(xy^2 + yz^2 + zx^2) + 13xyz = 5$$

Find the minium value of the expression:

$$P = (x^2 + 4xy + 5y^2)(y^2 + 4yz + 5z^2)(z^2 + 4zx + 5x^2) + 6x^2y^2z^2$$

Proposed Do Quoc Chinh – Vinh Phuc – Viet Nam

Solution by proposer

We will prove that $P \geq 15$. ***Since*** $y^2 + 4yz + 5z^2 = (y + 2z)^2 + z^2$. ***And***

$$(x^2 + 4xy + 5y^2)(z^2 + 4zx + 5x^2) = [(x + 2y)^2 + y^2][x^2 + (z + 2x)^2] =$$

$$= [x(x + 2y) + y(z + 2x)]^2 + [(x + 2y)(z + 2x) - xy]^2 =$$

$$= (x^2 + 4xy + yz)^2 + (2x^2 + 3xy + 2y + zx)^2$$

By the Cauchy-Schwarz inequality, we have:

$$P \geq [(y + 2z)(x^2 + 4xy + yz) + z(2x^2 + 3xy + 2yz + zx)]^2 =$$

$$= [x^2y + y^2z + z^2x + 4(xy^2 + yz^2 + zx^2) + 11xyz]^2 = (5 - 2xyz)^2$$

Thus, it suffies to show that: $(5 - 2xyz)^2 + 6x^2y^2z^2 \geq 15.$

This inequality is equivalent to $(xyz - 1)^2 \geq 0$

The equality holds for $\begin{cases} x^2y + y^2z + z^2x = \frac{16}{5} \\ xy^2 + yz^2 + zx^2 = \frac{-14}{5} \\ xyz = 1 \end{cases}$

JP.114. Let a, b, c be positive real numbers. Prove that:

(a) $ln^3\left(\frac{ab}{c}\right) + ln^3\left(\frac{bc}{a}\right) + ln^3\left(\frac{ca}{b}\right) + 24\ln a \cdot \ln b \cdot \ln c = ln^3(abc)$

(b) $ln(ab) \cdot ln(bc) \cdot ln(ca) = \frac{ln^3(abc) - ln^3 a - ln^3 b - ln^3 c}{3}$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Ravi Prakash-New Delhi-India

Put $x = \ln a, y = \ln b, z = \ln c$. ***With this (a) can be written as***

$(x + y - z)^3 + (y + z - x)^3 + (x + z - y)^3 + 24xyz = (x + y + z)^3$ ***and (b) becomes***

$$(x + y)(y + z)(z + x) = \frac{1}{3}[(x + y + z)^3 - x^3 - y^3 - z^3]$$

(a) Consider

$$(x+y+z)^3+(x+y-z)^3+\big(z-(x-y)\big)^3+\big(z+x(x-y)\big)^3$$
$$=2(x+y)^3+6(x+y)z^2+2z^3+6(x-y)^2z$$
$$=2(x+y+z)^3-6(x+y)z(x+y+z)+6(x+y)z^2+6(x-y)^2z$$
$$=2(x+y+z)^3-6(x+y)^2z-6(x+y)z^2+6(x+y)z^2+6(x-y)^2z$$
$$=2(x+y+z)^3-6z[(x+y)^2-(x-y)^2]=2(x+y+z)^3-6z(4xy)$$
$$\Rightarrow (x+y-z)^3+(z+x-y)^3+(z+y-x)^3+24xyz=(x+y+z)^3$$

as desired.

(b) Consider

$$(x+y+z)^3-x^3-(y^3+z^3)=$$
$$=(x+y+z-x)[(x+y+z)^2+x(x+y+z)+x^2]-(y+z)(y^2-yz+z^2)$$
$$=(y+z)[(x+y+z)^2+x(x+y+z)+x^2-y^2+yz-z^2]$$
$$=(y+z)[(x+y+z-y)(x+y+z+y)+(x^2+xy+yz+zx)+(x-z)(x+z)]$$
$$=(y+z)[(x+z)\{(x+y)+(y+z)\}+(x+y)(x+z)+(x-z)(x+z)]$$
$$=(y+z)(x+z)[x+y+y+z+x+y+x-z]=3(x+y)(y+z)(z+x)$$

JP.115. If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$ then:

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\ge 2+\frac{18}{a+b+c}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

Solution 1 by Christos Eythimiou-Greece

$$a,b,c>0\wedge a^2+b^2+c^2=3\Rightarrow\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\ge 2+\frac{18}{a+b+c}$$

$$a,b,c>0\wedge a^2+b^2+c^2=3\Rightarrow\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)=$$

$$2+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}=2+\frac{a^2b+b^2c+c^2a+a^2c+b^2a+c^2b}{abc}=$$

$$2+\frac{(a+b+c)(ab+bc+ca)-3abc}{abc}=$$

$$2+\frac{(a+b+c)^2(ab+bc+ca)-3(abbc+bcca+caab)}{abc(a+b+c)}\geq$$

$$2+\frac{\left(a^2+b^2+c^2+2(ab+bc+ca)\right)(ab+bc+ca)-(ab+bc+ca)^2}{abc(a+b+c)}=$$

$$2+\frac{(1+1+1+ab+bc+ca)(ab+bc+ca)}{abc(a+b+c)}\geq$$

$$2+\frac{(6\sqrt[6]{1\cdot 1\cdot 1\cdot ab\cdot bc\cdot ca})(3\sqrt[3]{abbcca})}{abc(a+b+c)}=2+\frac{18}{a+b+c}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow \frac{(a+b)(b+c)(c+a)}{abc}\overset{(1)}{\geq} 2+\frac{6\sqrt{3\sum a^2}}{\sum a}\quad (\because \sum a^2=3)$

Let $a+b=x, b+c=y, c+a=z$. ***Then*** $x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ ***are three sides of a triangle with semiperiemeter, circumradius & inradius*** $=s,R,r$ ***respectively (say):*** $\therefore \sum a=s\Rightarrow c=s-x, a=s-y, b=s-z$. ***Using above subtitution, (1)*** $\Leftrightarrow \frac{xyz}{r^2S}-2\geq \frac{6}{s}\sqrt{3\sum(s-y)^2}\Leftrightarrow \frac{4Rrs}{r^2s}-2\geq\frac{6}{s}\sqrt{3\sum(s^2-2sy+y^2)}\Leftrightarrow$

$$\Leftrightarrow \frac{2R-r}{r}\geq\frac{3}{s}\sqrt{3\{3s^2-4s^2+2(s^2-4Rr-r^2)\}}\Leftrightarrow$$

$\Leftrightarrow \frac{(2R-r)^2}{r^2}\geq\frac{27}{s^2}(s^2-8Rr-2r^2)\Leftrightarrow (2R-r)^2s^2\overset{(2)}{\geq}27r^2(s^2-8Rr-2r^2)\because s^2\geq 27r^2\therefore$ ***it suffices to prove:*** $(2R-r)^2\geq s^2-8Rr-2r^2\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2$, ***which is true by Gerretsen*** $\Rightarrow$ ***(2) is true (hence proved)***

JP.116. If a,b,c are positive real numbers such that $abc=1$ then:

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\leq\frac{3}{2}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

Solution by Marian Ursarescu-Romania

Because $abc=1\Rightarrow a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$ ***with*** $x,y,z>0$. ***Inequality becomes:***

$$\sum\frac{\frac{x}{y}}{\frac{x^2}{y^2}+\frac{y}{z}\cdot\frac{z}{x}}\le\frac{3}{2}\Leftrightarrow\sum\frac{\frac{x}{y}}{\frac{x^3+y^3}{xy^2}}\le\frac{3}{2}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{x^2y}{x^3+y^3}\le\frac{3}{2}\quad\textbf{(1)}$$

But $x^3+y^3\ge xy(x+y)$ ***(because*** $x^3+y^3-x^2y-xy^2\ge 0\Leftrightarrow$

$\Leftrightarrow x^2(x-y)+y^2(x-y)\ge 0\Leftrightarrow(x-y)(x^2-y^2)\ge 0\Leftrightarrow(x-y)^2(x+y)\ge 0$ ***true)***

$$\Rightarrow\frac{1}{x^3+y^3}\le\frac{1}{xy(x+y)}\Rightarrow\frac{x^2y}{x^3+y^3}\le\frac{x}{x+y}\quad\textbf{(2)}$$

From (1) + (2) we must show this: $\frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x}\le\frac{3}{2}\Leftrightarrow$

$$\Leftrightarrow 2x(y+z)(z+x)+2y(x+y)(z+x)+2z(x+y)(y+z)\le$$

$$\le 3(x+y)(y+z)(z+x)\Leftrightarrow$$

$$\Leftrightarrow 6xyz+4x^2y+4xz^2+4y^2z+2xz^2+2y^2z+2yz^2\le$$

$$\le 6xyz+3x^2y+3xz^2+3y^2z+3x^2z+3xy^2+3yz^2\Leftrightarrow$$

$$\Leftrightarrow z^2y+x^2z+xy^2-x^2y-y^2z-z^2x\ge 0\Leftrightarrow$$

$$\left.\begin{array}{c}\Leftrightarrow xy(y-x)+xz(x-z)+zy(z-y)\Leftrightarrow\\ But\ z-y=z-x+x-y\end{array}\right\}\Leftrightarrow$$

$$\Leftrightarrow xy(y-x)+xz(x-z)+zy(z-x+x-y)\ge 0\Leftrightarrow$$

$$\Leftrightarrow(xy-zy)(y-x)+(xz-zy)(x-z)\ge 0$$

$$\Leftrightarrow(y-x)y(x-z)+(x-z)z(x-y)\ge 0\Leftrightarrow$$

$\Leftrightarrow(y-x)(x-z)(y-z)\ge 0$ ***which is true.***

JP.117. If a,b,c are positive real numbers then:

$$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\ge\frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

Solution by proposer

First, we prove: $\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$

Proof.

$$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\Leftrightarrow\frac{b(c+a)-bc}{c(c+a)}+\frac{c(a+b)-ca}{a(a+b)}+\frac{a(b+c)-ab}{b(b+c)}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$\Leftrightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{a+c}+\frac{b+c}{b+a}+\frac{c+a}{c+b}$***. Let*** $x=\frac{a}{b},y=\frac{b}{c},z=\frac{c}{a}\Rightarrow xyz=1$***. So,***

$$\frac{a+b}{a+c}=\frac{1+yz}{1+z}=y+\frac{1-y}{1+z};\ \frac{b+c}{b+a}=\frac{1+zx}{1+x}=z+\frac{1-z}{1+x}$$

$\frac{c+a}{c+b}=\frac{1+xy}{1+y}=x+\frac{1-x}{1+y}$***. We need to prove that:*** $\frac{x-1}{y+1}+\frac{y-1}{z+1}+\frac{z-1}{x+1}\geq 0$

$$\Leftrightarrow(x^2-1)(z+1)+(y^2-1)(x+1)+(z^2-1)(y+1)\geq 0$$

$$\Leftrightarrow x^2z+z^2y+y^2x+x^2+y^2+z^2\geq x+y+z+3\quad\textbf{(1)}$$

On the other hand, we have: $x^2z+z^2y+y^2x\geq 3xyz=3$***. And***

$x^2+y^2+z^2\geq\frac{1}{3}(x+y+z)^2\geq x+y+z$***. So (1) right! Or***

$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$***. Next, we prove:***

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

Proof. Assume

$$a\geq b\geq c\Rightarrow(a-c)^2=\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

We have: $\frac{a}{b+c}-\frac{1}{2}+\frac{b}{c+a}-\frac{1}{2}+\frac{c}{a+b}-\frac{1}{2}$

$$=\frac{1}{2}\left[\frac{(a-b)+(a-c)}{b+c}+\frac{(b-c)+(b-a)}{c+a}+\frac{(c-a)+(c-b)}{a+b}\right]$$

$$=\frac{1}{2}\left[\frac{(a-b)^2}{(a+c)(b+c)}+\frac{(b-c)^2}{(b+a)(c+a)}+\frac{(c-a)^2}{(c+b)(a+b)}\right]$$

$$\geq\frac{1}{2}\cdot\frac{(a-b+b-c+a-c)^2}{(a+c)(b+c)+(b+a)(c+a)+(c+b)(a+b)}$$

$$=\frac{1}{2}\cdot\frac{4(a-c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}\geq\frac{1}{2}\cdot\frac{4(a-c)^2}{(a+b+c)^2+\frac{1}{3}(a+b+c)^2}$$

$$=\frac{3(a-c)^2}{2(a+b+c)^2}=\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

So, $\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\ge\frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$

Equality occurs if and only if $a=b=c$.

JP.118. Let a,b,c be the three sides of a triangle. Prove that:

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\ge\frac{3(a^3+b^3+c^3)}{a+b+c}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\overset{(1)}{\ge}\frac{3\sum a^3}{\sum a}$$

$$\textbf{(1)}\Leftrightarrow\frac{ab^4+bc^4+ca^4}{abc}\ge\frac{3\sum a^3}{\sum a}$$

$$\Leftrightarrow\left(\sum a\right)(ab^4+bc^4+ca^4)\overset{(2)}{\ge}3abc\left(\sum a^3\right)$$

Let $s-a=x,s-b=y,s-c=z(x,y,z>0)$

$$\therefore s=x+y+z\Rightarrow a=y+z,b=z+x,c=x+y$$

By this substitution, (2) transforms into:

$$2\left((x+y)(y+z)^4+(y+z)(z+x)^4+(z+x)(x+y)^4\right)(x+y+z)\ge$$

$$\ge3(x+y)(y+z)(z+x)((x+y)^3+(y+z)^3+(z+x)^3)$$

$$\Leftrightarrow2\sum x^6+6\sum x^5y+7\sum x^4y^2+2\sum x^3y^3\ge3\sum x^2y^4+2xyz(\sum x^2y)+6xyz(\sum xy^2+$$

18x2y2z2 (a)

Now, $y^6+x^4y^2\overset{A-G}{\ge}2x^2y^4\rightarrow$ ***(1a)***

$z^6+y^4z^2\overset{A-G}{\ge}2y^2z^4\rightarrow$ ***(1b)***

$x^6+z^4x^2\overset{A-G}{\ge}2z^2x^4\rightarrow$ ***(1c)***

(1a)+(1b)+(1c)$\Rightarrow\sum x^6+\sum x^4y^2\ge2\sum x^2y^4$ ***(1)***

Again, $y^6+y^6+x^6\overset{A-G}{\ge}3x^2y^4\rightarrow$ ***(2a)***

$z^6+z^6+y^6\overset{A-G}{\ge}3y^2z^4\rightarrow$ ***(2b)***

$$x^6+x^6+z^6\overset{A-G}{\geq}3z^2x^4\to\textit{(2c)}$$

$$\textit{(2a)+(2b)+(2c)}\Rightarrow 3\sum x^6\geq 3\sum x^2y^2\Rightarrow\sum x^6\geq\sum x^2y^4\quad\textit{(2)}$$

$$\textit{Also, }x^4y^2+y^4z^2\overset{A-G}{\geq}2x^2y^3z\to\quad\textit{(3a)}$$

$$y^4z^2+z^4x^2\overset{A-G}{\geq}2y^2z^3x\to\quad\textit{(3b)}$$

$$z^4x^2+x^4y^2\overset{A-G}{\geq}2z^2x^3y\to\textit{(3c)}$$

$$\textit{(3a)+(3b)+(3c)}\Rightarrow 2\sum x^4y^2\geq 2xyz\left(\sum xy^2\right)\Rightarrow 6\sum x^4y^2\geq 6xyz\left(\sum xy^2\right)\to\textit{(3)}$$

$$\textit{Moreover, }x^3y^3+x^3y^3+z^3x^3\overset{A-G}{\geq}3x^3y^2z\to\textit{(4a)}$$

$$y^3z^3+y^3z^3+x^3y^3\overset{A-G}{\geq}3y^3z^2x\to\quad\textit{(4b)}$$

$$z^3x^3+z^3x^3+y^3z^3\overset{A-G}{\geq}3z^3x^2y\to\textit{(4c)}$$

$$\textit{(4a)+(4b)+(4c)}\Rightarrow 3\sum x^3y^3\geq 3xyz\left(\sum x^2y\right)\Rightarrow 2\sum x^3y^3\geq 2xyz\left(\sum x^2y\right)\to\quad\textit{(4)}$$

$$\textit{Lastly, }6\sum x^5y\overset{A-G}{\geq}18x^2y^2z^2\to\textit{(5)}$$

(1)+(2)+(3)+(4)+(5)⇒(a) is true (Proved)

JP.119. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a+b}{c^2(c^3+a+b)}+\frac{b+c}{a^2(a^3+b+c)}+\frac{c+a}{b^2(b^3+c+a)}\geq\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}$$

Proposed by Do Quoc Chinh – Vinh Phuc – Vietnam

Solution by proposer

The inequality need to prove to be equivalent to:

$$\frac{c^3+a+b-c^3}{c^2(c^3+a+b)}+\frac{a^3+b+c-a^3}{a^2(a^3+b+c)}+\frac{b^3+c+a-b^3}{b^2(b^3+c+a)}\geq\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}$$

$$\Leftrightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}+\frac{a}{a^3+b+c}+\frac{b}{b^3+c+a}+\frac{c}{c^3+a+b}$$

Applying the Hölder and AM-GM inequality, we have:

$$\sum\frac{a}{a^3+b+c}=\sum\frac{a(1+b+c)^2}{(a^3+b+c)(1+b+c)^2}\leq\sum\frac{a(1+b+c)^2}{(a+b+c)^3}$$

$$=\frac{\sum a\,(1+b^2+c^2+2bc+2b+2c)}{27}$$

$$=\frac{4(ab+bc+ca)+a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+6abc}{27}$$

$$=\frac{4(ab+bc+ca)+(a+b+c)(ab+bc+ca)+3abc+3}{27}$$

$$=\frac{7(ab+bc+ca)+3abc+3}{27}\le\frac{7\cdot\frac{(a+b+c)^2}{3}+3\cdot\frac{(a+b+c)^3}{27}+3}{27}=1$$

Therefore, we need to prove that: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}+1$

Applying the Cauchy-Schwarz inequality, we have:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{3}\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\ge\frac{3}{a+b+c}\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

We have: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1=\frac{1}{3}\cdot(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1=\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$

We need to prove that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\ge\frac{2\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$

We will prove that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$*. Which is equivalent to:*

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\ge\frac{3(a^2+b^2+c^2)}{\sqrt[3]{a^2b^2c^2}}\Leftrightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\frac{3(a^2+b^2+c^2)}{\sqrt[3]{a^2b^2c^2}}$$

Applying the AM-GM inequality, we have:

$$LHS=\sum\frac{a^2}{b^2}+\frac{a}{c}+\frac{a}{c}\ge3\sum\sqrt[3]{\frac{a^4}{b^2c^2}}=\frac{3(a^2+b^2+c^2)}{\sqrt[3]{a^2b^2c^2}}=RHS$$

Similarly, we have: $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\ge\frac{\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$

The proof of the inequality is complete. The equality holds for $a=b=c=1$.

