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SOLUTIONS

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SOLUTIONS

Proposed by

Daniel Sitaru – Romania

Nguyen Viet Hung – Hanoi – Vietnam

Marin Chirciu – Romania

Hoang Le Nhat Tung – Hanoi – Vietnam

Marian Ursărescu – Romania

Vasile Mircea Popa – Romania

D.M. Bătinețu – Giurgiu – Romania

Neculai Stanciu – Romania

Solutions by

Nassim Nicholas Taleb-USA, Daniel Sitaru-Romania

Orlando Irahola Ortega-Bolivia,Soumava Chakraborty-Kolkata-India

Mustafa Tarek-Cairo-Egypt, Tran Hong-Dong Thap-Vietnam

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Adrian Popa – Romania, Jalil Hajimir-Canada

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Florentin Vișescu-Romania,Boris Colakovic-Belgrade-Serbie,

Ruangkhaw Chaoka-Chiangrai-Thailand, Ali Jaffal-Lebanon

JP.226. Let a, b, c be positive real numbers. Find the k_{max} such that the inequality is true:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\geq k\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\geq k\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)\quad (*)$$

But: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\overset{Nesbit}{\geq}\frac{3}{2}$; $a^2+b^2+c^2\geq ab+bc+ca\to\frac{a^2+b^2+c^2}{ab+bc+ca}-1>0$

So, $k>0\to k_{max}>0$.

Because: (*) is true for all $a, b, c>0$

Hence, we let: $a=b=1; c=\frac{1}{n}(\forall n\in\mathbb{N}^*)$ then:

$$k\leq\frac{\frac{3}{1+\frac{1}{n}}-\frac{3}{2}}{\frac{2+\frac{1}{n^2}}{1+\frac{2}{n}}-1};(\forall n\in\mathbb{N}^*)$$

Let $n\to+\infty$ we have: $k\leq\frac{1}{2}$

Now, we show that: $k_{\max}=\frac{1}{2}$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\geq\frac{1}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)$$

$$\leftrightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{1}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)+1=\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

It is true because:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ca+cb}\overset{Schwarz}{\geq}\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Proved. Equality if and only if $a=b=c$.

JP.227. Prove that in any ABC triangle the following relationship holds:

$$3\left(r_a r_b^3 + r_b r_c^3 + r_c r_a^3\right) \ge r(4R + r)^3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

Because $r_a + r_b + r_c = 4R + r$ ***we must show:***

$$3\left(r_a r_b^3 + r_b r_c^3 + r_c r_a^3\right) \ge r(r_a + r_b + r_c)^3$$

$$\left.\begin{array}{c}3\left(r_a r_b^3 + r_b r_c^3 + r_c r_a^3\right) \ge r(r_a + r_b + r_C)^3 \\ But\ r_a = \frac{S}{s-a}, r = \frac{S}{s}, s = \frac{a+b+c}{2}\end{array}\right\} \Rightarrow \text{we must show:}$$

$$3\left(\frac{1}{(s-a)(s-b)^3} + \frac{1}{(s-b)(s-c)^3} + \frac{1}{(s-c)(s-a)^3}\right) \ge \frac{1}{s}\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)^3$$

$$\Leftrightarrow 3s\left((s-a)^2(s-c)^3 + (s-b)^2(s-a)^3 + (s-c)^2(s-b)^3\right) \ge$$

$$\ge \left((s-a)(s-b) + (s-b)(s-c) + (s-b)(s-c)\right)^3 \quad \textbf{(1)}$$

$$(s-a)^2(s-c)^3 + (s-b)^2(s-a)^3 + (s-c)^2(s-b)^3 =$$

$$= \frac{(s-a)^3(s-c)^3}{s-a} + \frac{(s-b)^3(s-a)^3}{s-b} + \frac{(s-c)^3(s-b)^3}{s-c} \overset{Holder}{\ge}$$

$$\ge \frac{\left((s-a)(s-c) + (s-b)(s-a) + (s-c)(s-b)\right)^3}{3(s-a+s-b+s-c)} \Leftrightarrow$$

$$\Leftrightarrow 3s\left((s-a)^2(s-c)^3 + (s-b)^2(s-a)^3 + (s-c)^2(s-b)^3\right) \ge$$

$$\ge \left((s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)\right)^3 \Rightarrow \textbf{(1) it is true.}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

We have: $r_a + r_b + r_c = 4R + r$***. And:*** $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$

Now, LHS$= 3\left(\frac{r_b^3}{\left(\frac{1}{r_a}\right)} + \frac{r_c^3}{\left(\frac{1}{r_b}\right)} + \frac{r_a^3}{\left(\frac{1}{r_c}\right)}\right) \overset{Holder}{\ge} 3 \cdot \frac{(r_b+r_c+r_a)^3}{3\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)} = \frac{(4R+r)^3}{\frac{1}{r}} = r(4r + r)^3$

Proved. Equality $\Leftrightarrow r_a = r_b = r_c \Leftrightarrow a = b = c.$

Solution 3 by Adrian Popa-Romania

$$LHS = 3\left(\frac{r_a r_b^3 r_c}{r_c}\right) + \left(\frac{r_b r_c^3 r_b}{r_c}\right) + \left(\frac{r_c r_a^3 r_b}{r_c}\right) = 3\Delta s\left(\frac{\left(r_a^2 + r_b^2 + r_c\right)^2}{r_a + r_b + r_c}\right) =$$

$$= 3\Delta s(r_a + r_b + r_c) = 3\Delta s(4R + r) \overset{?}{\geq} r(4R + r)^2$$

$$?: 3\Delta s \geq r(4R + r)^2 \leftrightarrow 3s^2 \geq (4R + r)^2$$

$$3s^2 \geq 3(16rR - 4r^2) \leftrightarrow 3s^2 - (4R + r^2) \geq 3(16rR - 5r^2) - (4R + r)^2 \leftrightarrow$$

$$\leftrightarrow 40Rr - 16R^2 - 16r^2 \geq 0$$

$$R \geq 2r;\ r = \frac{\Delta}{s};\ r_a r_b r_c = \Delta s$$

$$r_a + r_b + r_c = 4R + r$$

$$s^2 \geq 16rR - 5r^2$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\because m^2 + n^2 + p^2 \geq mn + np + pm, \forall m, n, p \in \mathbb{R}$$

$$\therefore x^2y^4 + y^2z^4 + z^2x^4 \geq xy^2 \cdot yz^2 + yz^2 \cdot zx^2 + zx^2 \cdot xy^2 = xyz\left(\sum x^2y\right)$$

$$\Rightarrow \frac{\sum x^2y^4}{xyz} \geq \sum x^2y \Rightarrow \sum \frac{xy^3}{z} \overset{(1)}{\geq} \sum x^2y$$

Now, $3\sum xy\left(\sum \frac{x^2}{y}\right) = 3\sum x^3 + 3\sum xy^2 + 2\sum \frac{xy^3}{z} \overset{by\,(1)}{\geq} 3\sum x^3 + 3\sum xy^2 + 3\sum x^2y$

$$\overset{A-G}{\geq} \sum x^3 + 6xyz + 3\sum xy^2 + 3\sum x^2y = \sum x^3 + 3\prod(x + y) = \left(\sum x\right)^3$$

$$\therefore 3\sum xy\left(\sum \frac{x^2}{y}\right) \overset{(2)}{\geq} \left(\sum x\right)^3$$

Choosing $x = r_a, y = r_b, z = r_c$ ***in (2),***

$$3\sum r_a r_b\left(\sum \frac{r_a^2}{r_b}\right) \overset{(3)}{\geq} \left(\sum r_a\right)^3$$

Now, $3\sum r_a r_b^3 = 3s\sum r_a^2(s - b)$ $(\because r_a r_b = s(s - c), etc)$

$$= 3s\sum r_a^2 \frac{rs}{r_b} = 3r(s^2)\left[\sum \frac{r_a^2}{r_b}\right] = 3r\left(\sum r_a r_b\right)\left(\sum \frac{r_a^2}{r_b}\right) \overset{by\,(3)}{\geq} r(4R + r)^3 \text{ (proved)}$$

JP.228. Prove that if a, b, c are the lenghts of the sides of a triangle, then:

$$\sqrt{\frac{a}{b+c-a}}+\sqrt{\frac{b}{c+a-b}}+\sqrt{\frac{c}{a+b-c}}\ge\frac{(a+b+c)^2}{ab+bc+ca}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

$$\sqrt{\frac{a}{b+c-a}}=\sqrt{\frac{a}{b+c-a}\cdot 1}\ge\frac{2}{\frac{b+c-a}{a}+1}=\frac{2a}{b+c}\Rightarrow \text{we must show:}$$

$$2\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)\ge\frac{(a+b+c)^2}{ab+bc+ca}\quad (1)$$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=\frac{a^2}{ab+ac}+\frac{b^2}{ab+bc}+\frac{c^2}{ac+bc}\overset{Bergstrom}{\ge}\frac{(a+b+c)^2}{2(ab+ac+bc)}\Rightarrow \text{(1) it is true.}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$LHS=\frac{a^2}{a\sqrt{ba+ca-a^2}}+\frac{b^2}{b\sqrt{cb+ab-b^2}}+\frac{c^2}{c\sqrt{ac+bc-a^2}}\overset{Schwarz}{\ge}$$

$$\ge\frac{(a+b+c)^2}{a\sqrt{ba+ca-a^2}+b\sqrt{cb+ab-b^2}+c\sqrt{ac+bc-c^2}}=A$$

$$a\sqrt{bc+ca-a^2}\overset{AM-GM}{\le}\frac{a^2+(bc+ca-a^2)}{2}=\frac{bc+ca}{2}$$

$$b\sqrt{cb+ab-b^2}\overset{AM-GM}{\le}\frac{b^2+(cb+ab-b^2)}{2}=\frac{cb+ab}{2}$$

$$c\sqrt{ac+bc-c^2}\overset{AM-GM}{\le}\frac{c^2+(ac+bc-c^2)}{2}=\frac{ab+bc}{2}$$

$$\Rightarrow a\sqrt{bc+ca-a^2}+b\sqrt{cb+ab-b^2}+c\sqrt{ac+bc-c^2}\le ab+bc+ca$$

$$\Rightarrow A\ge\frac{(a+b+c)^2}{ab+bc+ca}=RHS$$

Proved. Equality $\Leftrightarrow a=b=c.$

Solution 3 by Michael Sterghiou-Greece

$$\sum_{cyc}\sqrt{\frac{a}{b+c-a}}\ge\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}ab}\quad (1)$$