JP.120. Let a, b, c be positive real numbers and $k \in [1; 3]$. Prove that:

$$\frac{1}{a^2 + ab + ca + kbc} + \frac{1}{b^2 + bc + ab + kca} + \frac{1}{c^2 + ca + bc + kab} \le \frac{9}{(k+3)(ab + bc + ca)}$$

Proposed by Do Quoc Chinh – Vinh Phuc – Vietnam

Solution by proposer

Put $\frac{1}{a} = x, \frac{1}{b} = y, \frac{1}{c} = z$. *The inequality need to prove to be equivalent to:*

$$\frac{1}{a^2 + ab + ca + kbc} + \frac{1}{b^2 + bc + ab + kca} + \frac{1}{c^2 + ca + bc + kab} \le \frac{9}{(k+3)(ab + bc + ca)}$$

$$\Leftrightarrow \frac{1}{\frac{1}{x^1} + \frac{1}{xy} + \frac{1}{zx} + \frac{k}{yz}} + \frac{1}{\frac{1}{y^2} + \frac{1}{yz} + \frac{1}{xy} + \frac{k}{zx}} + \frac{1}{\frac{1}{z^2} + \frac{1}{zx} + \frac{1}{yz} + \frac{k}{xy}} \le \frac{9}{(k+3)\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right)}$$

$$\Leftrightarrow \frac{x}{kx^2 + xy + yz + zx} + \frac{y}{ky^2 + xy + yz + zx} + \frac{z}{kz^2 + xy + yz + zx} \le \frac{9}{(k+3)(x + y + z)}$$

$$\Leftrightarrow \frac{x(xy + yz + zx)}{kx^2 + xy + yz + zx} + \frac{y(xy + yz + zx)}{ky^2 + xy + yz + zx} + \frac{z(xy + yz + zx)}{kz^2 + xy + yz + zx} \le \frac{9(xy + yz + zx)}{(k+3)(x + y + z)}$$

$$\Leftrightarrow x - \frac{kx^3}{kx^2 + xy + yz + zx} + y - \frac{ky^3}{ky^2 + xy + yz + zx} + z - \frac{kz^3}{kz^2 + xy + yz + zx} \le \frac{9(xy + yz + zx)}{(k+3)(x + y + z)}$$

$$\Leftrightarrow k\left(\frac{x^3}{kx^2 + xy + yz + zx} + \frac{y^3}{ky^2 + xy + yz + zx} + \frac{z^3}{kz^2 + xy + yz + zx}\right) + \frac{9(xy + yz + zx)}{(k+3)(x + y + z)} \le x + y + z$$

Applying the Cauchy-Schwarz inequality, we have:

$$\sum \frac{x^3}{kx^2 + xy + yz + zx} \ge \frac{\left(\sum x^2\right)^2}{\sum x\,(kx^2 + xy + yz + zx)} = \frac{\left(\sum x^2\right)^2}{3kxyz + \left(\sum x\right)\left[k\left(\sum x^2\right) - (k-1)\sum xy\right]}$$

Note that we have: $(xy + yz + zx)^2 \ge 3xyz(x + y + z) \Leftrightarrow 3xyz \le \frac{(xy+yz+zx)^2}{x+y+z}$

Therefore, we have: $\sum \frac{x^3}{kx^2+xy+yz+zx} \ge \frac{\left(\sum x^2\right)^2\left(\sum x\right)}{k\left(\sum xy\right)^2+\left(\sum x\right)^2\left[k\sum x^2-(k-1)\sum xy\right]}$

Therefore, we need to prove that: $\frac{k\left(\sum x^2\right)^2\left(\sum x\right)}{k\left(\sum xy\right)^2+\left(\sum x\right)^2\left[k\sum x^2-(k-1)\sum xy\right]} + \frac{9\left(\sum xy\right)}{(k+3)\left(\sum x\right)} \ge \sum x$

$$\Leftrightarrow \frac{k\left(\sum x^2\right)^2}{k\left(\sum xy\right)^2 + \left(\sum x\right)^2\left[k\sum x^2 - (k-1)\sum xy\right]} + \frac{9\left(\sum xy\right)}{(k+3)\left(\sum x\right)^2} \ge 1$$

Put $t = \frac{x^2+y^2+z^2}{xy+yz+zx}$ $(t \ge 1)$. ***The inequality is equivalent to:*** $\frac{kt^2}{k+(t+2)(kt-k+1)} + \frac{9}{(k+3)(t+2)} \ge 1$

$$\Leftrightarrow \frac{kt^2(k+3)}{kt^2+kt+t-k+2}-k+\frac{9}{t+2}-3\geq 0 \Leftrightarrow (t-1)\left[\frac{k(3t+2-k)}{kt^2+kt+t-k+2}-\frac{3}{t+2}\right]\geq 0$$

$$\Leftrightarrow k(3t+2-k)(t+2)\geq 3[kt^2+kt+t-k+2]$$

$$\Leftrightarrow k(3t^2-tk-2k+8t+4)\geq 3kt^2+3kt+3t-3k+6$$

$$\Leftrightarrow k^2(t+2)-k(5t+7)+3(t+2)\leq 0 \Leftrightarrow k^2-k\cdot\frac{5t+7}{t+2}+3\leq 0$$

We see $\frac{5t+7}{t+2}=5-\frac{3}{t+2}\geq 5-\frac{3}{1+2}=4$***. Therefore, we have:***

$k^2-k\cdot\frac{5t+7}{t+2}+3\leq k^2-4k+3=(k-1)(k-3)\leq 0$***. The equality for*** $a=b=c$***.***

SP.106. In ΔABC the following relationship holds:

$$(a\cot 20^\circ+b\cot 40^\circ+c\cot 80^\circ)^3>9\sqrt{3}r\left(\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\cot 20^\circ 40^\circ \cot 80^\circ=\frac{\cos 20^\circ\cos 40^\circ\cos 80^\circ}{\sin 20^\circ\sin 40^\circ\sin 80^\circ}=\frac{\cos 80^\circ}{4\sin 20^\circ\cdot\frac{1}{2}(\cos 20^\circ-\cos 60^\circ)}=$$

$$=\frac{\cos 80^\circ}{2\sin 20^\circ\cos 20^\circ-\sin 20^\circ}=\frac{\cos 80^\circ}{2\sin 10^\circ\cos 30^\circ}=\frac{1}{2\cdot\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{3}$$

$$\cot 20^\circ\cot 40^\circ\cot 80^\circ=\frac{\sqrt{3}}{3}\quad \textit{(1)}$$

$$r\sum\frac{a^3}{r_a}=r\sum\frac{a^3(s-a)}{rs}=\frac{1}{s}\sum a^3(s-a)\quad \textit{(2)}$$

We prove that: $\sum a^3(s-a)\leq abcs \Leftrightarrow \sum a^3(b+c-a)\leq abc(a+b+c)$

$\Leftrightarrow \sum a^2(a-b)(a-c)\geq$ ***(by Schur's inequality)***

By (2): $r\sum\frac{a^3}{r_a}\leq abc$ ***(3)***

$$(a\cot 20^\circ+b\cot 40^\circ+c\cot 80^\circ)^3\overset{AM-GM}{>}27abc\cot 20^\circ\cot 40^\circ\cot 80^\circ\overset{(1)}{=}$$

$$=27\cdot\frac{\sqrt{3}}{3}abc\overset{(3)}{\geq}9\sqrt{3}r\sum\frac{a^3}{r_a}$$

SP.107. Prove that:

$$\left(\int_0^1 arctan^2 x\,dx\right)\left(\int_0^1 \frac{dx}{arctan^2\left(\frac{1}{x^2-x+1}\right)}\right) > \frac{1}{4}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

We know $x \geq \tan^{-1} x$ ***and*** $\tan^{-1} x \geq x + \frac{x^3}{3}$ ***for all*** $x \geq 0$

$$\left(\int_0^1 (\tan^{-1} x)^2 dx\right)\left(\int_0^1 \frac{dx}{\left(\tan^{-1}\frac{1}{x^2-x+1}\right)^2}\right)$$

$$\geq \left(\int_0^1 \left(x + \frac{x^3}{3}\right)^2 dx\right)\left(\int_0^1 (x^2-x+1)^2 dx\right)$$

$$= \left(\frac{1}{3} + \frac{1}{63} + \frac{2}{15}\right)\left(\frac{1}{5} + \frac{1}{3} + 1 - \frac{1}{2} - 1 + \frac{2}{3}\right) = \frac{152}{315} \cdot \frac{7}{10} > \frac{1}{4}\ \textit{(Proved)}$$

SP.108. If $a, b, c > 0, a + b + c = abc$ **then:**

$$\frac{4(a+b)(a+c)}{(b+c)^2} + \frac{4(b+c)(b+a)}{(c+a)^2} + \frac{4(c+a)(c+b)}{(a+b)^2} \leq 3 + a^2 + b^2 + c^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

The above expression can be written as

$$\left\{\frac{4(a+b)(a+c)}{(b+c)^2} - 1\right\} + \left\{\frac{4(b+c)(b+a)}{(c+a)^2} - 1\right\} + \left\{\frac{4(c+a)(c+b)}{(a+b)^2} - 1\right\} \leq a^2 + b^2 + c^2$$

Let us take $\frac{4(a+b)(a+c)}{(b+c)^2} - 1 = \frac{4(a^2+ac+ab+bc)-(b+c)^2}{(b+c)^2}$

$$= \frac{4a^2 + 4ac + 4ab + 4bc - b^2 - c^2 - 2bc}{(b+c)^2} = \frac{4a^2 + 4ac + 4ab + 2bc - b^2 - c^2}{(b+c)^2}$$

$$= \frac{4a^2 + 4ac + 4ab - (b-c)^2}{(b+c)^2} = \frac{4a(a+b+c)}{(b+c)^2} - \left(\frac{b-c}{b+c}\right)^2$$

$$= \frac{4a(abc)}{(b+c)^2} - \left(\frac{b-c}{b+c}\right)^2 \{\because a+b+c=abc\}$$

$$\because b,c>0, AM-GM$$

$$\frac{b+c}{2} \ge \overline{bc} \Rightarrow (b+c)^2 \ge 4bc$$

$$\le \frac{4a^2(bc)}{(4bc)} - \left(\frac{b-c}{b+c}\right)^2 \le a^2 - \left(\frac{b-c}{b+c}\right)^2 \le a^2$$

$$\therefore \frac{4(a+b)(a+c)}{(b+c)^2} - 1 \le a^2 \Rightarrow \frac{4(a+b)(a+c)}{(b+c)^2} \le a^2+1 \ \textit{(i)}$$

Similarly, we have

$$\frac{4(b+c)(b+a)}{(c+a)^2} \le b^2+1 \quad \textit{(ii)}$$

$$\textit{and} \frac{4(c+a)(c+b)}{(a+b)^2} \le c^2+1 \ \textit{(iii)}$$

Adding (i), (ii), (iii) we get the result

$$\frac{4(a+b)(a+c)}{(b+c)^2} + \frac{4(b+c)(b+a)}{(c+a)^2} + 4\left(\frac{(c+a)(c+b)}{(a+b)^2}\right) \le a^2+b^2+c^2+3 \ \textit{(proved)}$$

Solution 2 by Boris Colakovic-Belgrade-Serbia

$$b+c \overset{AM-GM}{\ge} 2\sqrt{bc} \Rightarrow (b+c)^2 \ge 4bc \Rightarrow \frac{1}{(b+c)^2} \le \frac{1}{4bc}$$

$$\frac{4(a+b)(a+c)}{(b+c)^2} \le \frac{(a+b)(a+c)}{bc} = \frac{a+b}{b}\cdot\frac{a+c}{c} = \left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right) = 1 + \frac{a^2+ab+ac}{bc} \quad \textit{(1)}$$

Similarly

$$\frac{4(b+c)(b+a)}{(c+a)^2} \le \frac{(b+c)(b+a)}{ca} = 1 + \frac{b^2+ab+bc}{ca} \quad \textit{(2)}$$

$$\frac{4(c+a)(c+b)}{(a+b)^2} \le 1 + \frac{c^2+ac+bc}{ab} \quad \textit{(3)}$$

$$\textit{(1)+(2)+(3)} \Rightarrow LHS \le 3 + \frac{a^2+ab+ac}{bc} + \frac{b^2+ab+bc}{ca} + \frac{c^2+ac+bc}{ab} =$$

$$= 3 + \frac{(a^3+a^2b+a^2c)+(b^3+b^2a+b^2c)+(c^3+c^2a+c^2b)}{abc} =$$

$$= 3 + \frac{a^2(a+b+c)+b^2(a+b+c)+c^2(a+b+c)}{a+b+c} = 3+a^2+b^2+c^2$$

Solution 3 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

By Cauchy's inequality we get: $\sum\frac{4(a+b)(a+c)}{(b+c)^2} \le \sum\frac{4(a+b)(a+c)}{4bc} = \sum\frac{a(a+b+c)+bc}{bc}$

$$= \sum\frac{a^2bc+bc}{bc} = \sum(a^2+1) = 3+a^2+b^2+c^2 \Rightarrow \textit{Q.E.D.}$$

SP.109. If $a, b, c \ge 0; \Omega(a) = \int_0^a \sin\left(\frac{x}{x^2+1}\right)dx$ then:

$$e^{\pi(b\Omega(a)+c\Omega(b)+a\Omega(c))} \ge (a^2+1)^b(b^2+1)^c(c^2+1)^a$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

From Jordan's inequality: $\sin x \ge \frac{2x}{\pi}; x \ge 0 \Rightarrow \sin\left(\frac{x}{x^2+1}\right) \ge \frac{2x}{x^2+1}\cdot\frac{1}{\pi}$

$$\int_0^a \sin\left(\frac{x}{x^2+1}\right)dx \ge \frac{1}{\pi}\int_0^a \frac{2x}{x^2+1}dx = \frac{1}{\pi}\ln(a^2+1)$$

$$\pi\Omega(a) \ge \ln(a^2+1) \Rightarrow \pi b\Omega(a) \ge b\ln(a^2+1)$$

$$\sum \pi b\Omega(a) \ge \sum b\ln(a^2+1) = \sum(a^2+1)^b$$

$$\pi\sum b\Omega(a) \ge \ln\prod(a^2+1)^b; e^{\pi\sum b\Omega(a)} \ge \prod(a^2+1)^b$$

$$e^{\pi(b\Omega(a)+c\Omega(b)+a\Omega(c))} \ge (a^2+1)^b(b^2+1)^c(c^2+1)^a$$

Equality holds for $a = b = c = 0$.