(1) homogeneous so, WLOG $\sum_{cyc}a=p=3$***. Let*** $\sum_{cyc}ab=a\,,\ abc=r$

(1)$\rightarrow \sum_{cyc}\frac{a}{\sqrt{(b+c-a)a}} \geq \frac{a}{q}$ ***(2). The function*** $f(x) = \frac{1}{\sqrt{x}}$ ***is convex hence by the generalized Jensen inequality we have LHS of (2)*** $\geq (a+b+c)\cdot\frac{1}{\sqrt{\frac{\sum_{cyc} a\cdot a(b+c-a)}{a+b+c}}} \overset{?}{\geq} \frac{9}{q}$ ***(where*** a, b, c ***are "weights"). The last inequalities reduces to*** $\frac{3}{\sqrt{\frac{\sum_{cyc} a^2(3-2a)}{3}}} \geq \frac{9}{q}$ ***or***

$$q^2 - 36q + 18r + 81 \geq 0 \quad (3)$$

as $\sum_{cyc} a^2 = 9-2q, \sum_{cyc} a^3 = 27-9q+3r$***. As*** $q \leq \frac{p^3+9r}{4p}$ ***from Schur inequality 3rd degree we have*** $3r \geq 4q-9$ ***so (3)*** $\rightarrow q^2 - 12q + 27 \geq 0$ ***or*** $(3-q)(9-q) \geq 0$ ***which holds as*** $q \leq \frac{p^2}{3} = 3$

Solution 4 by Jalil Hajimir-Canada

$$\sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{1}{\frac{b+c}{a}-1}} \overset{(*)}{\geq} \frac{1}{1+\frac{1}{2}\left(\frac{b+c}{a}-1\right)} = \frac{2a}{b+c}$$

$$\left(\frac{b+c}{a}-1\right)^{\frac{1}{2}} = \left(1+\frac{b+c}{a}-2\right)^{\frac{1}{2}} \geq 1+\frac{1}{2}\left(\frac{b+c}{a}-2\right) \quad \textit{(*) Bernoulli}$$

$$\sum_{cyc}\sqrt{\frac{a}{b+c-a}} \geq \sum_{cyc}\frac{2a}{b+c} \quad (1)$$

$$\sum_{cyc}\frac{2a}{b+c} = 2\sum_{cyc}\frac{a^2}{ab+ac} \geq \frac{2(a+b+c)^2}{2(ab+bc+ca)} \quad (2)$$

From (1) and (2): $\sum_{cyc}\sqrt{\frac{a}{b+c-a}} \geq \frac{(a+b+c)}{(a^2+b^2+c^2)}$

Solution 5 by Marian Dinca-Romania

$$\sum_{cyc}\sqrt{\frac{a}{b+c-a}} = \sum_{cyc}\sqrt{\frac{a\cdot a}{(b+c-a)a}} = \sum_{cyc}\frac{a}{\sqrt{(b+c-a)a}} \geq$$

$$\geq \sum_{cyc}\frac{a}{\frac{(b+c-a)+a}{2}} = \sum_{cyc}\frac{2a}{b+c}$$

$$\sum_{cyc}\frac{2a}{b+c} = 2\cdot\sum_{cyc}\frac{a^2}{(b+c)a} \geq 2\sum_{cyc}\frac{(a+b+c)^2}{(b+c)a} = \frac{(a+b+c)^2}{ab+bc+ca}$$

Solution 6 by Anant Bansal-India

We know that the minimum value of $\sum \sin A \cdot \sin B$ *is at* $A = B = C$

$$\Rightarrow \sum \sin A \sin B \geq \frac{9}{4} > \frac{27}{1024} = \left(\frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{24\sqrt{3}}\right)^2 > \left(\frac{(\sum \sin A)^2}{24\sqrt{3}}\right)^2$$

$$\Rightarrow \frac{\left(24\sqrt{3}\right)^2}{\sum \sin A \cdot \sin B} > \frac{(\sum \sin A)^4}{(\sum \sin A \cdot \sin B)^2}$$

$$\Rightarrow 8\sqrt{\frac{3}{\sin A \cdot \sin B} + \frac{3}{\sin B \cdot \sin C} + \frac{3}{\sin A \cdot \sin C}} > \frac{(\sum \sin A)^2}{\sum \sin A \cdot \sin B}$$

Putting $\sin A = \frac{a}{2R}$

$$\frac{16\sqrt{3}R}{\sqrt{abc}} > \frac{(a+b+c)^2}{ab+bc+ca} = k$$

$$R = \frac{abc}{4\Delta}$$

$$\frac{4\sqrt{3abc}}{\Delta} > k$$

Using Heron's formula and $AM \geq GM$ *we get:*

$$\sqrt{\frac{a}{b+c-a}} + \sqrt{\frac{b}{a+c-b}} + \sqrt{\frac{c}{a+b-c}} > \frac{(a+b+c)^2}{ab+bc+ca}$$

Equality holds for $a = b = c$

Solution 7 by Soumava Chakraborty-Kolkata-India

$$b+c-a = 4R\cos\frac{A}{2}\cos\frac{B-C}{2} - 4R\sin\frac{A}{2}\cos\frac{A}{2} = 4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right)$$

$$= 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Rightarrow \sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{4R\sin^2\frac{A}{2}\cos\frac{A}{2}}{8R\cos\frac{A}{2}\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}} =$$

$$= \sin\frac{A}{2}\sqrt{\frac{1}{2\left(\frac{r}{4R}\right)}} \overset{(1)}{=} \sqrt{\frac{2R}{r}}\sin\frac{A}{2}$$

Similarly, $\sqrt{\frac{b}{c+a-b}} \overset{(2)}{=} \sqrt{\frac{2R}{r}}\sin\frac{B}{2}$ ***and*** $\sqrt{\frac{c}{a+b-c}} \overset{(3)}{=} \sqrt{\frac{2R}{r}}\sin\frac{C}{2}$

(1)+(2)+(3)⇒ $\sum\sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{R}{2r}}\sum\frac{2\sin\frac{A}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}} \overset{(4)}{\geq} \sqrt{\frac{R}{2r}}\sum\left(2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\right)$

$$\left(\because 0 < \cos\frac{B-C}{2} \leq 1\right)$$

$$= \sqrt{\frac{R}{2r}}\sum(\cos B + \cos C) = \sqrt{\frac{2R}{r}}\left(1 + \frac{r}{R}\right)$$

We shall now prove $\frac{\sum a^2}{\sum ab} \overset{(a)}{\leq} \sqrt{\frac{R}{2r}}$

(a)⇔ $R(s^2 + 4Rr + r^2)^2 \geq 8r(s^2 - 4Rr - r^2)^2$

$$\Leftrightarrow R\{s^4 + (4Rr + r^2)^2 + 2(4Rr + r^2)s^2\} \geq 8r\{s^4 + (4Rr + r^2)^2 - 2(4Rr + r^2)s^2\}$$

$$\Leftrightarrow (R - 2r)s^4 + (R - 8r)(4Rr + r^2)^2 + (2R + 16r)(4Rr + r^2)s^2 \overset{(b)}{\geq} 6rs^4$$

Now, LHS of (b) $\underset{(i)}{\overset{Gerretsen}{\geq}} (R - 2r)(16Rr - 5r^2)s^2 + (R - 8r)(4Rr + r^2)^2 +$

$+(2R + 16r)(4Rr + r^2)s^2$ ***and RHS of (b)*** $\underset{(ii)}{\overset{Gerretsen}{\leq}} 6rs^2(4R^2 + 4Rr + 3r^2)$

(i)(ii)⇒ in order to prove (b), it suffices to prove:

$$s^2\{(R - 2r)(16Rr - 5r^2) + (2R + 16r)(4Rr + r^2) - 6r(4R^2 + 4Rr + 3r^2)\} +$$
$$+(R - 8r)(4Rr + r^2)^2 \geq 0$$

$$\Leftrightarrow s^2(5R + 8r) + (R - 8r)(4R + r)^2 \overset{(c)}{\geq} 0$$

Now, LHS of (c) $\overset{Gerretsen}{\geq} (5R + 8r)(16Rr - 5r^2) + (R - 8r)(4R + r)^2 \overset{?}{\geq} 0$

$$\Leftrightarrow 2t^3 - 5r^2 + 5t - 6 \overset{?}{\geq} 0 \ \left(t = \frac{R}{r}\right)$$

$\Leftrightarrow (t - 2)\{2t(t - 2) + 3t + 3\} \overset{?}{\geq} 0 \to$ ***true*** $\because t \overset{Euler}{\geq} 2$

⇒(c) ⇒(b) ⇒(a) is true. Now, $\frac{(\sum a)^2}{\sum ab} = \frac{\sum a^2 + 2\sum ab}{\sum ab} = 2 + \frac{\sum a^2}{\sum ab} \underset{(5)}{\overset{(a)}{\leq}} 2 + \sqrt{\frac{R}{2r}}$

(4), (5)⇒ it suffices to prove: $\sqrt{\frac{2R}{r}}\left(\frac{R+r}{R}\right) \geq 2 + \frac{1}{2}\sqrt{\frac{2R}{r}} \Leftrightarrow \sqrt{\frac{2R}{r}}\left(\frac{2R+2r-R}{2R}\right) \geq 2 \Leftrightarrow$

$$\Leftrightarrow \frac{2R}{r}\cdot\frac{(R+2r)^2}{4R^2} \geq 4 \Leftrightarrow (R+2r)^2 \geq 8Rr \Leftrightarrow R^2-4Rr+4r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)^2 \geq 0 \rightarrow \textit{true (Proved)}$$