SP.110. Let $m, x, y, z > 0$ be positive real numbers and F be the area of the triangle ABC. Prove that:

$$\frac{a^{2m+2}x^{m+1}}{(y+z)^{m+1}} + \frac{b^{2m+2}y^{m+1}}{(z+x)^{m+1}} + \frac{c^{2m+2}z^{m+1}}{(x+y)^{m+1}} \ge \frac{2^{m+1}}{(\sqrt{3})^{m-1}}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje

Solution by Soumitra Mandal-Chandar Nagore-India

We know $r(r+4R) \ge \sqrt{3}F$ *then* $\sum_{cyc}\frac{a^{2m+2}x^{m+1}}{(y+z)^{m+1}}$

$$\geq \frac{1}{3^m}\left\{\sum_{cyc}\frac{a^2x}{y+z}\right\}^{m+1} = \frac{1}{3^m}\left\{(x+y+z)\sum_{cyc}\frac{a^2}{y+z}-\sum_{cyc}a^2\right\}^{m+1}$$

$$\overset{Bergstrom}{\geq} \frac{1}{3^m}\left\{\frac{(a+b+c)^2}{2}-\sum_{cyc}a^2\right\}^{m+1} = \frac{1}{3^m}\left\{\frac{2(ab+bc+ca)-a^2-b^2-c^2}{2}\right\}^{m+1}$$

$$= \frac{1}{3^m}\left\{\frac{2(p^2+r^2+4Rr)-2(p^2-r^2-4Rr)}{2}\right\}^{m+1} = \frac{2^{m+1}}{3^m}(r^2+4Rr)^{m+1}$$

$$= \frac{2^{m+1}}{3^m}\left(\sqrt{3}F\right)^{m+1} = \frac{2^{m+1}}{\left(\sqrt{3}\right)^{m-1}}F^{m+1} \ \textit{(Proved)}$$

SP.111. Let $x,y,z>0$ be positive real numbers and F the area of the triangle ABC. Prove that:

$$\frac{(y+z)^2a^4}{x^2}+\frac{(z+x)^2b^4}{y^2}+\frac{(z+x)^2c^4}{z^2}\geq 64F^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS \overset{Chebyshev}{\geq} \frac{1}{3}\left\{\frac{(y+z)a^2}{x}+\frac{(z+x)b^2}{y}+\frac{(x+y)c^2}{z}\right\}^2$$

$$\geq \frac{1}{3}\left\{4F\sqrt{\frac{(y+z)(z+x)}{xy}+\frac{(z+x)(x+y)}{yz}+\frac{(x+y)(y+z)}{xy}}\right\}^2$$

(∵ in any $\Delta\, ABC, \forall m,n,p\in\mathbb{R}^+, a^2m+b^2n+c^2p\geq 4F\sqrt{mn+np+pn}$***)***

$$= \frac{16F^2}{3}\cdot\frac{\sum\{z(y+z)(z+x)\}}{xyz}\overset{?}{\geq} 64F^2 \Leftrightarrow \sum\left\{z^3+z\left(\sum xy\right)\right\}\overset{?}{\geq} 12xyz$$

$$\Leftrightarrow \sum x^3+\left(\sum xy\right)\left(\sum x\right)\overset{?}{\geq} 12xyz \quad \textit{(1)}$$

$$\textit{LHS of (1)} \overset{A-G}{\geq} 3xyz+\left(\sqrt[3]{xyz}\cdot\sqrt[3]{x^2y^2z^2}\right)\cdot 9 = 3xyz+9xyz = 12xyz$$

$\Rightarrow$ ***(1) is true (Proved)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know $r(r+4R) \ge \sqrt{3}F$ *and* $p^2 \ge 3\sqrt{3}F$ *then*

$$\sum_{cyc}\frac{(y+z)^2a^4}{x^2} \ge \frac{1}{3}\left(\sum_{cyc}\frac{(y+z)a^2}{x}\right)^2 = \frac{1}{3}\left\{(x+y+z)\sum_{cyc}\frac{a^2}{x} - \sum_{cyc}a^2\right\}^2$$

$$\overset{Bergstrom}{\ge} \frac{1}{3}\left\{(a+b+c)^2-\sum\nolimits_{cyc}a^2\right\}^2 = \frac{4}{3}\left(p^2+r(r+4R)\right)^2 \ge \frac{4}{3}\left(3\sqrt{3}F+\sqrt{3}F\right)^2 = 64F^2$$

(Proved)

SP.112. Let $x, y, z > 0$ be positive real numbers and F be the area of the triangle ABC with circumradius R. Prove that:

$$\frac{x}{y+z}sin^2\frac{A}{2} + \frac{y}{z+x}sin^2\frac{B}{2} + \frac{z}{x+y}sin^2\frac{C}{2} \ge \frac{2\sqrt{3}F}{R^2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

Solution by Marian Ursarescu-Romania

First step: $\sum\frac{x}{y+z}\cdot\sin^2\frac{A}{2} = \sum\left(\frac{x+y+z-y-z}{y+z}\right)\sin^2\frac{A}{2} =$

$= (x+y+z)\sum\frac{1}{y+z}\cdot\sin^2\frac{A}{2} - \sum\sin^2\frac{A}{2}$ *(1)*

But from Cauchy inequality: $\sum\frac{1}{y+z}\cdot\sin^2\frac{A}{2} \ge \frac{\left(\sum\sin\frac{A}{2}\right)^2}{2(x+y+z)}$ *(2)*

From (1)+(2)⇒ $\sum\frac{x}{y+z}\sin^2\frac{A}{2} \ge \frac{1}{2}\left(\sum\sin\frac{A}{2}\right)^2 - \sum\sin^2\frac{A}{2}$ *(3)*

From (3) inequality becomes: $\frac{1}{2}\left(\sum\sin\frac{A}{2}\right)^2 - \sum\sin^2\frac{A}{2} \ge \frac{F}{2\sqrt{3}R^2}$

But $F = pr$ *(4)*

$\sum\sin^2\frac{A}{2} = 1 - \frac{r}{2R}$ *(5)*

And $\sin^n\left(\frac{A}{2}\right) + \sin^n\left(\frac{B}{2}\right) + \sin^n\left(\frac{C}{2}\right) \ge \frac{3}{2^n}, n \in \mathbb{N}^*$

In our case: $\sum\sin\frac{A}{2} \ge \frac{3}{2}$ *(6)*

From (4)+(5)+(6) inequality becomes: $\frac{9}{8} - 1 + \frac{r}{2R} \ge \frac{pr}{2\sqrt{3}R^2}$ *(7)*

But $p \le \frac{3\sqrt{3}}{2}R \Rightarrow \frac{pr}{2\sqrt{3}R^2} \le \frac{3r}{4R}$ *(8)*

From (7)+(8) we must show: $\frac{3r}{4R} \le \frac{1}{8} + \frac{r}{2R} \Leftrightarrow \frac{r}{4R} \le \frac{1}{8} \Leftrightarrow 2r \le R$ *(true)*

SP.113. If $x, y, z > 0$ then:

$$4xyz(x^3 + y^3 + z^3) \le (x^2 + y^2)(x^4 + y^4) + (y^2 + z^2)(y^4 + z^4) + (z^2 + x^2)(z^4 + x^4)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $xy(x^2 + y^2) \le x^4 + y^4 \Leftrightarrow (x - y)^2(x^2 + xy + y^2) \ge 0$ *or* $x^3y + xy^3 \le x^4 + y^4$

and $2xz + 2yz - 2z^2 \le x^2 + y^2 \Leftrightarrow (x - z)^2 + (y - z)^2 \ge 0 \Rightarrow$

$xy(x^2 + y^2)(2xz + 2yz - 2z^2) \le (x^2 + y^2)(x^4 + y^4)$ *or*

$$2x^4yz + 2x^2y^3z + 2x^3y^2z + 2xy^4z - 2x^3yz^2 - 2xy^3z^2 \le (x^2 + y^2)(x^4 + y^4) \Rightarrow$$

$$4xyz(x^3 + y^3 + z^3) = \sum_{cyclic}(2x^4yz + 2x^2y^3z + 2x^3y^2z + 2xy^4z - 2x^3yz^2 - 2xy^3z^2) \le$$

$$\le \sum_{cyclic}(x^2 + y^2)(x^4 + y^4)$$

SP.114. If $x, y, z > 0$ then:

$$2\left(\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + \left(z + \frac{1}{z}\right)^2\right) \le \frac{3}{2} + \frac{1}{xyz}\left((x^4 + y^4)z + (y^4 + z^4)x + (z^4 + x^4)y\right) +$$

$$+ \frac{x^2 + y^2}{z} + \frac{y^2 + z^2}{x} + \frac{z^2 + x^2}{y}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $2x + 2y - 2z \le \frac{x^2+y^2}{z} \Leftrightarrow (x - z)^2 + (y - z)^2 \ge 0$ *and* $x^2 + y^2 \le \frac{x^3}{y} + \frac{y^3}{x} \Leftrightarrow$

$\Leftrightarrow (x - y)^2(x^2 + xy + y^2) \ge 0$ *therefore* $\sum_{cyclic}(2x + 2y - 2z) + \sum_{cyclic}(x^2 + y^2) \le$

$\le \sum_{cyclic}\frac{x^2+y^2}{z} + \sum_{cyclic}\left(\frac{x^3}{y} + \frac{y^3}{x}\right)$ *or* $2\left(\sum x^2 + \sum x\right) \le \frac{1}{xyz}\sum(x^4 + y^4)\,z +$

$$+ \sum\frac{x^2 + y^2}{z} \Rightarrow 2\sum\left(x + \frac{1}{2}\right)^2 \le \frac{3}{2} + \frac{1}{xyz}\sum(x^4 + y^4)z + \sum\frac{x^2 + y^2}{z}$$

SP.115. Let a, b, c be the lengths of the sides of a triangle with inradius r and circumradius R. Let r_a, r_b, r_c be the exradii of triangle. Prove that:

$$1728\cdot r^5\le\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_c}\le 108R^4(R-r)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursărescu – Romania

$\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_c}\ge 3\sqrt[3]{\frac{(abc)^6}{r_ar_br_c}}\Rightarrow$ we must show: $3\sqrt[3]{\frac{(abc)^6}{r_ar_br_c}}\ge 2^6\cdot 3^3r^5\Leftrightarrow\frac{(abc)^6}{r_ar_br_c}\ge 2^{18}3^6r^{15}$ (1)

But $abc=4sRr$ and $r_ar_br_c=s^2r$ (2). From (1)+(2) we must show:

$$\frac{2^{12}s^6R^6r^6}{s^2r}\ge 2^{18}\cdot 3^6\cdot r^{15}\Leftrightarrow s^4R^6\ge 6^6r^{10}\Leftrightarrow s^2R^3\ge 6^3r^5\ \ (3)$$

But $R\ge 2r\Rightarrow R^3\ge 8r^3$ and $s^2\ge 27r^2\Rightarrow s^2R^3\ge 2^3\cdot 3^3\cdot r^5\Rightarrow$ (3) its true. Now we use Schur inequality: $a^5(a-b)(a-c)+b^5(b-c)(b-a)+c^5(c-a)(c-b)\ge 0\Leftrightarrow$

$$\Leftrightarrow a^5({}^2-ab-ac+bc)+\ b^5(b^2-ab-bc+ac)+c^5\left(c^2-4c-ac+a^b\right)\ge 0\Leftrightarrow$$

$$\Leftrightarrow a^6(a-b-c)+b^6(b-c-a)+c^6(c-a-b)+a^5bc+ab^5c+abc^5\ge 0\Leftrightarrow$$

$$\Leftrightarrow a^6(b+c-a)+b^6(a+c-b)+c^6(a+b-c)\le abc(a^4+b^4+c^4)\Leftrightarrow$$

$\Leftrightarrow 2a^6(s-a)+2b^6(s-b)+2c^6(s-c)\le abc(a^4+b^4+c^4)$. But $r_a=\frac{S}{s-a}\Rightarrow$

$\Rightarrow s-a=\frac{S}{r_a}\Rightarrow 2S\left(\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_b}\right)\ge abc(a^4+b^4+c^4)$ (4). But $S=\frac{abc}{4R}$ (5) $\Rightarrow$ From

(4)+(5) $\Rightarrow\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_b}\le 2R(a^4+b^4+c^4)$ (6). From (6) we must show:

$$2R(a^4+b^4+c^4)\le 108R^4(R-r)\Leftrightarrow a^4+b^4+c^4\le 54R^3(R-r)\ \ (7)$$

But $a^4+b^4+c^4=2(s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2)$ (8)

From (7)+(8) we must show: $s^4-2s^2\left(4Rr+3r^2\right)+r^2(4R+r)^2-27R^3(R-r)\le 0$ (9)

Now, let $f(s^2)$ a polygon of second degree $\Rightarrow f(p^2)=(p^2-x_1)(p^2-x_2)$, (9) its equivalent with $\left[p^2-r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}\right]\cdot$

$$\cdot\left[s^2-r(4R+3r)+\sqrt{8r^3(2R+r)+27R^3(R-r)}\right]\le 0\ \ (10)$$

(10) its true if $x_1\le p^2\le x_2, x_1, x_2$ its square, then we must show:

$$r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}\le s^2\ \ (1)$$

and $s^2 \le r(4R+3r) - \sqrt{8r^3(2R+r)+27R^3(R-r)}$ ***(2)***

For (1) using Gerretsen's inequality: $s^2 \ge 16Rr - 5r^2$ ***(11)*** $\Leftrightarrow$

$$\Leftrightarrow 4Rr+3r^2-\sqrt{8r^3(2R+r)+27R^3(R-r)} \le 16Rr-5r^2 \Leftrightarrow$$

$$\Leftrightarrow 8r^2 = 12Rr \le \sqrt{8r^3(2R+r)+27R^3(R-r)} \Leftrightarrow$$

$\Leftrightarrow (8r^2-12Rr)^2 \le 8r^3(2R+r)+27R^3(R-r) \Leftrightarrow R \ge 2r$ ***(Euler). For (12) using again Gerretsen's inequality*** $s^2 \le 4R^2+3r^2+4Rr$

(12) $\Leftrightarrow 4R^2+3r^2+4Rr \le r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)} \Leftrightarrow$

$\Leftrightarrow 8r^4+16Rr^3-27R^3r+11R^4 \ge 0$***. Let*** $x=\frac{R}{2r}\ge 1$ ***(Euler)*** $\Rightarrow$ ***we must show:***

$22x^4-27x^3+4x+1 \ge 0$ ***and with Horner and Rolle sequence*** $\Rightarrow$

$(x-1)\left(11x^3+(x-1)(11x^2+6x+1)\right) \ge 0$ ***true.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$1728r^5 \overset{(a)}{\le} \sum\frac{a^6}{r_a} \overset{(b)}{\le} 108R^4(R-r)$$

$$\sum\frac{a^6}{r_a} \le 108R^4(R-r) \Leftrightarrow \sum a^6(s-a) \le 108r^4\Delta(R-r) \Leftrightarrow \sum a^6(s-a) \overset{(1)}{\le} 108R^5\cdot$$

$$\cdot\Delta\left(1-\frac{r}{R}\right) = 108\frac{a^5b^5c^5}{1024\Delta^5}\cdot\Delta\left(1-\frac{\Delta}{s}\cdot\frac{4\Delta}{abc}\right) =$$

$=\frac{27}{256}\cdot\frac{a^4b^4c^4(abc-4(s-a)(s-b)(s-c))}{s^2(s-a)^2(s-b)^2(s-c)^2}$***. Let*** $s-a=x, s-b=y, s-c=z \therefore s=x+y+z \Rightarrow$