JP.229. Let a,b,c be positive real numbers. Find the k_{max} such that the inequality is true:

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}-1 \geq k\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}-1 \geq k\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right) \quad (*)$$

Using Inequality: $x^2+y^2+z^2 \geq xy+xz+yz$ ***we have:***

$$a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2 \rightarrow \frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}-1 > 0$$

$$a^2+b^2+c^2 \geq ab+bc+ca \rightarrow \frac{a^2+b^2+c^2}{ab+bc+ca}-1 > 0$$

So, we only check case: $k > 0 \rightarrow k_{max} > 0$***. Because: (*) is true for all*** $a,b,c > 0$

Hence, we let: $a=b=1; c=\frac{1}{n} (\forall n \in \mathbb{N}^*)$ ***then:*** $k \leq \dfrac{\frac{2+\frac{1}{n^4}}{\frac{2}{n^2}+1}-1}{\frac{2+\frac{1}{n^2}}{1+\frac{2}{n}}-1}; (\forall n \in \mathbb{N}^*)$

Let $n \rightarrow +\infty$ ***we have:*** $k \leq 1$***. Now, we show that:*** $k_{max}=1$

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}-1 \geq 1\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)$$

$$\leftrightarrow \frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2} \geq \frac{a^2+b^2+c^2}{ab+bc+ca} \leftrightarrow abc(a^3+b^3+c^3)+\sum_{cyc}(a^5b+ab^5)$$

$$\geq 3a^2b^2c^2+\sum\nolimits_{cyc}(a^2b^4+a^4b^2)\text{. Because:}$$

$$a^3+b^3+c^3 \overset{AM-GM}{\geq} 3abc \rightarrow abc(a^3+b^3+c^3) \geq 3a^2b^2c^2 \quad (1)$$

$$a^5b+ab^5 = ab(a^4+b^4) \overset{(a^4+b^4)\geq(ab^3+ba^3)}{\geq} ab(ab^3+ba^3) = (a^2b^4+a^4b^2)$$

$$\rightarrow \sum_{cyc}\left(a^5b+ab^5\right) \ge \sum_{cyc}\left(a^2b^4+a^4b^2\right) \quad (2)$$

From (1)+(2) we have: $abc\left(a^3+b^3+c^3\right)+\sum_{cyc}\left(a^5b+ab^5\right)$

$\ge 3a^2b^2c^2+\sum_{cyc}\left(a^2b^4+a^4b^2\right)$. *Proved. Equality holds if and only if* $a=b=c$

JP.230. Prove that for any ABC triangle the following relationship holds:

$$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}+\frac{3\sqrt[4]{3}}{\sqrt{2}} \le 2\left(\sqrt{\cos\frac{A}{2}}+\sqrt{\cos\frac{B}{2}}+\sqrt{\cos\frac{C}{2}}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by Marian Ursărescu – Romania

Let $f:(0,+\infty)\rightarrow\mathbb{R}$, $f(x)=\sqrt{x}$, *because* $f''(x)<0 \Rightarrow$ *we can use Tiberiu Popoviciu's inequality:*

$$\frac{1}{3}\left(f(x)+f(y)+f(z)\right)+f\left(\frac{x+y+z}{3}\right) \le \frac{2}{3}\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+z}{2}\right)\right)$$

$$x=\sin A,\ y=\sin B,\ z=\sin C \Rightarrow$$

$$\Rightarrow \frac{1}{3}\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)+\sqrt{\sin\left(\frac{\pi}{3}\right)} \le \frac{2}{3}\sum\sqrt{\frac{\sin A+\sin B}{2}} \Rightarrow$$

$$\Rightarrow \sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}+\frac{3\sqrt[4]{3}}{\sqrt{2}} \le \frac{2}{3}\sum\sqrt{\frac{\sin A+\sin B}{2}} \quad (1)$$

$$\frac{\sin A+\sin B}{2}=\frac{2\sin\left(\frac{A+B}{2}\right)\cos\frac{A-B}{2}}{2}=\cos\frac{C}{2}\cdot\frac{A-B}{2} \quad (2)$$

From (1)+(2) $\Rightarrow \sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}+\frac{3\sqrt[4]{3}}{\sqrt{2}} \le 2\sum\sqrt{\cos\frac{C}{2}\cdot\cos\frac{A-B}{2}}$ *(3)*

But $\cos\frac{A-B}{2} \le 1$, *with equality for* $A=B$ *(4)*

From (3)+(4) $\Rightarrow \sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}+\frac{3\sqrt[4]{3}}{\sqrt{2}} \le 2\left(\sqrt{\cos\frac{A}{2}}+\sqrt{\cos\frac{B}{2}}+\sqrt{\cos\frac{C}{2}}\right)$

Equality for $A=B=C=\frac{\pi}{3}$

JP.231. Prove that for any positive real numbers a, b, c the following relationship holds:

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}\ge\frac{a+b+c}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

$$\frac{a^3+abc-abc}{a^2+bc}+\frac{b^3+abc-abc}{b^2+ca}+\frac{c^3+abc-abc}{c^2+ab}\ge\frac{a+b+c}{2}\Leftrightarrow$$

$$\Leftrightarrow\frac{a(a^2+bc)}{a^2+bc}-\frac{abc}{a^2+bc}+\frac{b(b^2+ac)}{b^2+ca}-\frac{abc}{b^2+ca}+\frac{c(c^2+ab)}{c^2+ab}-\frac{abc}{c^2+ab}\ge\frac{a+b+c}{2}\Leftrightarrow$$

$$\Leftrightarrow\frac{a+b+c}{2}\ge\frac{abc}{a^2+bc}+\frac{abc}{b^2+ca}+\frac{abc}{c^2+ab}\quad(1)$$

$$a^2+bc\ge 2a\sqrt{bc}\Leftrightarrow\frac{1}{a^2+bc}\le\frac{1}{2a\sqrt{bc}}\Leftrightarrow\frac{abc}{a^2+bc}\le\frac{\sqrt{bc}}{2}\Rightarrow$$

$$\frac{abc}{a^2+bc}+\frac{abc}{b^2+ca}+\frac{abc}{c^2+ab}\le\frac{\sqrt{ab}+\sqrt{ac}+\sqrt{bc}}{2}\quad(2)$$

From (1)+(2) we must show:

$$\frac{a+b+c}{2}\ge\frac{\sqrt{ab}+\sqrt{ac}+\sqrt{bc}}{2}\Leftrightarrow a+b+c\ge\sqrt{ab}+\sqrt{ac}+\sqrt{bc}\quad\textit{true.}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\frac{a^3}{a^2+b\cdot c}+\frac{b^3}{b^2+c\cdot a}+\frac{c^3}{c^2+a\cdot b}\overset{?}{\ge}\frac{a+b+c}{2}$$

$$\left(\frac{a^3}{a^2+b\cdot c}-\frac{a}{2}\right)+\left(\frac{b^3}{b^2+c\cdot a}-\frac{b}{2}\right)+\left(\frac{c^3}{c^2+a\cdot b}-\frac{c}{2}\right)\overset{?}{\ge}0$$

Let be the function: $f(x)=\frac{x^3}{x^2+\alpha}-\frac{x}{2}=\frac{2x^3-x^3-\alpha x}{2(x^2+\alpha)}$

$$f(x)=\frac{x^3-\alpha\cdot x}{2(x^2+\alpha)},x>0,\alpha>0$$

$$\lim_{n\to 0}f(x)=0,\lim_{n\to 0}f(x)=+\infty$$

$$f'(x)=\frac{(3x^2-\alpha)(2x^2+2\alpha)-4x(x^3-\alpha x)}{4(x^2+\alpha)^2}=$$

$$= \frac{6x^4 + 6\alpha x^2 - 2\alpha x^2 - 2\alpha^2 - 4x^4 + 4\alpha x^2}{4(x^2 + \alpha)^2}$$

$$f'(x) = \frac{2x^4 + 8\alpha x^2 - 2\alpha^2}{4(x^2 + \alpha)^2}, f'(x) = 0 \Rightarrow 2x^4 - 8\alpha x^2 - 2\alpha^2 = 0 \div (2)$$

$$x^4 - 4\alpha x^2 - \alpha^2 = 0$$

$$\Delta = 16\alpha^2 - 4(1)(-\alpha^2) = 16\alpha^2 + 4\alpha^2 = 20\alpha^2 > 0$$

x	0 $\qquad\qquad$ $+\infty$
$f'(x)$	+ + + + + + + + + + + + + + + + + +
$f(x)$	0 $\qquad\qquad$ $+\infty$

So: $\forall x \in]0, +\infty[: f(x) > 0$

$$\frac{a^3}{a^2 + bc} - \frac{a}{2} > 0, \frac{b^3}{b^2 + ca} - \frac{b}{2} > 0, \frac{c^3}{c^2 + ab} - \frac{c}{2} > 0$$

So: $\frac{a^3}{a^2+bc} - \frac{a}{2} + \frac{b^3}{b^2+ca} - \frac{b}{2} + \frac{c^3}{c^2+ab} - \frac{c}{2} > 0$

$$\frac{a^3}{a^2 + bc} + \frac{b^3}{b^2 + ca} + \frac{c^3}{c^2 - ab} > \frac{a + b + c}{2}$$

and the inequality is true when $a = b = c$.