$\Rightarrow a=y+z, b=z+x, c=x+y$***. Using this substitution, (1) transforms into:***

$$27\{(x+y)(y+z)(z+x)-4xyz\}(x+y)^4(y+z)^4(z+x)^4-256x^2y^2z^2(x+y+z)^2\cdot$$

$$\cdot\{x(y+z)^6+y(z+x)^6+z(x+y)^6\} \ge 0 \Leftrightarrow 27\left(\sum x^{10}y^5+\sum x^5y^{10}\right)+$$

$$+135\left(\sum x^9y^6+\sum x^6y^9\right)+135xyz\left(\sum x^9y^3+\sum x^3y^9\right)+$$

$$+14x^2y^2z^2\left(\sum x^8y+\sum xy^8\right)+702xyz\left(\sum x^8y^4+\sum x^4y^8\right)+$$

$$+1081x^2y^2z^2\left(\sum x^7y^2+\sum x^2y^7\right)+270\left(\sum x^8y^7+\sum x^7y^8\right)+$$

$$+1593xyz\left(\sum x^7y^5+\sum x^5y^7\right)+3929x^2y^2z^2\left(\sum x^6y^3+\sum x^3y^6\right)+$$

$$+6480x^2y^2z^2\left(\sum x^5y^4+\sum x^4y^5\right)+2052xyz\left(\sum x^6y^6\right)+8076x^5y^5z^5\overset{(2)}{\ge}$$

$$\ge 2044x^3y^3z^3\left(\sum x^6\right)+4272x^3y^3z^3\left(\sum x^5y+\sum xy^5\right)+$$

$$+4100x^3y^3z^3\left(\sum x^4y^2+\sum x^2y^4\right)+7356x^4y^4z^4\left(\sum x^3\right)+$$

$$+3772x^3y^3z^3\left(\sum x^3y^3\right)+1780x^4y^4z^4\left(\sum x^2y+\sum xy^2\right)$$

$$\sum x^7y^2+\sum x^2y^7=\sum x^7(y^2+z^2)\overset{A-G}{\ge}\sum x^7\cdot 2yz=2xyz\left(\sum x^6\right)$$

$$\therefore 1022x^2y^2z^2\left(\sum x^7y^2+\sum x^2y^7\right)\overset{(i)}{\ge}2044x^3y^3z^3\left(\sum x^6\right)$$

$$\sum x^6y^3+\sum x^3y^6=\sum x^6(y^3+z^3)\ge\sum x^6yz\,(y+z)=xyz\left(\sum x^5y+\sum xy^5\right)$$

$$\therefore 3929x^2y^2z^2\left(\sum x^6y^3+\sum x^3y^6\right)\overset{(ii)}{\ge}3929x^3y^3z^3\left(\sum x^5y+\sum xy^5\right)$$

$$\sum x^7y^2+\sum x^2y^7=\sum x^7(y^2+z^2)\overset{A-G}{\ge}\sum x^7\cdot 2yz=xyz\left(2\sum x^6\right)=$$

$$=xyz\left(\sum(x^6+y^6)\right)\overset{Chebyshev}{\ge}xyz\cdot\frac{1}{2}\cdot\sum(x^2+y^2)(x^4+y^4)\overset{A-G}{\ge}xyz\sum xy\,(x^4+y^4)$$

$$=xyz\left(\sum x^5y+\sum xy^5\right)\Rightarrow 59x^2y^2z^2\left(\sum x^7y^2+\sum x^2y^7\right)\overset{(iii)}{\ge}$$

$$\ge 59x^3y^3z^3\left(\sum x^5y+\sum xy^5\right)$$

$$\sum x^9y^3+\sum x^3y^9=\sum z^3\,(x^9+y^9)\overset{Chebyshev}{\ge}\frac{1}{2}\sum z^3\,(x^2+y^2)(x^7+y^7)\overset{A-G}{\ge}$$

$$\ge\sum z^3\,xy(x^7+y^7)=xyz\left\{\sum z^7(x^2+y^2)\right\}\overset{A-G}{\ge}xyz\left(\sum z^7\cdot 2xy\right)=$$

$$=x^2y^2z^2\left(2\sum x^6\right)=x^2y^2z^2\left(\sum(x^6+y^6)\right)\overset{Chebyshev}{\ge}\frac{1}{2}x^2y^2z^2\sum(x^2+y^2)\,(x^4+y^4)$$

$$\overset{A-G}{\ge}x^2y^2z^2\sum xy(x^4+y^4)=x^2y^2z^2\left(\sum x^5y+\sum xy^5\right)\Rightarrow$$

$$\Rightarrow 135xyz\left(\sum x^9y^3+\sum x^3y^9\right)\overset{(iv)}{\ge}135x^3y^3z^3\left(\sum x^5y+\sum xy^5\right)$$

$$\sum x^8y+\sum xy^8=\sum z(x^8+y^8)\overset{Chebyshev}{\ge}\frac{1}{2}\sum z(x^2+y^2)(x^6+y^6)\overset{A-G}{\ge}$$

$$\ge \sum xyz\,(x^6+y^6) \overset{Chebyshev}{\ge} \frac{1}{2}xyz\cdot\sum(x^2+y^2)(x^4+y^4) \overset{A-G}{\ge}$$

$$\ge xyz\sum xy\,(x^4+y^4) = xyz\left(\sum x^5y+\sum xy^5\right)\Rightarrow$$

$$\Rightarrow 14x^2y^2z^2\left(\sum x^8y+\sum xy^8\right)\overset{(v)}{\ge}14x^3y^3z^3\left(\sum x^5y+\sum xy^5\right)$$

$$\sum x^8y^4+\sum x^4y^8=\sum z^8(x^4+y^4)\overset{Chebyshev}{\ge}\frac{1}{2}\sum z^8\,(x^2+y^2)^2\overset{A-G}{\ge}$$

$$\ge 2\sum z^8\,x^2y^2 = x^2y^2z^2\cdot\sum(x^6+y^6)\overset{Chebyshev}{\ge}\frac{1}{2}x^2y^2z^2\cdot\sum(x^2+y^2)(x^4+y^4)\ge$$

$$\overset{A-G}{\ge} x^2y^2z^2\cdot\sum xy\,(x^4+y^4) = x^2y^2z^2\left(\sum x^5y+\sum xy^5\right)\Rightarrow$$

$$\Rightarrow 135xyz\left(\sum x^8y^4+\sum x^4y^8\right)\overset{(vi)}{\ge}135x^3y^3z^3\left(\sum x^5y+\sum xy^5\right)$$

$$\sum x^6y^6+3x^4y^4z^4\overset{Schur}{\ge}x^2y^2z^2\left(\sum x^4y^2+\sum x^2y^4\right)\Rightarrow$$

$$\Rightarrow 2052xyz\left(\sum x^6y^6\right)+6156x^5y^5z^5\overset{(vii)}{\ge}2052x^3y^3z^3\left(\sum x^4y^2+\sum x^2y^4\right)$$

$$1024\left(\sum x^7y^5+\sum x^5y^7\right)\overset{A-G}{\ge}2048\sum x^6y^6\Rightarrow 1024\left(\sum x^7y^5+\sum x^5y^7\right)+$$

$$+6144x^5y^5z^5\ge 2048\left(\sum x^6y^6+3x^5y^5z^5\right)\underset{(viii)}{\overset{Schur}{\ge}}2048x^3y^3z^3\left(\sum x^4y^2+\sum x^2y^4\right)$$

$$\sum x^5y^4+\sum x^4y^5=\sum x^5(y^4+z^4)\overset{A-G}{\ge}2x^2y^2z^2\left(\sum x^3\right)\Rightarrow$$

$$\Rightarrow 3678x^2y^2z^2\left(\sum x^5y^4+\sum x^4y^5\right)\overset{(ix)}{\ge}7356x^4y^4z^4\left(\sum x^3\right)$$

$$\sum x^5y^4+\sum x^4y^5=\sum x^4(y^5+z^5)\ge\frac{1}{2}\sum x^4\,(y^2+z^2)(y^3+z^3)\overset{A-G}{\ge}$$

$$\ge\sum x^4yz\,(y^3+z^3)=xyz\sum(x^3y^3+x^3z^3)=2xyz\left(\sum x^3y^3\right)\Rightarrow$$

$$\Rightarrow 1886x^2y^2z^2\left(\sum x^5y^4+\sum x^4y^5\right)\overset{(x)}{\ge}3772x^3y^3z^3\left(\sum x^3y^3\right)$$

$\sum x^5y^4+\sum x^4y^5\ge xyz(2\sum x^3y^3)$ ***(proved above)***

$$\overset{Schur+A-G}{\ge} x^2y^2z^2\left(\sum x^2y+\sum xy^2\right)\Rightarrow$$

$$\Rightarrow 916x^2y^2z^2\left(\sum x^5y^4+\sum x^4y^5\right)\overset{(xi)}{\geq}916x^4y^4z^4\left(\sum x^2y+\sum xy^2\right)$$

$$\sum x^7y^5+\sum x^5y^7\overset{A-G}{\geq}2\sum x^6y^6=\sum(x^6y^6+y^6z^6)=\sum y^6(x^6+z^6)\overset{A-G}{\geq}$$

$$\geq\sum y^6\cdot 2x^3z^3=x^3y^3z^3\left(2\sum x^3\right)\overset{Schur+A-G}{\geq}x^3y^3z^3\left(\sum x^2y+\sum xy^2\right)\Rightarrow$$

$$\Rightarrow 569xyz\left(\sum x^7y^5+\sum x^5y^7\right)\overset{(xii)}{\geq}569x^4y^4z^4\left(\sum x^2y+\sum xy^2\right)$$

$$\sum x^8y^4+\sum x^4y^8\overset{A-G}{\geq}2\sum x^6y^6\overset{Schur+A-G}{\geq}x^2y^2z^2\left(\sum x^4y^2+\sum x^2y^4\right)\geq$$

$$\overset{A-G}{\geq}x^2y^2z^2\left(\sum 2x^3y^3\right)\overset{Schur+A-G}{\geq}x^3y^3z^3\left(\sum x^2y+\sum xy^2\right)\Rightarrow$$

$$\Rightarrow 295xyz\left(\sum x^8y^4+\sum x^4y^8\right)\overset{(xiii)}{\geq}295x^4y^4z^4\left(\sum x^2y+\sum xy^2\right)$$

Lastly, $27(\sum x^{10}y^5+\sum x^5y^{10})+272xyz(\sum x^8y^4+\sum x^4y^8)+135(\sum x^9y^6+\sum x^6y^9)+$

$$+270\left(\sum x^8y^5+\sum x^7y^8\right)\underset{(xiv)}{\overset{A-G}{\geq}}704\cdot 6x^5y^5z^5=4224x^5y^5z^5$$

(i)+(ii) +(iii) +(iv) +(v) +(vi) +(vii) +(viii)

(ix) +(x) +(xi) +(xii) +(xiii) +(xiv)

⇒ (2) is true ⇒ (b) is true (*)

Also, $\sum\frac{a^6}{r_a}\overset{A-G}{\geq}\frac{3a^2b^2c^2}{\sqrt[3]{rs^2}}\overset{r\leq\frac{s}{3\sqrt{3}}}{\geq}\frac{3\sqrt{3}\cdot 16R^2r^2s^2}{s}\overset{s\geq 3\sqrt{3}r}{\geq}27\cdot 16R^2r^3\overset{R\geq 2r}{\geq}27\cdot 64r^5=1728r^5\Rightarrow$

⇒ (a) is true (*) (Done)

SP.116. A triangle with side lenghts a,b,c has perimeter equal to 3.

Prove that: $a^3+b^3+c^3+a^4+b^4+c^4\geq 2(a^2b^2+b^2c^2+c^2a^2)$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$1=\frac{\sum a}{3}$ ∴ ***given inequality*** $\Leftrightarrow(\sum a)(\sum a^3)+3\sum a^4\geq 6\sum a^2b^2$

$$\Leftrightarrow\sum a^4+\sum a^3b+\sum ab^3+3\sum a^4\geq 6\sum a^2b^2$$

$$\Leftrightarrow 4\sum a^4+\sum a^3b+\sum ab^3\geq 6\sum a^2b^2\quad\textbf{(1)}$$

Now, $\sum a^3b + \sum ab^3 \underset{(a)}{\overset{A-G}{\geq}} 2\sum a^2b^2$***. Also,*** $4\sum a^4 \overset{(b)}{\geq} 4\sum a^2b^2$

(a)+(b)⇒(1) is true (Proved)

SP.117. Let ABC be a triangle with inradius r and circumradius R. Prove that:

a. $\frac{8\sqrt{3}}{3} \leq \frac{1}{cos^3\frac{A}{2}} + \frac{1}{cos^3\frac{B}{2}} + \frac{1}{cos^3\frac{C}{2}} \leq \frac{2\sqrt{3}}{3}\left(\frac{R}{2}\right)^2$

b. $9\left(\frac{r}{R}\right)^2 \leq cos^2\frac{A}{2} + cos^2\frac{B}{2} + cos^2\frac{C}{2} \leq \frac{9}{4}$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\left(\cos^3\frac{A}{2}\right)^{-1} = \sum\frac{8\cos^3\frac{B-C}{2}}{\left(2\sin\frac{B+2}{2}\cos\frac{B-C}{2}\right)^3} \leq \sum\frac{8}{(\sin B + \sin C)^3}$$

$$\left(\because 0 < \cos\frac{B-C}{2} \leq 1\ as -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2}\right)$$

$$= \sum\frac{64R^3}{(b+c)^3} = \sum\frac{64R^3}{b^3+c^3+3bc(b+c)} \overset{(1)}{\leq} \sum\frac{64R^3}{bc(b+c)+3bc(b+c)}$$

$$= \sum\frac{16R^3}{bc(b+c)} = \frac{16R^3\sum a(c+a)(a+b)}{abc(a+b)(b+c)(c+a)} = \frac{16R^3\sum a(\sum ab + a^2)}{4Rrs(a+b)(b+c)(c+a)} =$$

$= \frac{4R^2}{rs}\left(\frac{2s\sum ab+\sum a^3}{\pi(a+b)}\right)$***. Now,*** $\sum a^3 = 3abc + 2s(\sum a^2 - \sum ab) =$

$$= 12Rrs + 2s(s^2 - 12Rr - 3r^2) \overset{(a_1)}{=} 2s(s^2 - 6Rr - 3r^2)\ \&$$

$$\prod(a+b) = 2abc + \sum ab\,(2s-c) = 2s(s^2 + 4Rr + r^2) - 4Rrs \overset{(a_2)}{=}$$

$$= 2s(s^2 + 2Rr + r^2)$$

(1), (a₂), (a₃) $\Rightarrow \sum\left(\cos^3\frac{A}{2}\right)^{-1} \leq \frac{4R^2}{rs}\cdot\frac{2s(2s^2-4Rr-2r^2)}{2s(s^2+2Rr+r^2)} \overset{(?)}{\leq} \frac{2R^2}{\sqrt{3}r^2} \Leftrightarrow$

$$\Leftrightarrow 4\sqrt{3}r(s^2 - Rr - r^2) \overset{(?)}{\leq} s(s^2 + 2Rr + r^2) \Leftrightarrow s^2(s^2 + 2Rr + r^2)^2 \overset{(?)}{\geq}$$

$$\geq 48r^2(s^2 - Rr - r^2)^2 \Leftrightarrow s^2\left(s^4 + r^2(2R+r)^2 + 2s^2(2Rr + r^2)\right) \overset{(?)}{\geq}$$

$$\geq 48r^2\left(s^4+r^2(R+r)^2-2s^2(Rr+r^2)\right)\Leftrightarrow s^6+2s^4(2Rr+r^2)+s^2r^2(2R+r)^2+$$

$$+96s^2r^2(Rr+r^2)\underset{(a_3)}{\overset{?}{\geq}}48r^2s^4+48r^4(R+r)^2$$

Now, LHS of (a₃) $\overset{Gerretsen}{\geq} s^4\left(20Rr-3r^2\right)+s^2r^2\{(2R+r)^2+96\left(Rr+r^2\right)\}\overset{(?)}{\geq}48r^2s^4+$

$$+48r^4(R+r)^2\Leftrightarrow s^4(20Rr-40r^2)+s^2r^2\{(2R+r)^2+96(Rr+r^2)\}\underset{(a_4)}{\overset{?}{\geq}}$$

$$\geq\frac{1}{s^4r^2}+48r^4(R+r)^2$$

Now, LHS of (a₄) $\underset{(i)}{\overset{Gerretsen}{\geq}} s^2r^2(16R-5r)(20R-40r)+s^2r^2\{(2R+r)^2+$

96Rr+r2 and also, RHS of (a₄)) ≥iiGerretsen11s2r24R2+4Rr+3r2+48r4R+r2

(i) & (ii) ⇒ in order to prove (a₄), it suffices to prove:

$$s^2\{(16R-5r)(20R-40r)+(2R+r)^2+96(Rr+r^2)-11(4R^2+4Rr+3r^2)\}\geq$$

$$\geq 48r^2(R+r)^2\Leftrightarrow s^2(70R^2-171Rr+66r^2)\overset{(a_5)}{\geq}$$

$$\geq 12r^2(R+r)^2\because 70R^2-171Rr+66r^2=(R-2r)(70R-31r)+4r^2>0$$

$\therefore$ ***LHS of (a₅)*** $\overset{Gerretsen}{\geq}(16Rr-5r^2)(70R^2-171Rr+66r^2)\overset{?}{\geq}12r^2(R+r)^2\Leftrightarrow$

$$\Leftrightarrow 1120t^3-3098t^2+1887t-342\overset{?}{\geq}0\quad\left(t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(1120t^2-858t+171)\overset{?}{\geq}0\rightarrow$ ***true*** $\because t\overset{Euler}{\geq}2\Rightarrow\sum\left(\cos^3\frac{A}{2}\right)^{-1}\leq\frac{2\sqrt{3}}{3}\left(\frac{R}{r}\right)^2$

Also, $\sum\frac{1}{\cos^3\frac{A}{2}}\overset{Radon}{\geq}\frac{3^4}{\left(\sum\cos\frac{A}{2}\right)^3}\overset{Jensen}{\geq}\frac{3^4}{\left(\frac{3\sqrt{3}}{2}\right)^3}$

(∵ $f(x)=\cos\frac{x}{2}$ ***is concave on*** $(0,\pi)$***)***

$$=\frac{3^4\cdot 8}{3^3\cdot 3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}\therefore\sum\frac{1}{\cos^3\frac{A}{2}}\geq\frac{8\sqrt{3}}{3}$$

$\therefore$ ***both bounds of (a) are proved. Now,*** $\sum\cos^2\frac{A}{2}=\frac{1}{2}\sum(1+\cos A)=\frac{1}{2}\left(3+1+\frac{r}{R}\right)\overset{(b_1)}{=}$

$$=\frac{1}{2}\left(\frac{4R+r}{R}\right)$$

$$\textbf{(b}_1\textbf{)}\Rightarrow\sum\cos^2\frac{A}{2}\geq\frac{9r^2}{R^2}\Leftrightarrow\frac{4R+r}{2R}\geq\frac{9r^2}{R^2}\Leftrightarrow R(4R+r)\geq 18r^2\Leftrightarrow 4R^2+Rr-18r^2\geq 0$$

$$\Leftrightarrow(R-2r)(4R+9r)\overset{?}{\geq}0\rightarrow\textit{true}\because R\overset{Euler}{\geq}2r\therefore\sum\cos^2\frac{A}{2}\geq 9\left(\frac{r}{R}\right)^2.\textit{ Also, }\textbf{(b}_1\textbf{)}\Rightarrow$$

$$\Rightarrow\sum\cos^2\frac{A}{2}\leq\frac{9}{4}\Leftrightarrow\frac{4R+r}{2R}\leq\frac{9}{4}\Leftrightarrow 9R\geq 8R+2r\Leftrightarrow R\geq 2r\rightarrow\textit{true (Euler)}$$

$$\therefore\sum\cos^2\frac{A}{2}\leq\frac{9}{4}\therefore\textit{both bounds of (b) are proved (Done)}$$

SP.118. Let $a, b, c > 0$ such that: $a^2 + b^2 + c^2 = 3$. Find the minimum of the expression:

$$P=\frac{a^3}{\sqrt[4]{\frac{b^8+c^8}{2}}+5bc}+\frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}}+5ca}+\frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}}+5ab}+\frac{(a+b)(b+c)(c+a)}{16}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

**** By Cauchy-Schwarz's inequality we have:***

$$\left(\sqrt{2(b^8+c^8)}+2b^2c^2\right)^2\leq 2\left(2(b^8+c^8)+4b^4c^4\right)=4\left(b^8+2b^4c^4+c^8\right)=4\left(b^4+c^4\right)^2$$

$$\Rightarrow\sqrt{2(b^8+c^8)}+2b^2c^2\leq 2(b^4+c^4)\Leftrightarrow\sqrt{2(b^8+c^8)}\leq 2(b^4-b^2c^2+c^4)\Leftrightarrow$$

$$\Leftrightarrow\sqrt[4]{\frac{b^8+c^8}{2}}\leq\sqrt{b^4-b^2c^2+c^4}\Rightarrow$$

$$\Rightarrow\sqrt[4]{\frac{b^8+c^8}{2}}\leq\sqrt{(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)\cdot(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)}\leq$$

$$\leq\frac{(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)+(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)}{2}=2b^2-3bc+2c^2$$

$$\Leftrightarrow\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\leq 2(b^2+bc+c^2)\Leftrightarrow\frac{a^3}{\sqrt[4]{\frac{b^8+c^8}{2}}+5bc}\geq\frac{a^3}{2(b^2+bc+c^2)}$$

$$+\textit{ Similar: }\frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}}+5ca}\geq\frac{b^3}{2(c^2+ca+a^2)};\frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}}+5ab}\geq\frac{c^3}{2(a^2+b^2+ab)}$$

$$\Rightarrow P=\frac{a^3}{\sqrt[3]{\frac{b^8+c^8}{2}}+5bc}+\frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}}+5ca}+\frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}}+5ab}+\frac{(a+b)(b+c)(c+a)}{16}$$

$$\geq\frac{a^3}{2(b^2+bc+c^2)}+\frac{b^3}{2(c^2+ca+a^2)}+\frac{c^3}{2(a^2+b^2+ab)}+\frac{(a+b)(b+c)(c+a)}{16}$$

+ ***Using inequality:*** $9(x+y)(y+z)(z+x)\geq 8(x+y+z)(xy+yz+zx)$

$$\Rightarrow P\geq\frac{1}{2}\sum\frac{a^3}{b^2+bc+c^2}+\frac{\prod(b+c)}{16}\geq\frac{1}{2}\sum\frac{a^4}{ab^2+abc+ac^2}+\frac{\frac{8}{9}(\sum a)(\sum ab)}{16}$$

$$\Rightarrow P\geq\frac{1}{2}\cdot\frac{\left(\sum a^2\right)^2}{\sum(ab^2+abc+ac^2)}+\frac{(\sum a)(\sum ab)}{18}=\frac{9}{2(\sum a)(\sum ab)}+\frac{(\sum a)(\sum ab)}{18}\ \textit{(Cauchy-Schwarz)}$$

$$\Rightarrow P\geq 2\cdot\sqrt{\frac{9}{2(\sum a)(\sum ab)}\cdot\frac{(\sum a)(\sum ab)}{18}}=2\cdot\sqrt{\frac{1}{4}}=1\Rightarrow P_{\min}=1$$

(Because by AM-GM inequality and $a^2+b^2+c^2=3$***)***

$$\Rightarrow P_{\min}=1\Leftrightarrow\begin{cases}a=b=c\\a^2+b^2+c^2=3\\(\sum a)(\sum ab)=9\end{cases}\Leftrightarrow a=b=c=1.$$

SP.119. Let $a,b,c>0$ such that: $a+b+c=3$. Find the minimum of the expression:

$$P=\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{(a+b)(b+c)(c+a)}{24}$$

Proposed by Hoang Le Nhat Tung – Hanoi - VietNam

Solution by proposer:

*** We have:*** $b^6+c^6=(b^2+c^2)(b^4-b^2c^2+c^4)=(b^2+c^2)(b^2-bc\sqrt{3}+c^2)(b^2+bc\sqrt{3}+c^2)$

- Therefore, by AM-GM inequality:

$$\sqrt[3]{4(b^6+c^6)}=2\cdot\sqrt[3]{\frac{b^2+c^2}{2}(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)}\leq$$

$$\leq 2\cdot\frac{\frac{b^2+c^2}{2}+(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)+(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)}{3}=3b^2-4bc+3c^2$$

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc} \geq \frac{a}{3b^2-4bc+3c^2+7bc} = \frac{a}{3(b^2+bc+c^2)}$$

+ ***Similar:*** $\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca} \geq \frac{b}{3(c^2+ca+a^2)}; \frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab} \geq \frac{c}{3(a^2+b^2+ab)}$

$$\Rightarrow P = \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc} + \frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca} + \frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab} + \frac{(a+b)(b+c)(c+a)}{24} \geq$$

$$\geq \frac{a}{3(b^2+bc+c^2)} + \frac{b}{3(c^2+ca+a^2)} + \frac{c}{3(a^2+b^2+ab)} + \frac{(a+b)(b+c)(c+a)}{24}$$

+ ***Using inequality:*** $9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy+yz+zx)$

$$\Rightarrow P \geq \frac{1}{3}\sum \frac{a}{b^2+bc+c^2} + \frac{\prod(b+c)}{24} \geq \frac{1}{3}\sum \frac{a^2}{ab^2+abc+ac^2} + \frac{\frac{8}{9}(\sum a)(\sum ab)}{24}$$

$\Rightarrow P \geq \frac{1}{3} \cdot \frac{(\sum a)^2}{\sum(ab^2+abc+ac^2)} + \frac{(\sum ab)}{9} = \frac{3}{(\sum a)(\sum ab)} + \frac{(\sum ab)}{9}$ ***(Cauchy-Schwarz)***

$\Rightarrow P \geq \frac{1}{\sum ab} + \frac{\sum ab}{9} \geq 2 \cdot \sqrt{\frac{1}{\sum ab} \cdot \frac{\sum ab}{9}} = \frac{2}{3}$ ***(Because by AM-GM and*** $a+b+c=3$***)***

$$\Rightarrow P_{\min} = \frac{2}{3} \Leftrightarrow \begin{cases} a=b=c \\ a+b+c=3 \end{cases} \Leftrightarrow a=b=c=1.$$

SP.120. In ΔABC the following relationship holds:

$$\sqrt[3]{a^2B} + \sqrt[3]{b^2C} + \sqrt[3]{c^2A} \leq \sqrt[3]{4\pi s^2}$$

s – semiperimeter; a, b, c – length's sides; A, B, C – angled's measures

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

$$B\sqrt[3]{\frac{a^2}{B^2}} + C\sqrt[3]{\frac{b^2}{C^2}} + A\sqrt[3]{\frac{c^2}{A^2}} \leq \sqrt[3]{\pi(a+b+c)^2} \Leftrightarrow$$

$$\frac{B}{\pi}\sqrt[3]{\frac{a^2}{B^2}} + \frac{C}{\pi}\sqrt[3]{\frac{b^2}{C^2}} + \frac{A}{\pi}\sqrt[3]{\frac{c^2}{A^2}} \leq \sqrt[3]{\frac{(a+b+c)^2}{\pi^2}} \quad \textit{(1)}$$

Let $f:(0,\infty) \to \mathbb{R}; f(x) = \sqrt[3]{x^2};\ f'(x) = \left(x^{\frac{2}{3}}\right)' = \frac{2}{3}x^{-\frac{1}{3}}; f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} < 0 \Rightarrow$

from Jensen's inequality $\Rightarrow$

$p_1 f(x_1) + p_2 f(x_2) + p_3 f(x_3) \leq f(p_1 x_1 + p_2 x_2 + p_3 x_3)$ ***with*** $p_1 + p_2 + p_3 = 1$

$$p_1 = \frac{B}{\pi}, p_2 = \frac{C}{\pi}, p_3 = \frac{A}{\pi}$$

$$x_1 = \frac{a}{B}, x_2 = \frac{b}{C}, x_3 = \frac{c}{A} \Rightarrow \frac{B}{\pi}\sqrt[3]{\left(\frac{a}{B}\right)^2} + \frac{C}{\pi}\sqrt[3]{\left(\frac{b}{c}\right)^2} + \frac{A}{\pi}\sqrt[3]{\left(\frac{c}{A}\right)^2} \le \sqrt[3]{\left(\frac{a+b+c}{A+B+C}\right)^2} \Rightarrow$$

$\frac{B}{\pi}\sqrt[3]{\left(\frac{a}{B}\right)^2} + \frac{C}{\pi}\sqrt[3]{\left(\frac{b}{C}\right)^2} + \frac{A}{\pi}\sqrt[3]{\left(\frac{c}{A}\right)^2} \le \sqrt[3]{\frac{(a+b+c)^2}{\pi^2}}$ ***then (1) is true***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Applying Hölder's Inequality $(a+b+c)^2(A+B+C) \ge \left(\sqrt[3]{a^2B} + \sqrt[3]{b^2C} + \sqrt[3]{c^2A}\right)^3$

$\sqrt[3]{4\pi p^2} \ge \sqrt[3]{a^2B} + \sqrt[3]{b^2C} + \sqrt[3]{c^2A}$ ***(proved)***

UP.106. Prove that:

$$\cos\frac{A}{4} + \cos\frac{B}{4} + \cos\frac{C}{4} \le \frac{\sqrt{3}-1}{4\sqrt{6}}t + \frac{11\sqrt{3}+37}{8\sqrt{6}}, t = \frac{r}{R}$$

Proposed by Vadim Mitrofanov-Kiev-Ukraine

Solution by proposer

We make some transformations, let $k_a = \tan\frac{A}{4} \Rightarrow \cos\frac{A}{4} = \frac{1}{\sqrt{k_a^2+1}}$ ***and we know***

$$\tan\frac{A}{4} = \frac{AI-(s-a)}{r} = \frac{\sqrt{2Rr}\sqrt{\frac{2(s-a)}{a}}-(s-a)}{r} = |Ravi| =$$

$$= \frac{2\sqrt{\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}\cdot\sqrt{\frac{xyz}{x+y+z}}\cdot x} - x\sqrt{y+z}}{\sqrt{\frac{xyz}{x+y+z}}\cdot\sqrt{y+z}} = \frac{\sqrt{(x+y)(x+z)}-\sqrt{x(x+y+z)}}{\sqrt{yz}} =$$

$$= \frac{\sqrt{yz}}{\sqrt{(x+y)(x+z)}+\sqrt{x(x+y+z)}}$$

$$\Rightarrow \sum_{cyc}\cos\frac{A}{4} = \sum_{cyc}\frac{1}{\sqrt{\left(\frac{\sqrt{yz}}{\sqrt{(x+y)(x+z)}+\sqrt{x(x+y+z)}}\right)^2+1}} =$$

$$=\sum_{cyc}\frac{1}{\sqrt{\frac{\left(\sqrt{(x+y)(x+z)}-\sqrt{x(x+y+z)}\right)^2}{yz}+1}}$$

$$=\sum_{cyc}\sqrt{\frac{yz}{(x+y+z)x+(x+y)(x+z)+yz-2\sqrt{x(x+y+z)(x+y)(x+z)}}}$$

$$=\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{\frac{yz}{(x+y)(x+z)-\sqrt{x(x+y+z)(x+y)(x+z)}}}$$

$$=\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{\frac{\sqrt{(x+y)(x+z)}+\sqrt{x(x+y+z)}}{\sqrt{(x+y)(x+z)}}}=\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{1+\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}}$$

$$\le\sqrt{\frac{3}{2}\left(3+\sum_{cyc}\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}\right)}\le\sqrt{\frac{3}{2}\left(3+\sqrt{3\sum_{cyc}\frac{x(x+y+z)}{(x+y)(x+z)}}\right)}$$

$$=\sqrt{\frac{3}{2}\left(3+\sqrt{6\left(1+\frac{xyz}{(x+y)(y+z)(x+z)}\right)}\right)}=\sqrt{\frac{9}{2}+3\sqrt{\frac{3}{2}+\frac{3}{8}\cdot\frac{r}{R}}}$$

Hence we need to prove: $\sqrt{\frac{9}{2}+3\sqrt{\frac{3}{2}+\frac{3}{8}\cdot t}}\le\frac{\sqrt{3}-1}{4\sqrt{6}}t+\frac{11\sqrt{3}+37}{8\sqrt{6}}$

$$\Leftrightarrow\left(\left(\frac{\sqrt{3}-1}{4\sqrt{6}}t+\frac{11\sqrt{3}+37}{8\sqrt{6}}\right)^2-\frac{9}{2}\right)^2-\frac{27}{2}\left(1+\frac{1}{4}t\right)\ge 0$$

$$\Leftrightarrow\frac{(2t-1)^2\left(t^2\left(28-16\sqrt{3}\right)+t\left(208\sqrt{3}-316\right)+713-1628\sqrt{3}\right)}{36864}\ge 0\Leftrightarrow$$

$$\left(t^2\left(28-16\sqrt{3}\right)+t\left(208\sqrt{3}-316\right)-713+1628\sqrt{3}\right)\ge 0\quad (***)$$

$D_t=\left(208\sqrt{3}-316\right)^2-4\left(-713+1628\sqrt{3}\right)\left(28-16\sqrt{3}\right)=13824\left(45-26\sqrt{3}\right)<0\Rightarrow$ ***(***)*** $\forall\, t$

Done!

UP.107. Let ABC be a triangle with inradius r and circumradius R. Prove that

$$2\sqrt{3}\left(\frac{r}{R}\right)^2 \le \frac{\sum_{cyc}\sin^4 A}{\sum_{cyc}\sin^3 A} \le \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1-\frac{r}{R}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$2\sqrt{3}\left(\frac{r}{R}\right)^2 \overset{(1)}{\le} \frac{\sum\sin^4 A}{\sum\sin^3 A} \overset{(2)}{\le} \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1-\frac{r}{R}\right)$$

Firstly, $\sum a^4 = \left(\sum a^2\right)^2 - 2\{\left(\sum ab\right)^2 - 2abc(2s)\}$

$$= 4(s^2-4Rr-r^2)^2 - 2(s^2+4Rr+r^2)^2 + 32Rrs^2$$

$$= 2\left(s^4 + r^2(4R+r)^2 - 2s^2(4Rr+r^2)\right) + 2(2s^2)(-8Rr-2r^2) + 32Rrs^2$$

$$= 2s^4 + 2r^2(4R+r)^2 - 4s^2(4Rr+r^2) - 8s^2r^2$$

$$\overset{(i)}{=} 2s^4 + 2r^2(4R+r)^2 - 4s^2(4Rr+3r^2)$$

Also, $\sum a^3 = 3abc + 2s(s^2-12Rr-3r^2) \overset{(ii)}{=} 2s(s^2-6Rr-3r^2)$

Using $\sin A = \frac{a}{2R}$ ***etc,*** $\frac{\sum\sin^4 A}{\sum\sin^3 A} = \left(\frac{1}{2R}\right)\left(\frac{\sum a^4}{\sum a^3}\right) \overset{by\ (ii)}{=} \frac{\sum a^4}{4Rs(s^2-6Rr-3r^2)} \overset{?}{\le} \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1-\frac{r}{R}\right)$

$$\Leftrightarrow r^2\sum a^4 \underset{(2a)}{\overset{?}{\le}} \sqrt{3}R\cdot R(R-r)s(s^2-6Rr-3r^2)$$

RHS of (2a) $\overset{Mitrinovic}{\ge} \frac{2s^2(s^2-6Rr-3r^2)R(R-r)}{3} \overset{?}{\ge} r^2\sum a^4$

$$\Leftrightarrow 6r^2s^4 + 6r^4(4R+r)^2 - 12s^2r^2(4Rr+3r^2) \underset{by\ (i)}{\overset{?}{\le}} 2s^2(s^2-6Rr-3r^2)R(R-r)$$

$$\Leftrightarrow s^4(R^2-Rr-3r^2) + 6s^2r^3(4R+3r) \overset{?}{\ge} 3s^2Rr(R-r)(2R+r) + 3r^4(4R+r)^2$$

$$\Leftrightarrow s^4(R^2-Rr-2r^2) + 6s^2r^3(4R+3r) \underset{(2b)}{\overset{?}{\ge}} r^2s^4 + 3s^2Rr(R-r)(2R+r) + 3r^4(4R+r)^2$$