Solution 3 by Jalil Hajimir-Canada

Lemma x, y, z, a, b ***and*** c ***are positive real numbers. Prove:***

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a + b + c)^3}{3(x + y + z)}$$

Proof:

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{\left(a^{\frac{3}{2}} + b^{\frac{3}{2}} + a^{\frac{3}{2}}\right)^2}{x + y + z} \geq \frac{(a + b + c)^3}{3(x + y + z)}$$

$$* \left(\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}} \geq \frac{a+b+c}{3} \leftrightarrow \left(a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}}\right)^2 \geq \frac{(a+b+c)^3}{3}$$

**** Power Mean***

Proof based on the lemma:

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ac}+\frac{c^3}{c^2+ba}\ge\frac{(a+b+c)^3}{a^2+bc+b^2+ac+c^2+ab}\ge$$

$$\ge\frac{(a+b+c)^3}{3(a+b+c)^2}=\frac{a+b+c}{3}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{a^3}{a^2+bc}=\sum\frac{a^4}{a^3+abc}\overset{Bergstrom}{\ge}\frac{(\sum a^2)^2}{\sum a^3+3abc}$$

$$\overset{?}{\ge}\frac{\sum a}{2}\Leftrightarrow 2\left(\sum a^2\right)^2\overset{?}{\ge}\left(\sum a\right)\left(\sum a^3+3abc\right)$$

$$\Leftrightarrow\sum a^4+4\sum a^2b^2\underset{(1)}{\overset{?}{\ge}}\sum a^3b+\sum ab^3+3abc\left(\sum a\right)$$

Now, $\sum a^2b^2\ge ab\cdot bc+bc\cdot ca+ca\cdot ab=abc(\sum a)$

$$\Rightarrow 3\sum a^2b^2\overset{(a)}{\ge}3abc\left(\sum a\right)$$

Also, $\frac{a^4+a^2b^2}{2}\underset{(b)}{\overset{A-G}{\ge}}a^3b,\frac{b^4+a^2b^2}{2}\underset{(c)}{\overset{A-G}{\ge}}ab^3,\frac{b^4+b^2c^2}{2}\underset{(d)}{\overset{A-G}{\ge}},\frac{c^4+b^2c^2}{2}\underset{(e)}{\overset{A-G}{\ge}}bc^3,$

$$\frac{c^4+c^2a^2}{2}\underset{(f)}{\overset{A-G}{\ge}}c^3a,\frac{a^4+c^2a^2}{2}\underset{(g)}{\overset{A-G}{\ge}}ca^3$$

(a)+(b) +(c) +(d) +(e) +(f) +(g)⇒(1) is true (proved)

Solution 5 by Tran Hong-Dong Thap-Vietnam

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}\ge\frac{a+b+c}{2}$$

$$\leftrightarrow 2\left(a^3(b^2+ca)(c^2+ab)+b^3(a^2+bc)(c^2+ab)+c^3(a^2+bc)(b^2+ca)\right)$$

$$\ge(a+b+c)(a^2+bc)(b^2+ca)(c^2+ab)$$

$$\leftrightarrow 2\left[abc(a^4+b^4+c^4)+a^2b^2c^2(a+b+c)+a^4b^3+b^4c^3+c^4a^3+a^3b^4+b^3c^4+c^3a^4\right]$$

$$\ge abc(a^4+b^4+c^4)+a^4b^3+b^4c^3+c^4a^3+a^3b^4$$

$$+b^3c^4+c^3a^4+2a^2b^2c^2(a+b+c)$$

$$+abc(ba^3+ac^3+cb^3)+abc(ab^3+bc^3+ca^3)$$

$$+abc(a^2b^2+b^2c^2+c^2a^2)$$

$$\leftrightarrow abc(a^4+b^4+c^4)+a^4b^3+b^4c^3+c^4a^3+a^3b^4+b^3c^4+c^3a^4$$

$$\ge abc(ba^3+ac^3+cb^3+ab^3+bc^3+ca^3)+abc(a^2b^2+b^2c^2+c^2a^2)$$

Because:

$$a^4+b^4+c^4\ge a^2b^2+b^2c^2+c^2a^2\to abc(a^4+b^4+c^4)\ge$$

$$\ge abc(a^2b^2+b^2c^2+c^2a^2)\ \textit{(*)}$$

$$a^4b^3+a^4b^3+c^3a^4\overset{AM-GM}{\ge}3\sqrt[3]{(a^4)^3(b^3)^2c^3}=3a^4b^2c=3abc\cdot ba^3\quad (1)$$

$$b^4c^3+b^4c^3+a^3b^4\overset{AM-GM}{\ge}3abc\cdot cb^3\quad (2)$$

$$c^4a^3+c^4a^3+b^3c^4\overset{AM-GM}{\ge}3abc\cdot ac^3\quad (3)$$

$$a^3b^4+a^3b^4+c^3b^4\overset{AM-GM}{\ge}3abc\cdot ab^3\quad (4)$$

$$b^3c^4+b^3c^4+c^4a^3\overset{AM-GM}{\ge}3abc\cdot bc^3\quad (5)$$

$$c^3a^4+c^3a^4+a^4b^3\overset{AM-GM}{\ge}3abc\cdot ca^3\quad (6)$$

(1)+(2)+(3) +(4) +(5) +(6)→

$$3(a^4b^3+b^4c^3+c^4a^3+a^3b^4+b^3c^4+c^3a^4)\ge$$

$$\ge 3abc(ba^3+ac^3+cb^3+ab^3+bc^3+ca^3)$$

$$\leftrightarrow a^4b^3+b^4c^3+c^4a^3+a^3b^4+b^3c^4+c^3a^4\ge$$

$$\ge abc(ba^3+ac^3+cb^3+ab^3+bc^3+ca^3)\ \textit{(**)}$$

$$\textit{(*)+(**)}\to abc(a^4+b^4+c^4)+a^4b^3+b^4c^3+c^4a^3+a^3b^4+b^3c^4+c^3a^4\ge$$

$$\ge abc(ba^3+ac^3+cb^3+ab^3+bc^3+ca^3)+abc(a^2b^2+b^2c^2+c^2a^2)$$

Proved. Equality if and only if $a=b=c$.

JP.232. Prove that in any ABC triangle the following relationship holds:

$$\sqrt{\frac{r_a}{r_br_c}}+\sqrt{\frac{r_b}{r_cr_a}}+\sqrt{\frac{r_c}{r_ar_b}}\ge\sqrt{\frac{3}{r}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{r_a}{\sqrt{r_ar_br_c}}=\frac{4R+r}{S\sqrt{r}}\overset{Trucht}{\ge}\frac{\sqrt{3}S}{S\sqrt{r}}=\sqrt{\frac{3}{r}}$$

Solution 2 by Mustafa Tarek-Cairo-Egypt

$$\sum_{cyc}\sqrt{\frac{r_a}{r_br_c}}\geq\sqrt{\frac{3}{r}}$$

First, we will prove the following inequality:

$$\sum_{cyc}\sqrt{\frac{x}{yz}}\geq\sqrt{3\sum_{cyc}\frac{1}{x}}\ \textit{where}\ x,y,z>0\quad (1)$$

$$(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

$$x^2+y^2+z^2\geq xy+yz+xz$$

$$(x+y+z)^2\geq 3(xy+yz+xz)$$

$$\frac{x+y+z}{\sqrt{xyz}}\geq\frac{\sqrt{3(xy+yz+xz)}}{\sqrt{xyz}}\Rightarrow\sum_{cyc}\sqrt{\frac{x}{yz}}\geq\sqrt{3\sum_{cyc}\frac{1}{x}}$$

(1) it is true

In (1) let $x=r_a, y=r_b, z=r_c$ and using the identity

$\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$ ***then:***

$$\sum_{cyc}\sqrt{\frac{r_a}{r_br_c}}\geq\sqrt{\frac{3}{r}}$$

Solution 3 by Jalil Hajimir-Canada

$$\sqrt{\frac{r_a}{r_br_c}}+\sqrt{\frac{r_b}{r_cr_a}}+\sqrt{\frac{r_c}{r_ar_b}}=\frac{r_a+r_b+r_c}{\sqrt{r_ar_br_c}}\geq\sqrt{3}\sqrt{\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}}=\sqrt{\frac{3}{r}}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$$

$$\frac{x+y+z}{\sqrt{3xyz}}\geq\sqrt{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}};\ x,y,z>0$$

Solution 4 by Bogdan Fustei-Romania

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\quad \textit{(Blundan Inequality)}$$

(RMM – Famous Inequalities Marathon 1 – 100)

$$\Rightarrow s^2\cdot\frac{2(2R-r)}{R}\le(4R+r)^2\le s^2\cdot\frac{(4R-2r)}{R}\le(4R+r)^2$$

$$s^2\left(4-\frac{2r}{R}\right)\le(4R+r)^2\Rightarrow s\sqrt{4-\frac{2r}{R}}\le 4R+r$$

$$\sum\sqrt{\frac{r_a}{r_br_c}}=\sum\sqrt{\frac{r_a^2}{r_ar_br_c}};r_ar_br_c=Ss=s\cdot r\cdot s=s^2r$$

$$\sum\sqrt{\frac{r_a}{r_br_c}}=\sum\frac{r_a}{s\sqrt{r}}=\frac{r_a+r_b+r_c}{s\sqrt{r}}\ge\frac{s\sqrt{4-\frac{2r}{R}}}{s\sqrt{r}}=\sqrt{\frac{4}{R}-\frac{2}{R}}$$

Now, we will prove that $\frac{4}{r}-\frac{2}{R}\ge\frac{3}{r}\Rightarrow\frac{4}{r}-\frac{3}{r}\ge\frac{2}{R}\Rightarrow\frac{1}{r}\ge\frac{2}{R}\Rightarrow R\ge 2r$ ***(Euler)***

So, finally we have the following: $\sum\sqrt{\frac{r_a}{r_br_c}}\ge\sqrt{\frac{4}{r}-\frac{2}{R}}\ge\sqrt{\frac{3}{r}}$ ***Q.E.D.***