LHS of (2b) $\overset{Gerretsen}{\ge} s^2(16Rr-5r^2)(R^2-Rr-2r^2) + 6s^2r^3(4R+3r)$

Also, RHS $\overset{Gerretsen}{\le} r^2s^2(4R^2+4Rr+3r^2) + 3s^2Rr(R-r)(2R+r) + 3r^4(4R+r)^2$

$\therefore$ ***in order to prove (2b), it suffices to prove:***

$$s^2(16R-5r)(R^2-Rr-2r^2) + 6s^2r^2(4R+3r) \ge$$
$$rs^2(4R^2+4Rr+3r^2) + 3s^2R(R-r)(2R+r) + 3r^3(4R+r)^2$$

$$\Leftrightarrow s^2\{(16R-5r)(R^2-Rr-2r^2)-3R(R-r)(2R+r)+6r^2(4R+3r)-r(4R^2+4Rr+3r^2)\}$$

$$\Leftrightarrow s^2(10R^3-22R^2r-4Rr^2+25r^3)\overset{(2c)}{\geq}3r^3(4R+r)^2$$

$$\because 10R^3-22R^2r-4Rr^2+25r^3=(R-2r)\{(R-2r)(10R+18r)+28r^2\}+9r^3>0$$

as $R\geq 2r$ (Euler),

$\therefore$ LHS of (2c) $\overset{Gerretsen}{\geq}(16Rr-5r^2)(10R^3-22R^2r-4Rr^2+25r^3)\overset{?}{\geq}3r^3(4R+r)^2$

$\Leftrightarrow 80t^4-201t^3-t^2+198t-64\overset{?}{\geq}0$ ***(where $t=\frac{R}{r}$)***

$\Leftrightarrow (t-2)\{(t-2)(80t^2+119t+155)+342\}\overset{?}{\geq}0\rightarrow$ ***true*** $\because t\overset{Euler}{\geq}2$

$\Rightarrow$ (2a) is true $\Rightarrow$ (2) is true

Also, $\frac{\sum\sin^4 A}{\sum\sin^3 A}=\left(\frac{1}{2R}\right)\left(\frac{s^4+r^2(4R+r)^2-2s^2(4Rr+3r^2)}{s(s^2-6Rr-3r^2)}\right)\geq 2\sqrt{3}\left(\frac{r}{R}\right)^2$

$$\Leftrightarrow s^4+r^2(4R+r)^2-2s^2(4Rr+3r^2)\overset{(1a)}{\geq}\frac{4s(s^2-6Rr-3r^2)\sqrt{3}r^2}{R}$$

Now, RHS of (1a) $\overset{Mitrinovic}{\leq}18r^2(s^2-6Rr-3r^2)\overset{?}{\leq}s^4+r^2(4R+r)^2-2s^2(4Rr+3r^2)$

$$\Leftrightarrow s^4+r^2(4R+r)^2+54r^3(2R+r)\underset{(1b)}{\overset{?}{\geq}}18r^2s^2+s^2(8Rr+6r^2)$$

Now, LHS of (1b) $\overset{Gerretsen}{\geq}s^2(16Rr-5r^2)+r^2(4R+r)^2+54r^3(2R+r)\overset{?}{\geq}s^2(8Rr+24r^2)$

$$\Leftrightarrow s^2(8R-29r)+r(4R+r)^2+54r^2(2R+r)\overset{?}{\geq}0$$

$$\Leftrightarrow s^2(8R-16r)+r(4R+r)^2+54r^2(2R+r)\underset{(1c)}{\overset{?}{\geq}}13rs^2$$

Now, LHS of (1c) $\overset{Gerretsen}{\geq}(16Rr-5r^2)(8R-16r)+r(4R+r)^2+54r^2(2R+r)$

Also, RHS of (1c) $\overset{Gerretsen}{\leq}13r(4R^2+4Rr+3r^2)$

$\therefore$ in order to prove (1c), it suffices to prove:

$$(16R-5r)(8R-16r)+(4R+r)^2+54r(2R+r)\geq 13(4R^2+4Rr+3r^2)$$

$\Leftrightarrow 23R^2-58Rr+24\geq 0\Leftrightarrow (R-2r)(23R-12r)\geq 0\rightarrow$ ***true*** $\because R\geq 2r$

$\Rightarrow$ (1a) is true $\Rightarrow$ (1) is true

UP.108. Let ABC be a triangle with circumradius R and inradius r. Prove that

$$\frac{3}{16} \le \cos^4 A + \cos^4 B + \cos^4 C \le 6\left(\frac{r}{R}\right)^2 - \frac{123}{8}\cdot\frac{r}{R} + \frac{51}{8}$$

Proposed by George Apostolopoulos - Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3}{16} \overset{(1)}{\le} \sum \cos^4 A \overset{(2)}{\le} 6\left(\frac{r}{R}\right)^2 - \left(\frac{123}{8}\right)\left(\frac{r}{R}\right) + \frac{51}{8}$$

$$\cos A \cos B \cos C = \frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{2bc\cdot 2ca\cdot 2ab} \rightarrow \quad \textit{(i)}$$

$$\textit{Numerator} = \left(\sum a^2 - 2a^2\right)\left(\sum a^2 - 2b^2\right)\left(\sum a^2 - 2c^2\right)$$

$$= \left(\sum a^2\right)^3 - 2\left(\sum a^2\right)^2\left(\sum a^2\right) + 4\left(\sum a^2\right)\left(\sum a^2b^2\right) - 8a^2b^2c^2$$

$$= -\left(\sum a^2\right)^3 + 4\left(\sum a^2\right)\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\} - 128R^2r^2s^2$$

$$= \left(\sum a^2\right)\left\{4\left(\sum ab\right)^2 - \left(\sum a^2\right)^2 - 16sabc\right\} - 128R^2r^2s^2$$

$$= 4\left(\sum a^2\right)\{(s^2+4Rr+r^2)^2 - s(s^2-4Rr-r^2)^2 - 16Rrs^2\} - 128R^2r^2s^2$$

$$= 4\left(\sum a^2\right)\{2s^2(8Rr+2r^2) - 16Rrs^2\} - 128R^2r^2s^2$$

$$= 32r^2s^2(s^2-4Rr-r^2) - 128R^2r^2s^2 = 32r^2s^2(s^2-4R^2-4Rr-r^2) \rightarrow \quad \textit{(ii)}$$

$$\textit{(i), (ii)} \Rightarrow \prod \cos A \overset{(iii)}{=} \frac{s^2-4R^2-4Rr-r^2}{4R^2}$$

$$\textit{(2)} \Leftrightarrow \sum(1-\sin^2 A)^2 \le \frac{51R^2-123Rr+48r^2}{8R^2}$$

$$\Leftrightarrow \sum(1-2\sin^2 A + \sin^4 A) \le \frac{51R^2-123Rr+48r^2}{8R^2}$$

$$\Leftrightarrow \sum(\cos 2A) + \sum(\sin^4 A) \le \frac{51R^2-123Rr+48r^2}{8R^2}$$

$$\Leftrightarrow 1 - 4\left(\prod \cos A\right) + \sum(\sin^4 A) \le \frac{51R^2-123Rr+48r^2}{8R^2}$$

$$\Leftrightarrow -1 - \frac{s^2-(2R+r)^2}{R^2} + \sum(\sin^4 A) \overset{by\ (iii)}{\le} \frac{51R^2-123Rr+48r^2}{8R^2}$$

$$\Leftrightarrow \sum(\sin^4 A) \le \frac{27R^2 - 155Rr + 40r^2 + 8s^2}{8R^2}$$

$$\Leftrightarrow \frac{\sum a^4}{16R^4} \le \frac{27R^2 - 155Rr + 40r^2 + 8s^2}{8R^2}$$

$$\Leftrightarrow \sum a^4 \overset{(a)}{\le} 54R^4 - 310R^3r + 80R^2r^2 + 16R^2s^2$$

Now, $\sum a^4 = (\sum a^2)^2 - 2\{(\sum ab)^2 - 2abc(2s)\}$

$$= 4(s^2 - 4Rr - r^2)^2 - 2(s^2 + 4Rr + r^2)^2 + 32Rrs^2$$

$$= 2\left(s^4 + r^2(4R + r)^2 - 2s^2(4Rr + r^2)\right) + 2(2s^2)(-8Rr - 2r^2) + 32Rrs^2$$

$$= 2s^4 + 2r^2(4R + r)^2 - 4s^2(4Rr + r^2) - 8s^2r^2$$

$$= 2s^4 + 2r^2(4R + r)^2 - 4s^2(4Rr + 3r^2) \overset{?}{\le} 54R^4 - 310R^3r + 80R^2r^2 + 16R^2s^2$$

$$\Leftrightarrow s^4 \overset{?}{\le} 27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 + 2s^2(4R^2 + 4Rr + 3r^2)$$

Now, LHS of (b) $\overset{Gerretsen}{\le} s^2(4R^2 + 4Rr + 3r^2) \overset{?}{\le}$

$$27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 + 2s^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(4R^2 + 4Rr + 3r^2) + 27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 \underset{(c)}{\overset{?}{\ge}} 0$$

Now, LHS of (c) $\overset{Gerretsen}{\ge} 27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 +$

$$+(16Rr - 5r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\ge} 0$$

$\Leftrightarrow 27t^4 - 91t^3 + 68t^2 + 20t - 16 \overset{?}{\ge} 0$ ***(where*** $t = \frac{R}{r}$***)***

$\Leftrightarrow (t - 2)\{(t - 2)(27t^2 + 17t + 28) + 64\} \overset{?}{\ge} 0 \rightarrow$ ***true*** $\because t \overset{Euler}{\ge} 2$

⇒ ***(a) is true*** ⇒ ***(2) is true***

Also, $\sum \cos^4 A \overset{Chebyshev}{\ge} \left(\frac{1}{3}\right)(\sum \cos^2 A)^2 \overset{?}{\ge} \frac{3}{16} \Leftrightarrow \sum \cos^2 A \overset{?}{\ge} \frac{3}{4} \Leftrightarrow \sum \sin^2 A \overset{?}{\le} \frac{9}{4}$

$\Leftrightarrow \sum a^2 \overset{?}{\le} 9R^2 \rightarrow$ ***true by Leibniz*** ⇒ ***(1) is true***

UP.109. Let a, b, c and d be positive real numbers. Prove or disprove that:

$$\frac{(a+b+c+d)^3}{abc+bcd+cda+dab} \geq 16$$

Proposed by George Apostolopoulos - Messolonghi – Greece

Solution 1 by Catinca Alexandru-Romania

$$\frac{(a+b+c+d)^3}{\sum abc} \geq 16 \Leftrightarrow (a+b+c+d)^3 \geq 16 \cdot 4[1,1,1,0];$$

$$\Leftrightarrow \sum a^3 + 3\sum_{sym} a^2b + 6\sum abc \geq 16[1,1,1,0]4;$$

$$\Leftrightarrow [3,0,0,0] \cdot 4 + 3 \cdot [2,1,0,0] \cdot 12 + 6[1,1,1,0] \cdot 4 \geq 16 \cdot 4[1,1,1,0]$$

$$\Leftrightarrow 4[3,0,0,0] + 36[2,1,0,0] \geq 40[1,1,1,0] \quad \textbf{\textit{(1)}}$$

$4[3,0,0,0] \geq 4[1,1,1,0]$ ***as*** $(3,0,0,0) > (1,1,1,0)$ ***Muirhead***

$36[2,1,0,0] \geq 36[1,1,1,0]$ ***as*** $(2,1,0,0) > (1,1,1,0)$ ***Muirhead***

$$\overline{\qquad\qquad\qquad\qquad\qquad} \; +$$

$4[3,0,0,0] + 36[2,1,0,0] \geq 40[1,1,1,0] \Rightarrow$ ***(1) is True***$\Rightarrow \frac{(\sum a)^3}{\sum abc} \geq 16$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

1. $a^3 + b^2 + c^2 + d^3 \geq a^2b + b^2c + c^2d + d^2a$

$a^3 + b^3 + c^3 + d^3 \geq a^2c + c^2a + b^2d + d^2b$

2. $(a^2b + b^2c + c^2a) + (b^2c + c^2d + d^2b) + (c^2d + d^2a + a^2c) + (d^2a + a^2b + b^2d) \geq$

$\geq 3(abc + bcd + cda + dab)$

Hence $a^3 + b^2 + c^2 + d^3 \geq a^2b + b^2c + c^2d + d^2a \geq abc + bcd + cda + dab$

$a^3 + b^2 + c^2 + d^3 \geq a^2c + c^2a + b^2d + d^2b \geq abc + bcd + cda + dab$

That is $(a+b+c+d)^3 \geq 4^2(abc + bcd + cda + dab)$

Therefore $\frac{(a+b+c+d)^3}{abc+bcd+cda+dab} \geq 16$ ***is to be true***

UP.110. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that:

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \le \frac{R}{2r^2}$$

where R, r are the circumradius and inradius respectively of $\Delta\, ABC$.

Proposed by George Apostolopoulos - Messolonghi – Greece

Solution 1 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

Lemma: $m_a \ge h_a \Rightarrow \frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \le \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{a+b+c}{2S} = \frac{2p}{2pr} = \frac{1}{r} = \frac{2r}{r^2} \le \frac{R}{2r^2} \Rightarrow$ *Q.E.D.*

Solution 2 by Mehmet Sahin-Ankara-Turkey

By using the Tereshin Inequality: $4Rm_a \ge b^2 + c^2 \ge 2bc$

$$m_a \ge \frac{bc}{2R} \Rightarrow \frac{1}{m_a} \le \frac{2R}{bc}$$

$$\sum\frac{1}{m_a} \le 2R\left(\frac{a+b+c}{abc}\right) = \frac{1}{r} \le \frac{R}{2r^2} \text{ (Euler)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{R}{2r} \ge 1 \Leftrightarrow \frac{R}{2r^2} \ge \frac{1}{2} = \frac{p}{p\cdot r} = \frac{p}{\Delta}$$

$$\frac{R}{2r^2} \ge \frac{p}{\Delta} = \frac{\sum\frac{a}{2}}{\Delta} = \frac{\sum\frac{(p-b)+(p-c)}{2}}{\Delta} \overset{AM\ge GM}{\ge}$$

$$\ge \frac{\sum\sqrt{(p-b)(p-c)}}{\sqrt{p(p-a)(p-b)(p-c)}} = \sum\frac{1}{\sqrt{p(p-a)}} \ge \sum\frac{1}{m_a}$$

Solution 4 by Marian Ursarescu-Romania

In any $\Delta\, ABC$ *we have:* $m_a \ge \frac{b^2+c^2}{4R}$ *(1), because:*

$$m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = \frac{2(b^2+c^2)-(b^2+c^2-2bc\sin A)}{4} =$$

$$\frac{b^2+c^2+2bc\cos A}{4} = \frac{b^2+c^2-2bc\cos(B+C)}{4} =$$

$$= \frac{(b\cos B-c\cos C)^2+(b\sin B+c\sin C)^2}{4} \ge \frac{(b\sin B+c\sin C)^2}{4} \Rightarrow 1 \text{ is true}$$

From (1) $\Rightarrow \frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \le \frac{R}{2r^2}$

$$\Rightarrow m_a \geq \frac{b^2+c^2}{4R} \geq \frac{2bc}{4R} = \frac{bc}{2R} \Rightarrow \frac{1}{m_a} \leq \frac{2R}{bc} \Rightarrow$$

$$\sum \frac{1}{m_a} \leq 2R\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)$$

Now we show this $\Rightarrow 2R\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) \leq \frac{R}{2r^2} \Leftrightarrow$

$\Leftrightarrow \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \leq \frac{1}{4r^2}$ *which its true.*

P.S. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \leq \frac{1}{4r^2}$ *its true because* $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} = \frac{1}{2Rr} \leq \frac{1}{4r^2} \Leftrightarrow R \geq 2r$ *true.*

UP.111. For an acute triangle ABC and a positive integer n, prove that:

$$\left(\sum (\sin A \sin B \sin C)^{\frac{1}{n}}\right)^n \leq \frac{3^{n+1}}{8}$$

where the sum is over all cyclic permutations of (A, B, C).