JP.233. Find the maximum and minimum possible value of:

$$\frac{1}{\sin^4 x+\cos^2 x}+\frac{1}{\cos^4 x+\sin^2 x}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Sohini Mondal-India

$$\frac{1}{\sin^4 x+\cos^2 x}+\frac{1}{\cos^4 x+\sin^2 x}$$

We know, $\sin^4 x+\cos^2 x=\sin^4 x+1-\sin^2 x=1-\sin^2 x(1-\sin^2 x)$

$$=1-\sin^2 x\cos^2 x=1-\frac{1}{4}\sin^2 2x$$

and $\cos^4 x+\sin^2 x=1-\cos^2 x+\cos^4 x=1-\cos^2 x(1-\cos^2 x)$

$$=1-\cos^2 x\sin^2 x=1-\frac{1}{4}\sin^2 2x$$

$$\therefore\frac{1}{\sin^4 x+\cos^2 x}+\frac{1}{\cos^4 x+\sin^2 x}=\frac{2}{1-\frac{1}{4}\sin^2 2x}$$

$$\because\frac{2}{1}\le\frac{2}{1-\frac{1}{4}\sin^2 2x}\le\frac{2}{1-\frac{1}{4}}\Rightarrow 2\le\frac{2}{1-\frac{1}{4}\sin^2 2x}\le\frac{8}{3}$$

$\therefore$ ***min value*** $= 2$; ***max value*** $= \frac{8}{3}$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\textit{Suppose } f(x) = \frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^4 x}$$

$$f(x) = \frac{1}{\sin^4 x + 1 - \sin^2 x} + \frac{1}{\cos^4 x + 1 - \cos^2 x}$$

$$f(x) = \frac{1}{\sin^4 x - \sin^2 x + 1} + \frac{1}{\cos^4 x - \cos^2 x + 1}$$

$$\textit{Suppose: } t = \sin^2 x, \cos^2 x = 1 - t, 0 \le t \le 1$$

$$f(x) = \frac{1}{t^2 - t + 1} + \frac{1}{t^2 - t + 1} = \frac{2}{t^2 - t + 1}$$

$$f(0) = 2, f(1) = 2, f'(x) = \frac{-2(2t-1)\cdot 1}{(t^2 - t + 1)^2}, f'(x) = 0 \Rightarrow t = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = \frac{2}{\frac{1}{4} - \frac{1}{2} + 1} = \frac{8}{3}$$

x	0	$\frac{1}{2}$	1
$f'(x)$	+ + + + + + + + + +		− − − − − − − − −
$f(x)$	2 ↗	$\frac{8}{3}$	↘ 2

$$\max(f(x)) = \frac{8}{3}, \min(f(x)) = 2$$

Solution 3 by Sudhir Jha-Kolkata-India

$$\Leftrightarrow \frac{1}{\sin^2 x + \cos^4 x} + \frac{1}{\sin^4 x + \cos^2 x} =$$

$$= \frac{1}{1 - \cos^2 x(1 - \cos^2 x)} + \frac{1}{1 - \sin^2 x(1 - \sin^2 x)}$$

$$= \frac{2}{1 - \sin^2 x \cdot \cos^2 x} = \frac{1}{1 - \frac{1}{4}(\sin 2x)^2}$$

$$\Leftrightarrow 0 \le (\sin 2x)^2 \le 1 \Rightarrow 0 \le \frac{1}{4}(\sin 2x)^2 \le \frac{1}{4} \Rightarrow 1 \ge 1 - \frac{1}{4}(\sin 2x)^2 \ge 1 - \frac{1}{4}$$

$$\Rightarrow 2 \le \frac{2}{1 - \frac{1}{4}(\sin 2x)^2} \le \frac{8}{3} \Leftrightarrow \max = \frac{8}{3}, \min = 2 \textit{ (Answer)}$$

Solution 4 by Nani Gopal Saha-India

$$T=\frac{1}{\sin^4 x+\cos^2 x}+\frac{1}{\cos^4 x+\sin^2 x}=\frac{1}{\sin^4 x-\sin^2 x+1}+\frac{1}{\sin^4 x-\sin^2 x+1}$$

$$=\frac{2}{\sin^4 x-\sin^2 x+1}=\frac{2}{\left(\sin^2 x-\frac{1}{2}\right)^2+1-\frac{3}{4}}=\frac{2}{\left(\sin^2 x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

Clearly T will be maximum when $\sin^2 x=\frac{1}{2}\Rightarrow$ *i.e.* $\sin^2 x=\frac{1}{2}\therefore T_{\max}=\frac{2}{\frac{3}{4}}=\frac{8}{3}$

and T will be minimum when $\sin^2 x=1$

$$T_{\min}=\frac{2}{\left(\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{2}{\frac{1}{4}+\frac{3}{4}}=2$$

JP.234. Let x,y be positive real numbers such that $x+y\le 1$. Prove that:

$$\left(1-\frac{1}{x^4}\right)\left(1-\frac{1}{y^4}\right)\ge 225$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\because x,y>0 \text{ and } x+y\le 1,\therefore x,y<1$$

$$\Rightarrow x^4,y^4<1\Rightarrow (x^4-1),(y^4-1)<0\Rightarrow (x^4-1)(y^4-1)>0$$

$$\therefore \text{proposed inequality}\Leftrightarrow (x^4-1)(y^4-1)\ge 225x^4y^4$$

$$\Leftrightarrow 1\ge 225x^4y^4-x^4y^4+x^4+y^4\Leftrightarrow 1\overset{(1)}{\ge} 224x^4y^4+x^4+y^4$$

$$\because 1\ge x+y\therefore \text{LHS of (1)}\ge (x+y)^4\overset{?}{\ge} 224x^4y^4+x^4+y^4$$

$$\Leftrightarrow 2x^3y+2xy^3+3x^2y^2\underset{(2)}{\overset{?}{\ge}} 112x^4y^4$$

Now, $1\ge x+y\overset{A-G}{\ge}2\sqrt{xy}\Rightarrow 1\ge 4xy\Rightarrow 1\ge 16x^2y^2\Rightarrow x^2y^2\le\frac{1}{16}$

$$\therefore 112x^4y^4=112x^2y^2(x^2y^2)\le 112x^2y^2\left(\frac{1}{16}\right)=7x^2y^2\overset{?}{\le}2x^3y+2xy^3+3x^2y^2$$

$\Leftrightarrow x^3y+xy^3-2x^2y^2\overset{?}{\ge}0\Leftrightarrow xy(x-y)^2\overset{?}{\ge}0\to$ *true* $\therefore$ *(2)* $\Rightarrow$ *(1)* $\Rightarrow$ *proposed inequality is true (Proved)*

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y > 0$ *and* $x + y \leq 1$, *we have:* $\frac{1}{x} + \frac{1}{y} \leq \frac{1}{xy}$ *and* $\frac{1}{x} + \frac{1}{y} \geq 4$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{2}{xy} \leq \frac{1}{(xy)^2} \text{ and } \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) \geq 1$$

Consider $\left(1 - \frac{1}{x^4}\right)\left(1 - \frac{1}{y^4}\right) = \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{x^2}\right)\left(1 + \frac{1}{y^2}\right)$

$$= \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right)\left(1 + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{(xy)^2}\right)$$

$$\geq \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{x} + \frac{1}{y}\right)\left(1 + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{2}{xy}\right)$$

$$\geq \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 + 2\left(\frac{1}{x} + \frac{1}{y}\right)\right)\left(1 + \frac{\left(\frac{1}{x} + \frac{1}{y}\right)^2}{2} + \left(\frac{1}{x} + \frac{1}{y}\right)^2\right)$$

$$\geq \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)(1 + 8)(1 + 8 + 16) = \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) 9 \times 25$$

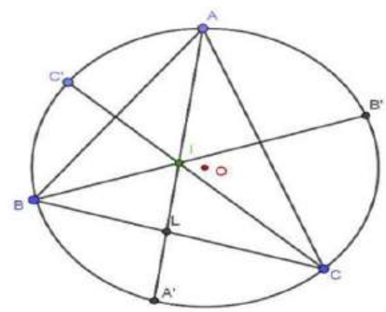
$\geq 1 \times 225 = 225$ *ok. Therefore, it is true.*

JP.235. In ΔABC; I – incenter; A', B', C' - lies on circumcircle such that: (A, I, A'); (B, I, B'); (C, I, C') are collinear. Prove that:

$$\frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} \geq \frac{a + b + c}{R}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam



Using Ptolemy's theorem with Cyclic quadrilateral $ABA'C$:

$$AA'\cdot BC = AB\cdot CA' + AC\cdot BA' \leftrightarrow AA'\cdot a = c\cdot CA' + b\cdot BA'$$

Because $\widehat{BAA'} = \widehat{CAA'} \to A'B = A'C \to AA'\cdot a = (b+c)A'B$

But: $\frac{A'B}{\sin\widehat{BAA'}} = 2R \to A'B = 2R\sin\widehat{BAA'} = 2R\sin\frac{A}{2}$***. So,*** $AA' = \frac{b+c}{a}\cdot 2R\sin\frac{A}{2} = \frac{b+c}{2\cos\frac{A}{2}}$

$$\to IA' = AA' - AI = \frac{b+c}{2\cos\frac{A}{2}} - \frac{b+c}{2s}\cdot\frac{2bc}{b+c}\cdot\cos\frac{A}{2} = \frac{b+c}{2\cos\frac{A}{2}} - \frac{bc}{s}\cos\frac{A}{2}$$

$$= \frac{s(b+c) - 2bc\cos^2\frac{A}{2}}{2s\cos\frac{A}{2}} = \frac{s(b+c) - 2bc\cdot\frac{s(s-a)}{bc}}{2s\cos\frac{A}{2}}$$

$$= \frac{b+c-2(s-a)}{2\cos\frac{A}{2}} = \frac{b+c-(b+c-a)}{2\cos\frac{A}{2}} = \frac{a}{2\cos\frac{A}{2}} \to \frac{a}{IA'} = 2\cos\frac{A}{2}$$

Similarly: $\frac{b}{IB'} = 2\cos\frac{B}{2}; \frac{c}{IC'} = 2\cos\frac{C}{2}$

$$\to \frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} = 2\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \overset{AM-GM}{\geq} 6\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 6\sqrt[3]{\frac{s}{4R}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let's us join AIA' ***and*** BA'. $\because$ ***angles*** $AA'B$ ***and*** ACB ***are angles on the same arc*** $AC'B$ ***and the same side of it,*** $\because \angle AA'B = \angle ACB = C$

$\therefore$ ***in*** $\Delta AA'B$, $\angle ABA' = 180^\circ - \left(\frac{A}{2} + C\right) = A + B - \frac{A}{2} = \frac{A+2B}{2} = \frac{B+(180^\circ - C)}{2}$

$$= 90^\circ + \frac{B-C}{2} \therefore \angle ABA' \overset{(1)}{=} 90^\circ + \frac{B-C}{2}$$

$\because$ ***the circumcircle of*** ΔABC ***coincides with the circumcircle of*** $\Delta AA'B$

$\therefore$ ***circumradius of*** $\Delta AA'B \overset{(2)}{=} R$

(1), (2) $\Rightarrow AA' \overset{sine\ rule}{=} 2R\sin\left(90^\circ + \frac{B-C}{2}\right) = 2R\cos\frac{B-C}{2}$

$$\Rightarrow IA' = AA' - IA = 2R\cos\frac{B-C}{2} - \frac{r}{\sin\frac{A}{2}} = 2R\cos\frac{B-C}{2} - \frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 2R\cos\frac{B-C}{2} - 2R\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = 2R\sin\frac{A}{2}$$

$$\Rightarrow \frac{a}{IA'} = \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{2R\sin\frac{A}{2}} = 2\cos\frac{A}{2} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}$$

$$\geq 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\left(\because 0 < \cos\frac{B-C}{2} \leq 1 \because -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2}\right)$$

$= \sin B + \sin C = \frac{b+c}{2R} \Rightarrow \frac{a}{IA'} \overset{(a)}{\geq} \frac{b+c}{2R}$. ***Similarly,*** $\frac{b}{IB'} \overset{(b)}{\geq} \frac{c+a}{2R}$ ***and*** $\frac{c}{IC'} \overset{(c)}{\geq} \frac{a+b}{2R}$

(a)+(b)+(c)$\Rightarrow \frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} \geq \frac{2(a+b+c)}{2R} = \frac{a+b+c}{R}$ ***(Proved)***

Solution 3 by Marian Dinca-Romania

$$IA\cdot IA' = IB\cdot IB' = IC\cdot IC' = R^2 - OI^2 = 2Rr$$

$$\frac{r}{IA} = \sin\frac{A}{2} \Rightarrow IA' = \frac{2Rr}{IA} = 2R\sin\frac{A}{2}$$

$$\frac{r}{IB} = \sin\frac{B}{2} \Rightarrow IB' = \frac{2Rr}{IB} = 2R\sin\frac{B}{2}$$

$$\frac{r}{IC} = \sin\frac{C}{2} \Rightarrow IC' = \frac{2Rr}{IC} = 2R\sin\frac{C}{2}$$

$$\frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} = \frac{2R\sin A}{2R\sin\frac{A}{2}} + \frac{2R\sin B}{2R\sin\frac{B}{2}} + \frac{2R\sin C}{2R\sin\frac{C}{2}} = 2\cos\frac{A}{2} + 2\cos\frac{B}{2} + 2\cos\frac{C}{2}$$

and: $\frac{a+b+c}{R} = 2(\sin A + \sin B + \sin C) = 4\left(\sin\frac{A}{2}\cdot\cos\frac{A}{2} + \sin\frac{B}{2}\cdot\cos\frac{B}{2} + \sin\frac{C}{2}\cdot\cos\frac{C}{2}\right) \leq$

$$\leq 4\cdot\frac{1}{3}\left(\sin\frac{A}{2} + \sin\frac{A}{2} + \sin\frac{A}{2}\right)\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \leq$$

$$\leq 4\sin\left(\frac{A+B+C}{6}\right)\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) =$$

$= 2\cos\frac{A}{2} + 2\cos\frac{B}{2} + 2\cos\frac{C}{2}$. ***Using Chebyshev and Jensen's inequality. We must show that:*** $6\sqrt[3]{\frac{s}{4R}} \geq \frac{a+b+c}{R} = \frac{2s}{R} \leftrightarrow 3\sqrt[3]{\frac{s}{4R}} \geq \frac{s}{R} \leftrightarrow 27\cdot\frac{s}{4R} \geq \left(\frac{s}{R}\right)^3 \leftrightarrow \frac{27}{4} \geq \left(\frac{s}{R}\right)^2$. ***It is true because:***

$s \leq \frac{3\sqrt{3}}{2}R \to \left(\frac{s}{R}\right)^2 \leq \left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4}$. ***Proved.***

JP.236. If $a, b, c, x, y, z > 0; a + b + c = 3$ then:

$$a^a\cdot b^b\cdot c^c\cdot(x+y+z)^3 \geq 27x^a y^b z^c$$

When does the equality holds?

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $(x+y+z)^3 \geq 27\left(\frac{x}{a}\right)^a \cdot \left(\frac{y}{b}\right)^b \cdot \left(\frac{z}{c}\right)^c \Leftrightarrow$

$$\frac{x+y+z}{3} \geq \left(\frac{x}{a}\right)^{\frac{a}{3}}\left(\frac{y}{b}\right)^{\frac{b}{3}}\left(\frac{z}{c}\right)^{\frac{c}{3}} \quad (1)$$

From inequality of generalized environments $\Rightarrow$

$\left(\frac{x}{a}\right)^{\frac{a}{3}}\left(\frac{y}{b}\right)^{\frac{b}{3}}\left(\frac{z}{c}\right)^{\frac{c}{3}} \leq \frac{a}{3}\cdot\frac{x}{a}+\frac{b}{3}\cdot\frac{y}{b}+\frac{c}{3}\cdot\frac{z}{c}=\frac{x+y+z}{3} \Rightarrow$ *(1) it is true, equality* $a=b=c=1$ *and* $x=y=z.$

Solution 2 by Sudhir Jha-Kolkata-India

Considering $\frac{x}{a},\frac{y}{b},\frac{z}{c}$ *with associated weights* a,b,c *respectively, we get, by* $AM \geq GM$

$$\frac{a\cdot\left(\frac{x}{a}\right)+b\left(\frac{y}{b}\right)+c\left(\frac{z}{c}\right)}{a+b+c} \geq \left[\left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c\right]^{\frac{1}{a+b+c}}$$

$$\Rightarrow \frac{x+y+z}{3} \geq \left[\left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c\right]^{\frac{1}{3}} \quad (\because a+b+c=3)$$

$$\Rightarrow \frac{(x+y+z)^3}{27} \geq \frac{x^ay^bz^c}{a^ab^bc^c} \Rightarrow a^ab^bc^c(x+y+z)^3 \geq 27x^ay^bz^c$$

(Proved)

Solution 3 by Daniel Văcaru – Romania

We have:

$$\left(\frac{x}{a}\right)^{\left(\frac{a}{a+b+c}\right)}\cdot\left(\frac{y}{b}\right)^{\left(\frac{b}{a+b+c}\right)}\cdot\left(\frac{z}{c}\right)^{\left(\frac{c}{a+b+c}\right)} \leq \left(\frac{a}{a+b+c}\right)\cdot\left(\frac{x}{a}\right)+\left(\frac{b}{a+b+c}\right)\cdot\left(\frac{y}{b}\right)+$$

$$+\left(\frac{c}{a+b+c}\right)\cdot\left(\frac{z}{c}\right)=\frac{x+y+z}{3} \Rightarrow$$

$$\left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c \leq \left(\frac{x+y+z}{3}\right)^3 \Rightarrow 27x^ay^bz^c \leq a^ab^bc^c(x+y+z)^3$$

JP.237. If $a,b,c,x,y,z>0; a+b+c \geq x+y+z$ **then:**

$a^a b^b c^c \geq x^a y^b z^c$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

$$a^a b^b c^c \geq x^a y^b z^c \Leftrightarrow a\ln a + b\ln b + c\ln c \geq a\ln x + b\ln y + c\ln z \Leftrightarrow$$

$$\Leftrightarrow a\ln\frac{a}{x} + b\ln\frac{b}{y} + c\ln\frac{c}{z} \geq 0 \Leftrightarrow \ln\left(\frac{a}{x}\right)^a + \ln\left(\frac{b}{y}\right)^b + \ln\left(\frac{c}{z}\right)^c \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \ln\left(\frac{a}{x}\right)^a\left(\frac{b}{y}\right)^b\left(\frac{c}{z}\right)^c \geq 0 \Leftrightarrow \left(\frac{a}{x}\right)^a\left(\frac{b}{y}\right)^b\left(\frac{c}{z}\right)^c \geq 1 \text{ true, because: by weighted GM-HM} \Rightarrow$$

$$\left(\frac{a}{x}\right)^a\left(\frac{b}{y}\right)^b\left(\frac{c}{z}\right)^c \geq \left(\frac{a+b+c}{\frac{a}{\frac{a}{x}} + \frac{b}{\frac{b}{y}} + \frac{c}{\frac{c}{z}}}\right)^{a+b+c} = \left(\frac{a+b+c}{x+y+z}\right)^{a+b+c} \geq 1^{a+b+c} = 1$$

Solution 2 by Sudhir Jha-Kolkata-India

Considering $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ *with associated weights* a, b, c *and applying weighted* $GM \leq$ *weighted* AM, *we get*

$$\left(\left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c\right)^{\frac{1}{a+b+c}} \leq \frac{a\left(\frac{x}{a}\right) + b\left(\frac{y}{b}\right) + c\left(\frac{z}{c}\right)}{a+b+c} \Rightarrow$$

$$\Rightarrow \left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c \leq \left(\frac{x+y+z}{a+b+c}\right)^{a+b+c}$$

$$\Rightarrow \left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c \leq \cdots (\because a+b+c \geq x+y+z)$$

$$\Rightarrow a^a b^b c^c \geq x^a y^b z^c \text{ (proved)}$$

Equality holds for $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 1$

Solution 3 by Michael Sterghiou-Greece

$$a^a b^b c^c \geq x^a y^b z^c \quad (1)$$

(1)→ $\left(\frac{x}{a}\right)^a \cdot \left(\frac{y}{b}\right)^b \cdot \left(\frac{z}{c}\right)^c \leq 1$ *or* $\sum_{cyc} a\ln\frac{x}{a} \leq 0$. *As* $f(t) = \ln t$ *is concave*

$\left(f''(t) = -\frac{1}{t^2} < 0\right)$ *using generalized Jensen we get:*

$\sum_{cyc} a\ln\frac{x}{a} \le \left(\sum_{cyc} a\right)\cdot\ln\left(\frac{\sum_{cyc} a\cdot\frac{x}{a}}{\sum_{cyc} a}\right) = \left(\sum_{cyc} a\right)\cdot\ln\frac{x+y+z}{a+b+c} \le 0$ ***as*** $x+y+z \le a+b+c$.