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Marian Ursărescu – Romania

From Hölder's inequality we have:

$$\left((\sin A+\sin B\cos C)^{\frac{1}{n}}\right)^n+\left((\sin A\cos B\sin C)^{\frac{1}{n}}\right)^n+\left((\cos A\sin B\sin C)^{\frac{1}{n}}\right)^n \geq$$

$$\geq \frac{\left((\sin A+\sin B\cos C)^{\frac{1}{n}}+(\sin A\cos B\sin C)^{\frac{1}{n}}+(\cos A\sin B\sin C)^{\frac{1}{n}}\right)^n}{3^{n-1}} \Leftrightarrow$$

$$\Leftrightarrow \left(\sum (\sin A\sin B\sin C)^{\frac{1}{n}}\right)^n \leq 3^{n-1}\cdot \sum \sin A\sin B\cos C \Rightarrow$$

We must show: $\sum \sin A\sin B\cos C \leq \frac{9}{8}$ ***(1)***

Now: $\cos 2A+\cos 2B-\cos 2C = 2\cos(A+B)\cos(A-B)-2\cos^2 C+1$

$$= -2\cos C\,(\cos(A-B)+\cos C)+1 = 1-2\cos C\cdot 2\cos\left(\frac{A-B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$$

$= 1-4\sin A\sin B\cos C \Rightarrow \sin A\sin B\sin C = \frac{1}{4}(1-\cos 2A-\cos 2B+\cos 2C)$ ***(2)***

From (1)+(2) we must show: $\frac{3-(\cos 2A+\cos 2B+\cos 2C)}{4} \leq \frac{9}{8}$ ***(3)***

But $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$ ***(4)***

From (3)+(4) we must show: $1+\cos A\cos B\cos C\le\frac{9}{8}\Leftrightarrow\cos A\cos B\cos C\le\frac{1}{8}$,

which its true.

UP.112. Solve for positive real numbers:

$$\begin{cases}\dfrac{x^2}{y}+\dfrac{y^2}{x}=\sqrt[8]{128(x^8+y^8)}\\ 4x^3-3y=\sqrt{\dfrac{1+\sqrt{1-xy}}{2}}\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^2}{y}+\frac{y^2}{x}\overset{(1)}{=}\sqrt[8]{128(x^8+y^8)},\ \ 4x^3-3y\overset{(2)}{=}\sqrt{\frac{1+\sqrt{1-xy}}{2}}$$

(1) $\Leftrightarrow(x^3+y^3)^8-128x^8y^8(x^8+y^8)=0\Leftrightarrow t^{24}+8t^{21}+28t^{18}-128t^{16}+56t^{15}+$

$$+70t^{12}+56t^9-128t^8+28t^6+8t^3+1=0\ \left(t=\frac{x}{y}\right)\Leftrightarrow$$

$$\Leftrightarrow(t-1)^2(t^{22}+2t^{21}+3t^{20}+12t^{19}+21t^{18}+30t^{17}+67t^{16}+104t^{15}+13t^{14}-$$

$$-22t^{13}-57t^{12}-92t^{11}-57t^{10}-22t^9+13t^8+104t^7+67t^6+30t^5+21t^4+$$

$$+12t^3+3t^2+2t+1)=0\Leftrightarrow(t-1)^2\cdot p=0\ \ \textit{(a) (say)}$$

Now, $12t^9-22t^{13}+104t^7=2t^7(6t^{12}-11t^6+52)=$

$=2t^7\left(6a_1^2-11a_1+52\right)[(a_1=t^6)]>0$ ***(i)*** $\because$ ***discriminant*** $\Delta=121-4\cdot6\cdot52<0$

Also, $21t^{18}-57t^{12}+67t^6=t^6(21t^{12}-57t^6+67)=t^6\left(21a_1^2-57a_1+67\right)>0$ ***(ii)***

$$\because\Delta=57^2-84\cdot67<0$$

Again, $67t^{16}-57t^{10}+21t^4=t^4(67t^{12}-57t^6+21)=$

$=t^4\left(67a_1^2-57a_1+21\right)>0$ ***(iii)*** $\because\Delta=57^2-84\cdot67<0$

Moreover, $104t^{15}-22t^9+12t^3$ ***(iv)*** $=2t^3(52t^{12}-11t^6+6)=$

$=2t^3\left(52a_1^2-11a_1+6\right)>0$ ***(v)*** $\because\Delta=11^2-24\cdot52<0$

Now, $30t^{17}+30t^5\underset{(vi)}{\overset{A-G}{\ge}}60t^{11},13t^4+13t^8\underset{(vii)}{\overset{A-G}{\ge}}26t^{11}$

$$3t^{20}+3t^{2}\overset{A-G}{\underset{(viii)}{\geq}}6^{11}\ \textit{\& of course, } t^{22}+2t^{21}+2t+1\overset{(ix)}{>}0\ \textit{as } t>0\left(t=\frac{x}{y}>0\right)$$

(i)+(ii)+(iii)+(iv)+(v)+(vi)+(vii)+(viii)+(ix) $\Rightarrow P>0 \therefore$ ***a*** $\Rightarrow t=1\Rightarrow x=y$

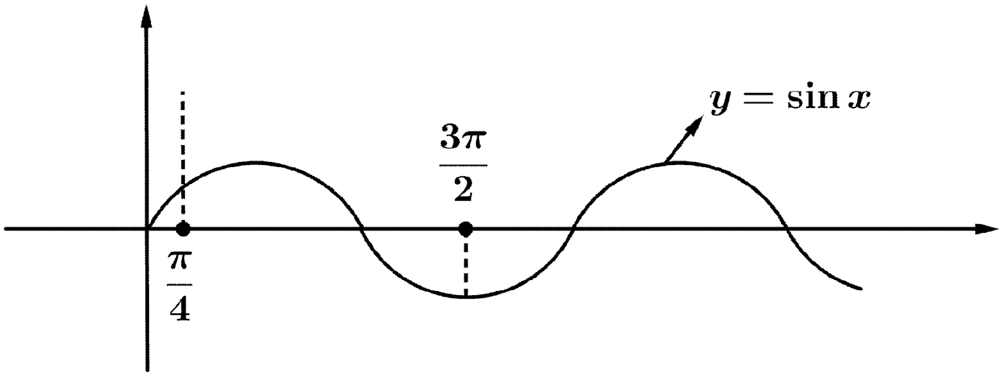
From (2), we have $1-xy=1-x^2\geq 0\Rightarrow x\leq 1\Rightarrow 0<x\leq 1$

$\therefore 0<x,y\leq 1$***. Let*** $x=y=\cos\theta\quad\left(0<\theta<\frac{\pi}{2}\right)$

$\therefore$ ***(2) becomes*** $4\cos^3\theta-3\cos\theta=\sqrt{\frac{1+\sin\theta}{2}}\Rightarrow\cos\theta=\sqrt{\frac{1+\sin\theta}{2}}$ ***(2a)*** $\Rightarrow$

$$\Rightarrow 2\cos^2(3\theta)=1+\sin\theta\Rightarrow 1+\cos 6\theta=1+\sin\theta\Rightarrow\cos 6\theta=\cos\left(\frac{\pi}{2}-\theta\right)\Rightarrow$$

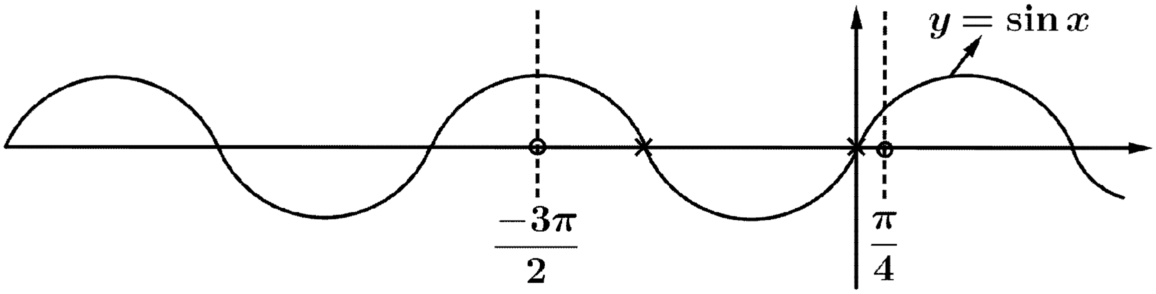
$\Rightarrow 2\sin\left(\frac{\pi}{4}+\frac{5\theta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{7\theta}{2}\right)=0$***. Now,*** $\frac{\pi}{4}<\frac{\pi}{4}+\frac{5\theta}{2}<\frac{3\pi}{2}$



$\therefore\sin\left(\frac{\pi}{4}+\frac{5\theta}{2}\right)=0\Rightarrow\frac{\pi}{4}+\frac{5\theta}{2}=\pi\Rightarrow\theta=\frac{3\pi}{10}$***. From (2a),*** $\cos 3\theta>0$ ***(*), but*** $\cos\frac{9\pi}{10}<0$

$\therefore x=y=\cos\frac{3\pi}{10}$ ***is not an acceptable solution.***

Also, $-\frac{3\pi}{2}<\frac{\pi}{4}-\frac{7\theta}{2}<\frac{\pi}{4}\therefore\sin\left(\frac{\pi}{4}-\frac{7\theta}{2}\right)=0$



$$\Rightarrow\frac{\pi}{4}-\frac{7\theta}{2}=0,-\pi\Rightarrow\theta=\frac{\pi}{14},\frac{5\pi}{14}$$

But $\cos\frac{15\pi}{14} < 0, \therefore x = y = \cos\frac{5\pi}{14}$ *is unacceptable*

$\therefore$ *only possible solution is:* $x = y = \cos\frac{\pi}{14}$

UP.113. Let x, y, z be positive real numbers. Prove that:

$$\sqrt{\frac{3x^2+yz}{y^2+z^2}}+\sqrt{\frac{3y^2+zx}{z^2+x^2}}+\sqrt{\frac{3z^2+xy}{x^2+y^2}} \ge \sqrt{\frac{103}{6}+\frac{20x^2y^2z^2}{3(x^2+y^2)(y^2+z^2)(z^2+x^2)}}$$

Proposed by Do Quoc Chinh – Vinh Phuc – Vietnam

Solution by proposer

By the AM-GM inequality, we have:

$$\sqrt{\frac{3x^2+yz}{y^2+z^2}}=\sqrt{\frac{3x^2+\frac{y^2z^2}{yz}}{y^2+z^2}} \ge \sqrt{\frac{3x^2+\frac{2y^2z^2}{y^2+z^2}}{y^2+z^2}}=\frac{\sqrt{2y^2z^2+3x^2y^2+3z^2x^2}}{y^2+z^2}$$

Similarly, we have: LHS $\ge \sum\frac{\sqrt{2x^2y^2+3y^2z^2+3z^2x^2}}{x^2+y^2}$

Put $x^2=a, y^2=b, z^2=c \ \ (a,b,c>0)$*. Thus, we need to prove:*

$$\sum\frac{\sqrt{2ab+3bc+3ca}}{a+b} \ge \sqrt{\frac{103}{6}+\frac{20abc}{3(a+b)(b+c)(c+a)}}$$

$$\Leftrightarrow \sum\frac{2ab+3bc+3ca}{(a+b)^2}+2\sum\frac{\sqrt{(2bc+3ca+3ab)(2ca+3ab+3bc)}}{(b+c)(c+a)} \ge$$

$$\ge \frac{103}{6}+\frac{20abc}{3(a+b)(b+c)(c+a)}$$

By the Cauchy – Schwarz inequality, we have:

$$\sum\frac{\sqrt{(2bc+3ca+3ab)(2ca+3ab+3bc)}}{(b+c)(c+a)} \ge \sum\frac{3ab+c\sqrt{(2a+3b)(3a+b)}}{(b+c)(c+a)}$$

$$=\sum\frac{3ab+c\cdot\frac{(2a+3b)(3a+2b)}{\sqrt{(2a+3b)(3a+2b)}}}{(b+c)(c+a)} \ge \sum\frac{3ab+c\cdot\frac{2(6a^2+13ab+6b^2)}{5(a+b)}}{(b+c)(c+a)}$$

$$\geq \sum \frac{3ab + c \cdot \dfrac{7a^2 + 16ab + 7b^2}{3(a+b)}}{(b+c)(c+a)} = \frac{16(a+b+c)(ab+bc+ca)}{3(a+b)(b+c)(c+a)}$$

By the Iran 1996 inequality, we have: $(ab + bc + ca)\left[\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right] \geq \frac{9}{4}$

Thus, we need to prove: $\frac{9}{2} + \sum \frac{a}{b+c} + \frac{32(a+b+c)(ab+bc+ca)}{3(a+b)(b+c)(c+a)} \geq \frac{103}{6} + \frac{20abc}{3(a+b)(b+c)(c+a)}$

$$\Leftrightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2$$

$$\Leftrightarrow a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a)$$

True by Schur inequality. The equality holds for $x = y = z$.

UP.114. Let a, b, c be the sides and R and r the circumradius and the inradius of a triangle ABC respectively. Prove that:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{9}{4r(4R+r)}$$

Proposed by Martin Lukarevski – Skopje – Macedonia

Solution by proposer

We use the substitution: $a = y + z, b = z + x, c = x + y$, *for three positive real numbers* $x, y, z > 0$. *Then* $x = s - a, y = s - b, z = s - c$, *with* s *the semiperimeter of the triangle. Hence:* $xy + yz + zx = -s^2 + ab + ab + bc + ca = r(4R + r)$, *be the well-known relations [1],* $ab + bc + ca = s^2 + 4Rr + r^2$. *The inequality is transformed to* $(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$, *and this is a famous inequality, [2].*

Remark 1. By Euler's inequality $R \geq 2r$, *we have* $(9R + 2r)(R - 2r) \geq 0$. *Hence* $\frac{9}{4r(4R+r)} \geq \frac{1}{R^2}$. *Thus the inequality is a sharpening of the well-known inequality*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{R^2}$$

References

[1] O. Bottema, R.Z. Djordjevic, R. R. Janic, D. S. Mitrinovic, P.M. Vasic, *Geometric inequalities., Groningen, Wolters-Noordhoff, 1969.*

[2] CRUX Mathematicorum, 1994, No. 4, p. 108, Problem 1940

UP.115. Evaluate:

$$\int_0^{\infty}\frac{\ln(x)\sin(x)}{(x)^{\frac{1}{2}}}dx$$

Proposed by Arafat Rahman Akib – Dahka – Bangladesh

Solution by Feti Sinani-Podujeve-Kosovo

$$\int_0^{+\infty}\frac{\ln(x)\sin x}{x^{\frac{1}{2}}}dx = Im\int_0^{+\infty} e^{xi}x^{-\frac{1}{2}}\ln(x)\,dx = \left[\frac{x}{i} = t\right] = Im\int_0^{+\infty} e^{-x}x^{-\frac{1}{2}}\ln(xi)\,i^{\frac{1}{2}}dx =$$

$$= Im\left(\int_0^{+\infty} e^{-x}x^{-\frac{1}{2}}\ln(x)\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}dx + Im\int_0^{+\infty} e^{-x}x^{-\frac{1}{2}}\ln\left(e^{i\frac{\pi}{2}}\right)\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}dx\right) =$$

$$= Im\left(\Gamma\left(\frac{1}{2}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + \Gamma\left(\frac{1}{2}\right)\frac{\pi}{2}i\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\Gamma\left(\frac{1}{2}\right) + \frac{\pi}{2}\Gamma\left(\frac{1}{2}\right)\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}}\left(\Psi\left(\frac{1}{2}\right) + \frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}\left(-\gamma - 2\ln 2 + \frac{\pi}{2}\right)$$

UP.116. Prove that:

$$\sum_{n=1}^{\infty}\left((-1)^{n-1}\frac{H_n^4 + 6H_n^2H_n^{(2)} + 3\left(H_n^{(2)}\right)^2 + 8H_nH_n^{(3)} + 6H_n^{(4)}}{n}\right) = 24Li_5\left(\frac{1}{2}\right)$$

Proposed by Ali Shather – An Nasiriyah – Iraq, Shivam Sharma – New Delhi – India

Solution by proposers

Apply the Stirling's second sum, we get,

$$\int_0^1 x^{n-1}\ln^k(1-x)\,dx = (-1)^k\frac{\frac{2}{k}(n)}{n} \quad \textbf{(1)}$$

where,

$$\frac{2}{k}(n) = \sum_{j=0}^{k-1}\binom{k-1}{j}(k-j-1)!\,H_n^{(k-j)}\frac{2}{j}(n)$$

Putting $k = 4$, we get,

$$\tfrac{2}{4}(n) = H_n^4 + 6H_n^2H_n^{(2)} + 3\left(H_n^{(2)}\right)^2 + 8H_nH_n^{(3)} + 6H_n^{(4)} \quad \textit{(2)}$$

then using (1) & (2), and then summing both sides, we get

$$-S = \sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n-1}\ln^4(1-x)\,dx \Rightarrow \int_0^1\sum_{n=1}^{\infty}(-x)^n\frac{\ln^4(1-x)}{x}dx$$

$$\Rightarrow \int_0^1\frac{\ln^4(1-x)}{1+x}dx$$

Replace, $x \to 1-x$

$$\Rightarrow \int_0^1\frac{\ln^4(x)}{2-x}dx \Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}\int_0^1\frac{x^n}{2^n}\ln^4(x)\,dx \Rightarrow \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 x^{n-1}\ln^4(x)\,dx$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^4}{\partial n^4}\left[\int_0^1 x^{n-1}\,dx\right] \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{x^n\ln^4(x)}{n} - \frac{4x^n}{n^2}\ln^3(x) + 12\frac{x^n}{n^3}\ln^2(x) - 24\frac{x^n}{n^4}\ln(x) + 24\frac{x^n}{n^5}\right]_0^1 \Rightarrow$$

$$\Rightarrow 24\sum_{n=1}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{n^5}\right) \Rightarrow 24Li_5\left(\frac{1}{2}\right)$$