Equality when $x = a, y = b, z = c$.

JP.238. If $a,b,c,d,x,y,z,t > 0$ **then:**

$$\frac{(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d}{(a+b+c+d)^{a+b+c+d}} \ge \left(\frac{xyzt}{xyz+xyt+xzt+yzt}\right)^{a+b+c+d}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $(ax)^a(by)^b(cz)^c(dt)^d \ge \left(\frac{a+b+c+d}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}}\right)^{a+b+c+d}$ ***(1)***

Let $\frac{1}{x} = m, \frac{1}{y} = n, \frac{1}{z} = p, \frac{1}{t} = q$ ***(2)***

From (1)+(2) we must show: $\left(\frac{a}{m}\right)^a\cdot\left(\frac{b}{n}\right)^b\cdot\left(\frac{c}{p}\right)^c\cdot\left(\frac{d}{q}\right)^d \ge \left(\frac{a+b+c+d}{m+n+p+q}\right)^{a+b+c+d} \Leftrightarrow$

$\Leftrightarrow a\ln\left(\frac{a}{m}\right) + b\ln\left(\frac{b}{n}\right) + c\ln\left(\frac{c}{p}\right) + d\ln\left(\frac{d}{q}\right)^d \ge (a+b+c+d)\ln\left(\frac{a+b+c+d}{m+n+p+q}\right)$***, which it is true, because it is Gibbs inequality.***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Applying Weighted GM≥ Weighted A.M:

$$\sqrt[a+b+c+d]{(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d} \ge \frac{a+b+c+d}{\frac{a}{ax}+\frac{b}{by}+\frac{c}{cz}+\frac{d}{dt}}$$

$$\Rightarrow \sqrt[a+b+c+d]{(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d} \ge \frac{a+b+c+d}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}}$$

$$\Rightarrow \frac{(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d}{(a+b+c+d)^{a+b+c+d}} \ge \left(\frac{xyzt}{xyz+xyt+xzt+yzt}\right)^{a+b+c+d} \quad \textit{(proved)}$$

Equality at $ax = by = cz = dt$.

JP.239. In ΔABC **the following relationship holds:**

$$4(m_a+m_b+m_c)\le 3\left(\frac{r_a}{\cos^2\frac{A}{2}}+\frac{r_b}{\cos^2\frac{B}{2}}+\frac{r_c}{\cos^2\frac{C}{2}}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$r_a=s\tan\frac{A}{2};r_b=s\tan\frac{B}{2};r_c=s\tan\frac{C}{2}$$

$$r_a+r_b+r_c=4R+r\ge m_a+m_b+m_c$$

$$\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}=\frac{s^2+(4R+r)^2}{s^2}$$

Suppose: $A\ge B\ge C\to\begin{cases}r_a\ge r_b\ge r_c\\ \cos\frac{A}{2}\le\cos\frac{B}{2}\le\cos\frac{C}{2}\end{cases}\to\begin{cases}r_a\ge r_b\ge r_c\\ \frac{1}{\cos^2\frac{A}{2}}\ge\frac{1}{\cos^2\frac{B}{2}}\ge\frac{1}{\cos^2\frac{C}{2}}\end{cases}$

By Chebyshev's inequality we have: RHS $\ge 3\cdot\frac{1}{3}(r_a+r_b+r_c)\left(\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}\right)$

$$\ge 3\cdot\frac{1}{3}(m_a+m_b+m_c)\cdot\frac{s^2+(4R+r)^2}{s^2}=(m_a+m_b+m_c)\cdot\frac{s^2+(4R+r)^2}{s^2}$$

We must show that: $(m_a+m_b+m_c)\cdot\frac{s^2+(4R+r)^2}{s^2}\ge 4(m_a+m_b+m_c)$

$\leftrightarrow s^2+(4R+r)^2\ge 4s^2\leftrightarrow(4R+r)^2\ge 3s^2\leftrightarrow 4R+r\ge\sqrt{3}s$ ***(true) (Proved)***

Solution 2 by Soumava Chakraborty-Kolkata-India

Firstly, $\sum\sec^2\frac{A}{2}=\sum\frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)}=\frac{\sum bc(s^2-s(2s-a)+bc)}{r^2s^2}=\frac{-s^2\sum ab+3sabc+(\sum ab)^2-2abc(2s)}{r^2s^2}$

$$=\frac{(s^2+4Rr+r^2)(4Rr+r^2)-4Rrs^2}{r^2s^2}=\frac{s^2r^2+r^2(4R+r)^2}{r^2s^2}\overset{(1)}{=}\frac{s^2+(4R+r)^2}{s^2}$$

Now, $3\sum\frac{r_a}{\cos^2\frac{A}{2}}=3\sum\left[\frac{rs}{s-a}\cdot\frac{bc}{s(s-a)}\right]=\frac{3r}{s}\sum\frac{bc(s-a+a)}{(s-a)^2}$

$$=3r\sum\frac{bc}{s(s-a)}+\frac{3r}{s}\cdot\frac{4Rrs}{s^2}\sum\frac{s^2}{(s-a)}=3r\sum\sec^2\frac{A}{2}+\frac{12R}{s^2}\sum\frac{r^2s^2}{(s-a)^2}$$

$$\overset{by\,(1)}{=}3r\left[\frac{s^2+(4R+r)^2}{s^2}\right]+\frac{12R}{s^2}\sum r_a^2=3r+\frac{3r(4R+r)^2}{s^2}+\frac{12R}{s^2}[(4R+r)^2-2s^2]$$

$$= 3r - 24R + \frac{3(4R+r)^3}{s^2} \Rightarrow 3\sum \frac{r_a}{\cos^2\frac{A}{2}} \overset{(2)}{=} 3r - 24R + \frac{3(4R+r)^3}{s^2}$$

Also, $4\sum m_a \le 4\sum r_a = 4(4R+r) \overset{?}{\le} 3\sum \frac{r_a}{\cos^2\frac{A}{2}} \overset{by\ (2)}{=} 3r - 24R + \frac{3(4R+r)^3}{s^2}$

$$\Leftrightarrow (40R+r)s^2 \underset{(3)}{\overset{?}{\le}} 3(4R+r)^3$$

Now, LHS of (3) $\overset{Geretsen}{\le} (40R+r)(4R^2+4Rr+3r^2) \overset{?}{\le} 3(4R+r)^3$

$\Leftrightarrow 8R^2 - 5Rr - 22r^2 \overset{?}{\ge} 0 \Leftrightarrow (8R+11r)(R-2r) \overset{?}{\ge} 0 \rightarrow$ *true* $\because R \overset{Euler}{\ge} 2r$

$\Rightarrow$ *(3)* $\Rightarrow$ *proposed inequality is true (Proved)*

JP.240. In ΔABC the following relationship holds:

$$3(m_a + m_b + m_c) \le r_a\cot^2\frac{A}{2} + r_b\cot^2\frac{B}{2} + r_c\cot^2\frac{C}{2}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\left.\begin{array}{c}\sum r_a\cot^2\frac{A}{2} = \sum \frac{s}{s-a}\cdot\frac{s(s-a)}{(s-b)(s-c)} = \sum \frac{Ss}{(s-b)(s-c)} \\ but \sum \frac{1}{(s-b)(s-c)} = \frac{1}{r^2}\end{array}\right\} \Rightarrow$$

$\sum r_a\cot^2\frac{A}{2} = \frac{s^2}{r}$ ***(1)*** $\Rightarrow$ ***we must show:*** $3(m_a+m_b+m_c) \le \frac{s^2}{r}$ ***(2)***

But $\frac{m_a}{h_a} \le \frac{R}{2r} \Rightarrow m_a \le \frac{R}{2r}h_a$ ***(3). From (2)+(3) we must show:***

$$\frac{3R}{2r}(h_a+h_b+h_c) \le \frac{s^2}{r} \Leftrightarrow \frac{3R}{2}(h_a+h_b+h_c) \le s^2 \quad (4)$$

But $h_a+h_b+h_c = \frac{s^2+r^2+4Rr}{2R}$ ***(5). From (4)+(5) we must show:***

$$\frac{3}{4}(s^2+r^2+4Rr) \le s^2 \Leftrightarrow 3s^2+3r^2+12Rr \le 4s^2 \Leftrightarrow$$

$$s^2 \ge 12Rr + 3r^2 \quad (6)$$

From Gerretsen's inequality: $s^2 \ge 16Rr - 5r^2$ ***(7)***

From (6)+(7) we must show:

$$16Rr-5r^2 \ge 12Rr+3r^2 \Leftrightarrow 4Rr \ge 8r^2 \Leftrightarrow R \ge 2r \text{ true (Euler)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$r_a = s\tan\frac{A}{2}; r_b = s\tan\frac{B}{2}; r_c = s\tan\frac{C}{2}$$

$$\to RHS = r_a\cdot\cot^2\frac{A}{2}$$

$$+r_b\cdot\cot^2\frac{B}{2}+r_c\cdot\cot^2\frac{C}{2}$$

$$= s\tan\frac{A}{2}\cdot\cot^2\frac{A}{2}+s\tan\frac{B}{2}\cdot\cot^2\frac{B}{2}+s\tan\frac{C}{2}\cdot\cot^2\frac{C}{2}$$

$$= s\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right) = s\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = s\cdot\frac{s}{r} = \frac{s^2}{r}$$