(OR) $-S = 24Li_5\left(\frac{1}{2}\right)$ ***(OR)*** $S = -24Li_5\left(\frac{1}{2}\right)$ ***(Answer)***

UP.117. Let $a, b, c \in \left[\frac{1}{2}; 3\right)$ be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30} + \frac{11}{40} \geq \frac{3(ab+bc+ca-2)}{2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)} \qquad \textit{(*)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

**** By AM-GM inequality we have:***

$$a^3+a^3+a^3+\sqrt[4]{a}+\sqrt[4]{a}+\sqrt[4]{a}+\sqrt[4]{a}+1+1+1\ge 10\cdot\sqrt[10]{a^3\cdot a^3\cdot a^3\cdot\left(\sqrt[4]{a}\right)^4\cdot 1\cdot 1\cdot 1}=$$

$$=10\cdot\sqrt[10]{a^{10}}=10a\Rightarrow 3a^3+4\cdot\sqrt[4]{a}+3\ge 10a\Rightarrow 4\cdot\sqrt[4]{a}\ge 10a-3a^3-3 \qquad (1)$$

- Other, because $a,b,c>0; a+b+c=3\Rightarrow a<3\Rightarrow (a-3)(a-1)^2\le 0\Leftrightarrow (a-3)(a^2-2a+1)\le 0$

$$\Leftrightarrow a^3-5a^2+7a-3\le 0\Leftrightarrow a^3\le 5a^2-7a+3 \qquad (2)$$

- Let (1), (2): $\Rightarrow 4\sqrt[4]{a}\ge 10a-3\left(5a^2-7a+3\right)-3=31a-15a^2-12\Rightarrow\sqrt[4]{a}\ge\frac{31a-15a^2-12}{4}$

+ Similar: $\sqrt[4]{b}\ge\frac{31b-15b^2-12}{4};\ \sqrt[4]{c}\ge\frac{31c-15c^2-12}{4}$

− Therefore: $\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\ge\frac{31(a+b+c)-15(a^2+b^2+c^2)-36}{4}=\frac{31\cdot 3-15(a^2+b^2+c^2)-36}{4}$

$$\Rightarrow\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30}\ge\frac{57-15[(a+b+c)^2-2(ab+bc+ca)]}{120}=\frac{30(ab+bc+ca)-78}{120}=\frac{ab+bc+ca}{4}-\frac{13}{20}$$

$$\Rightarrow\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30}+\frac{11}{40}\ge\frac{ab+bc+ca}{4}-\frac{3}{8} \qquad (3)$$

+ Other: $\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\ge 5\cdot\sqrt[5]{\left(\sqrt[3]{a}\right)^3\cdot a^2\cdot a^2}=5\cdot\sqrt[5]{a^5}=5a$

$\Leftrightarrow 3\cdot\sqrt[3]{a}+2a^2\ge 5a\Leftrightarrow\sqrt[3]{a}\ge\frac{5a-2a^2}{3}$***. Similar:*** $\sqrt[3]{b}\ge\frac{5b-2b^2}{3};\ \sqrt[3]{c}\ge\frac{5c-2c^2}{3}$

- Therefore: $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge\frac{5(a+b+c)-2(a^2+b^2+c^2)}{3}=\frac{5\cdot 3-2\left[(a+b+c)^2-2(ab+bc+ca)\right]}{3}$

$$\Rightarrow\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge\frac{15-2\cdot 3^2+4(ab+bc+ca)}{3}\Leftrightarrow\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge\frac{4(ab+bc+ca)-3}{3} \qquad (4)$$

- Because $a,b,c\in\left[\frac{1}{2};3\right)\Rightarrow\left(a-\frac{1}{2}\right)\left(b-\frac{1}{2}\right)+\left(b-\frac{1}{2}\right)\left(c-\frac{1}{2}\right)+\left(c-\frac{1}{2}\right)\left(a-\frac{1}{2}\right)\ge 0$

$$\Leftrightarrow ab+bc+ca\ge a+b+c-\frac{3}{4}=3-\frac{3}{4}=\frac{9}{4}\Rightarrow ab+bc+ca-2>0 \qquad (5)$$

+ Let (4), (5): $\Rightarrow\frac{3(ab+bc+ca-2)}{2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)}\le\frac{3(ab+bc+ca-2)}{\frac{8(ab+bc+ca)}{3}}=\frac{9(ab+bc+ca-2)}{8(ab+bc+ca)} \qquad$ ***(6)***

- Let (3), (6), (*). We need to prove: $\frac{ab+bc+ca}{4}-\frac{3}{8}\ge\frac{9(ab+bc+ca-2)}{8(ab+bc+ca)}\Leftrightarrow$

$\Leftrightarrow\frac{ab+bc+ca}{4}+\frac{9}{4(ab+bc+ca)}\ge\frac{3}{2}$ ***(True because by AM-GM inequality) and we get the result.***

+ Equality occurs if: $a=b=c=1.$

UP.118. Prove that:

$$\int_0^\infty x^p\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}\,dx = 2^{p+\frac{1}{2}}B\left(\frac{p+1}{2},-p-\frac{1}{2}\right);\ -1<p<-\frac{1}{2}$$

Proposed by Shivam Sharma - New Delhi – India

Solution 1 by Abdulrahman Hamed Balfaqih - Tarim – Yemen

$$\sqrt{1+x^2} = 2\tan^2 y+1 \Rightarrow I = \int_0^\infty (x^2)^{\frac{p}{2}}\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}\,dx =$$

$$= 2^{p+\frac{3}{2}}\int_0^{\frac{\pi}{2}} \sin^p y\cos^{-2p-2} y\,dx = 2^{p+\frac{1}{2}}\beta\left(\frac{p+1}{2},-p-\frac{1}{2}\right);\ -1<p<-\frac{1}{2}$$

Solution 2 by Khalef Ruhemi - Jarash – Jordan

$$\int_0^\infty x^{-\frac{1}{2}}\cdot e^{-x}\cdot\cos(Bx)\,dx = \Gamma\left(\frac{1}{2}\right)(1+\beta^2)^{-\frac{1}{4}}\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{\sqrt{1+B^2}}\right)\right)$$

$$= \frac{\sqrt{\pi}}{(1+\beta^2)^{\frac{1}{4}}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{1+\sqrt{1+B^2}}}{(1+B^2)^{\frac{1}{4}}} = \sqrt{\frac{\pi}{2}}\cdot\sqrt{\frac{1+\sqrt{1+B^2}}{1+B^2}}$$

$$\Rightarrow \sqrt{\frac{\pi}{2}}\int_0^\infty B^p\sqrt{\frac{1+\sqrt{1+B^2}}{1+B^2}}\cdot dB = \int_0^\infty\left(x^{-\frac{1}{2}}\cdot e^{-x}\cdot\int_0^\infty B^p\cdot\cos(Bx)\,dB\right)dx$$

$$= -\Gamma(1+p)\sin\left(\frac{\pi p}{2}\right)\int_0^\infty x^{-\frac{3}{2}-p}\cdot e^{-x}\,dx, -\frac{1}{2}>p>-1$$

$$= -\Gamma(1+p)\Gamma\left(-\frac{1}{2}-p\right)\sin\left(\frac{\pi p}{2}\right)\Rightarrow -1$$

$$\Rightarrow \int_0^\infty x^p\sqrt{\frac{1+\sqrt{1+B^2}}{1+B^2}}\cdot dB = -\sqrt{\frac{2}{\pi}}\cdot\Gamma(1+p)\Gamma\left(-\frac{1}{2}-p\right)\sin\left(\frac{\pi p}{2}\right)$$

$$= -\sqrt{\frac{2}{\pi}}\cdot\Gamma\left(-\frac{1}{2}-p\right)\sin\left(\frac{\pi p}{2}\right)x\cdot\frac{1}{\sqrt{\pi}}\cdot 2^p\cdot\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p}{2}+1\right)$$

$$= -\frac{(2)^{p+\frac{1}{2}}}{\pi}\cdot\Gamma\left(-\frac{1}{2}-p\right)\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p}{2}+1\right)x\,\frac{-\pi}{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{\pi}{2}+1\right)}$$

$$= (2)^{p+\frac{1}{2}}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(-\frac{1}{2}-p\right)}{\Gamma\left(-\frac{p}{2}\right)} = 2^{p+\frac{1}{2}}\cdot B\left(\frac{p+1}{2},-p-\frac{1}{2}\right)$$

$$\therefore \int_0^\infty x^p\cdot\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}\cdot dx = (2)^{p+\frac{1}{2}}\cdot B\left(\frac{p+1}{2},-p-\frac{1}{2}\right)$$

Solution 3 by Togrul Ehmedov-Baku-Azerbaidian

$$1+x^2 = y^2 \Rightarrow x = \sqrt{y^2-1} \Rightarrow dx = \frac{y}{\sqrt{y^2-1}}dy$$

$$I = \int_0^\infty x^p\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}dx = \int_0^\infty (y^2-1)^{\frac{p}{2}}\frac{dy}{\sqrt{y-1}} = \int_0^\infty \frac{(y-1)^{\frac{p}{2}}(y+1)^{\frac{p}{2}}}{(y-1)^{\frac{1}{2}}}dy$$

$$= \int_0^\infty (y-1)^{\frac{p-1}{2}}(y+1)^{\frac{p}{2}}dy$$

$$\frac{1}{y-1} = t \Rightarrow y = \frac{1}{t}-1 \Rightarrow dy = -\frac{1}{t^2}dt$$

$$I = \int_0^\infty t^{\frac{1-p}{2}}\left(\frac{1+2t}{t}\right)^{\frac{p}{2}}\frac{dt}{t^2} = \int_0^\infty t^{\frac{-2p-3}{2}}(1+2t)^{\frac{p}{2}}dt$$

$$2t = z \Rightarrow \frac{z}{2} \Rightarrow dt = \frac{dz}{2}$$

$$I = \int_0^\infty 2^{\frac{2p+1}{2}}z^{\frac{-2p-3}{z}}(1+z)^{\frac{p}{2}}dx = 2^{p+\frac{1}{2}}\int_0^\infty \frac{z^{\frac{-2p-3}{2}}}{(1+z)^{-\frac{p}{2}}}dz$$

$$= 2^{p+\frac{1}{2}}B\left(\frac{p+1}{2},-p-\frac{1}{2}\right)$$

Note: $B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt, Re(x) > 0; Re(y) > 0$

UP.119. Prove that:

$$\sum_{k=1}^{\infty}\left(\frac{H_k^{(2)}}{k^9}\right)=9\zeta(2)\zeta(9)+2\zeta(3)\zeta(8)+6\zeta(4)\zeta(7)+4\zeta(5)\zeta(6)-27\zeta(11)$$

Proposed by Ali Shather and Shivam Sharma

Solution by Ali Shather and Shivam Sharma

$$S=\sum_{k=1}^{\infty}\frac{H_n^{(3)}}{n^8}=\sum_{n=1}^{\infty}\frac{1}{n^8}\left(\zeta(3)-\sum_{k=1}^{\infty}\frac{1}{(n+k)^3}\right)=\zeta(3)\zeta(8)+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{-1}{n^8(n+k)^3}$$

$$S=\zeta(3)\zeta(8)+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\begin{array}{c}\frac{36}{k^{10}n}-\frac{36}{k^{10}(k+n)}-\frac{28}{k^9n^2}-\frac{8}{k^9(k+n)^2}+\frac{21}{k^8n^5}-\\ -\frac{1}{k^5(k+n)^3}-\frac{15}{k^7n^4}+\frac{10}{k^6n^5}-\frac{6}{k^5n^6}+\frac{3}{k^4n^7}-\frac{1}{k^3n^8}\end{array}\right]$$

$$S=\zeta(3)\zeta(8)+\sum_{k=1}^{n}\left[\begin{array}{c}\frac{36H_k}{k^{10}}-\frac{28\zeta(2)}{k^9}-\frac{8}{k^9}\left(\zeta(2)-H_k^{(2)}\right)+\frac{21\zeta(3)}{k^8}-\frac{35}{k^8}\left(\zeta(3)-H_k^{(3)}\right)-\\ -\frac{15\zeta(4)}{k^7}+\frac{10\zeta(5)}{k^6}-\frac{6\zeta(6)}{k^5}+\frac{3\zeta(7)}{k^4}-\frac{\zeta(8)}{k^3}\end{array}\right]$$

$$S=\zeta(3)\zeta(8)+\left[\begin{array}{c}36\left(6\zeta(11)-\zeta(2)\zeta(9)-\zeta(3)\zeta(8)-\zeta(4)\zeta(7)-\zeta(5)\zeta(6)\right)-\\ -28\zeta(2)\zeta(9)-8\zeta(2)\zeta(9)+8\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^8}+21\zeta(3)\zeta(8)-\zeta(3)\zeta(8)+\\ +S-15\zeta(4)\zeta(7)+10\zeta(5)\zeta(6)-6\zeta(6)\zeta(5)+3\zeta(7)\zeta(4)-\zeta(8)\zeta(3)\end{array}\right]$$

$$-8\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^9}=216\zeta(11)-72\zeta(2)\zeta(9)-16\zeta(3)\zeta(8)-48\zeta(4)\zeta(7)-32\zeta(5)\zeta(6)$$

Or

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^9}=9\zeta(2)\zeta(9)+2\zeta(3)\zeta(8)+6\zeta(4)\zeta(7)+4\zeta(5)\zeta(6)-27\zeta(11)$$

UP.120. Prove that in any acute-angled triangle ABC the following relationship holds:

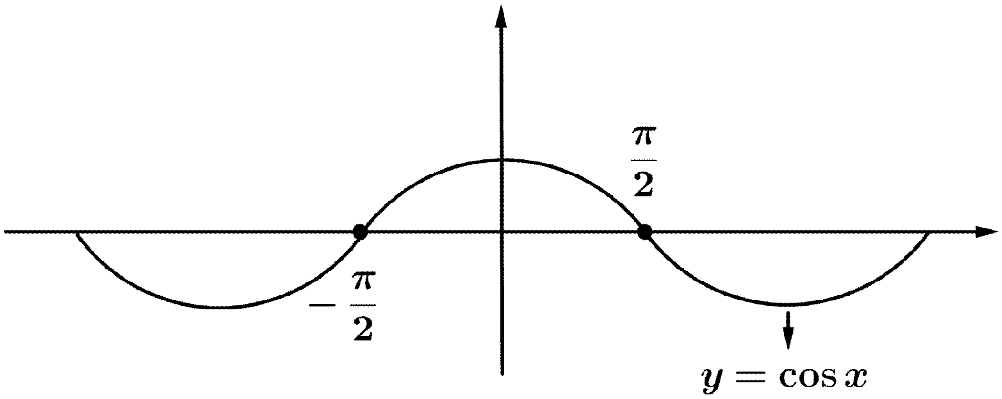
$$\sqrt{2}\sum(\sin A+\cos A)>\sum\sin C\left(1+\cos 2\,(A-B)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$\Delta\, ABC$ ***is acute-angled,*** $\therefore\ 0<A,B,C<\frac{\pi}{2}\ \therefore\ 0<A<\frac{\pi}{2}\ \&-\frac{\pi}{2}<-B<0$

Adding these last 2, $-\frac{\pi}{2} < A - B < \frac{\pi}{2}$



From the graph, we see $0 < \cos(A - B) \le 1$ ***(1)***

Similarly, $0 < \cos(B - C) \le 1$ ***(2)***

$0 < \cos(C - A) \le 1$ ***(3)***

$$\therefore \sum \sin C\,(1 + \cos 2(A - B)) = \sum \sin C \cdot 2\cos^2(A - B)$$

$$\overset{by\ (1),(2),(3)}{\le} \sum 2\sin C = 2\sum \sin A \overset{?}{<} \sqrt{2}\sum(\sin A + \cos A)$$

$$\Leftrightarrow \sqrt{2}\sum \cos A \overset{?}{>} (2 - \sqrt{2})\sum \sin A \Leftrightarrow \sqrt{2}\left(\frac{R + r}{R}\right) \overset{?}{>} (2 - \sqrt{2})\left(\frac{s}{R}\right)$$

$$\Leftrightarrow 2(R + r)^2 \overset{?}{>} (6 - 4\sqrt{2})s^2 \quad \textit{(a)}$$

$$\because 128 > 121 \therefore 8\sqrt{2} > 11 \Rightarrow 4\sqrt{2} > \frac{11}{2} = 6 - \frac{1}{2} \Rightarrow \frac{1}{2} > 6 - 4\sqrt{2} \quad \textit{(4)}$$

(4) $\Rightarrow$ ***in order to prove (a), it suffices to show:*** $2(R + r) > \frac{1}{2}s^2 \Leftrightarrow$

$$\Leftrightarrow s^2 < 4(R + r)^2 \quad \textit{(b)}$$

Now, LHS of (b) $\overset{Gerretsen}{\le} 4R^2 + 4Rr + 3r^2 \overset{?}{<} 4R^2 + 8Rr + 4r^2 \Leftrightarrow$

$\Leftrightarrow 4Rr + r^2 \overset{?}{>} 0 \rightarrow$ ***true*** $\Rightarrow$ ***(b) is true*** $\Rightarrow$ ***(a) is true***

(proved)