But: $m_a+m_b+m_c \overset{BCS}{\le} \sqrt{3(m_a^2+m_b^2+m_c^2)} = \sqrt{\frac{9}{4}(a^2+b^2+c^2)} =$

$$\frac{3}{2}\sqrt{(a^2+b^2+c^2)} \overset{Leibniz}{\le} \frac{3}{2}\cdot\sqrt{9R^2} = \frac{9R}{2} \to LHS = 3(m_a+m_b+m_c) \le \frac{27R}{2}$$

We must show that: $\frac{27R}{2} \le \frac{s^2}{r} \Leftrightarrow 2s^2 \ge 27Rr$

But: $s^2 \ge 16Rr-5r^2$. *We need to prove:* $2(16Rr-5r^2) \ge 27Rr$

$\leftrightarrow 5Rr \ge 10r^2 \leftrightarrow R \ge 2r$ *(Euler → true) Proved.*

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum r_a\cot^2\frac{A}{2} = \sum\left(s\tan\frac{A}{2}\cot^2\frac{A}{2}\right) = s\sum\cot\frac{A}{2} = s\sum\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= s\sum\sqrt{\frac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}} = \frac{s^2}{rs}\sum(s-a) \overset{(1)}{=} \frac{s^2}{r}$$

Now, Chu and Yang $\Rightarrow 3\sum m_a \le 3\sqrt{4s^2-16Rr+5r^2} \overset{?}{\le} \sum r_a\cot^2\frac{A}{2} \overset{by\,(1)}{=} \frac{s^2}{r}$

$$\Leftrightarrow s^4 \underset{(2)}{\overset{?}{\ge}} 9r^2(4s^2-16Rr+5r^2)$$

Now, LHS of (2) $\overset{Gerretsen}{\ge} (16Rr-5r^2)s^2 \overset{?}{\ge} 9r^2(4s^2-16Rr+5r^2)$

$$\Leftrightarrow (16R-41r)s^2+9r^2(16R-5r) \overset{?}{\ge} 0$$

$$\Leftrightarrow (16R - 32r)s^2 + 9r^2(16R - 5r) \underset{(3)}{\overset{?}{\geq}} 9rs^2$$

Now, LHS of (3) $\underset{(a)}{\overset{Gerretsen}{\geq}} (16R - 32r)(16Rr - 5r^2) + 9r^2(16R - 5r)$

and, RHS of (3) $\underset{(b)}{\overset{Gerretsen}{\leq}} 9r(4R^2 + 4Rr + 3r^2)$

(a), (b) $\Rightarrow$ ***in order to prove (3), it suffices to prove:***

$$(16R - 32r)(16Rr - 5r^2) + 9r^2(16R - 5r) - 9r(4R^2 + 4Rr + 3r^2) \geq 0$$

$\Leftrightarrow 55R^2 - 121Rr + 22r^2 \geq 0 \Leftrightarrow (R - 2r)(55R - 11r) \geq 0 \rightarrow$ ***true*** $\because R \overset{Euler}{\geq} 2r$

$\Rightarrow$ ***(3)*** $\Rightarrow$ ***(2)*** $\Rightarrow$ ***proposed inequality is true (proved)***

Solution 4 by Bogdan Fustei-Romania

$\sin\frac{A}{2} = \sqrt{\frac{rr_a}{bc}}$ *(and the analogs);* $\cos\frac{A}{2} = \sqrt{\frac{r_br_c}{bc}}$ *(and the analogs)*

$\cot^2\frac{A}{2} = \frac{r_br_c}{bc} \cdot \frac{bc}{rr_a} = \frac{r_br_c}{rr_a}$ *(and the analogs)* $\Rightarrow r_a\cot^2\frac{A}{2} = r_a \cdot \frac{r_br_c}{rr_a} = \frac{r_br_c}{r}$ *(and the analogs)*

$$r_ar_b + r_br_c + r_ar_c = s^2 \Rightarrow \sum r_a\cot^2\frac{A}{2} = \frac{s^2}{r}$$

$s^2 \geq 16R - 5r^2$ *(Gerretsen's Inequality)*

$$16Rr - 5r^2 \geq 3r(4R + r) = 12Rr + 3r^2 \Rightarrow 16Rr - 12Rr \geq 5r^2 + 3r^2 = 8r^2$$

$4Rr \geq 8r^2 \Rightarrow R \geq 2r$ ***(Euler). So, we will have the inequality:***

$s^2 \geq 3r(4R + r) \Rightarrow \frac{s^2}{r} \geq 3(4R + r) \Leftrightarrow \sum r_a\cot^2\frac{A}{2} \geq 3(4R + r)$ ***(1)***

$4R + r \geq m_a + m_b + m_c \Rightarrow 3(4R + r) \geq 3(m_a + m_b + m_c)$ ***(2)***

From (1) and (2) $\Rightarrow 3(m_a + m_b + m_c) \leq \sum r_a\cot^2\frac{A}{2}$***. Q.E.D.***

SP.226. If $a, b > 0$ then:

$$\left(\sqrt{ab} - \frac{a+b}{2}\right)\arctan\left(\frac{2ab}{a+b}\right) + \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\arctan(\sqrt{ab}) +$$
$$+\left(\frac{2ab}{a+b} - \sqrt{ab}\right)\arctan\left(\frac{a+b}{2}\right) \geq 0$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Marian Ursărescu-Romania

Let $\frac{2ab}{a+b}=x,\sqrt{ab}=y$, *and* $\frac{a+b}{2}=z\Rightarrow 0<x<y<z$

Let $f:(0,+\infty)\to\mathbb{R}; f(x)=\arctan x$. *We must show:*

$$(y-z)f(x)+(z-x)f(y)+(x-y)f(z)\geq 0 \quad (1)$$

Theorem: Let $f:I\to\mathbb{R}, I\subset\mathbb{R}, I=$ *interval*

f *its convex* $\Leftrightarrow\forall x,y,z\in I$ *with* $x<y<z$ *we have:* $\frac{f(x)-f(y)}{x-y}\leq\frac{f(y)-f(z)}{y-z}$

f *it is concave* $\Leftrightarrow\frac{f(x)-f(y)}{x-y}\geq\frac{f(y)-f(z)}{y-z}$

In our case f *it is concave (for* $(0,+\infty)$*)* $\Rightarrow\frac{f(x)-f(y)}{x-y}\geq\frac{f(y)-f(z)}{y-z}\Big|\cdot(x-y)(y-z)>0\Rightarrow$

$$\Rightarrow(y-z)\big(f(x)-f(y)\big)\geq(x-y)\big(f(y)-f(z)\big)\Leftrightarrow$$

$$\Leftrightarrow(y-z)f(x)+(z-x)f(y)+(x-y)f(z)\geq 0\Rightarrow$$ *(1) it's true.*

Solution 2 by Soumitra Mandal-Chandar Nagore-India

According to AM $\geq GM\geq HM$ *we have:* $\therefore\frac{a+b}{2}\geq\sqrt{ab}\geq\frac{2ab}{a+b}$

Let $f(x)=\tan^{-1}x$ *for all* $x\geq 0$ *then* $f'(x)=\frac{1}{1+x^2}, f''(x)=-\frac{2x}{(1+x^2)^2}\leq 0$ *for all* $x\geq 0$

hence f *is a concave function* $\therefore\frac{f\left(\frac{a+b}{2}\right)-f\left(\frac{2ab}{a+b}\right)}{\frac{a+b}{2}-\frac{2ab}{a+b}}\leq\frac{f(\sqrt{ab})-f\left(\frac{2ab}{a+b}\right)}{\sqrt{ab}-\frac{2ab}{a+b}}$

$$\Rightarrow\frac{\tan^{-1}\left(\frac{a+b}{2}\right)-\tan^{-1}\left(\frac{2ab}{a+b}\right)}{\frac{a+b}{2}-\frac{2ab}{a+b}}\leq\frac{\tan^{-1}(\sqrt{ab})-\tan^{-1}\left(\frac{2ab}{a+b}\right)}{\sqrt{ab}-\frac{2ab}{a+b}}$$

$$\Rightarrow\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\tan^{-1}(\sqrt{ab})-\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\tan^{-1}\left(\frac{2ab}{a+b}\right)$$

$$\geq\left(\sqrt{ab}-\frac{2ab}{a+b}\right)\tan^{-1}\left(\frac{a+b}{2}\right)-\left(\sqrt{ab}-\frac{2ab}{a+b}\right)\tan^{-1}\left(\frac{2ab}{a+b}\right)$$

$$\therefore\left(\sqrt{ab}-\frac{a+b}{2}\right)\tan^{-1}\left(\frac{2ab}{a+b}\right)+\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\tan^{-1}(\sqrt{ab})+$$

$$+\left(\frac{2ab}{a+b}-\sqrt{ab}\right)\tan^{-1}\left(\frac{a+b}{2}\right)\geq 0$$ *(Proved)*

SP.227. Prove that for any positive real numbers a,b,c:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{8(a+b+c)^3}{3(a+b)(b+c)(c+a)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Mustafa Tarek-Cairo-Egypt

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{8(a+b+c)^3}{3(a+b)(b+a)(c+a)}\quad (1)$$

(1) is equivalent to $\sum\frac{1}{a}\ge\frac{8(\sum a)^2}{3\prod(a+b)}$ *(2)*

Let $x=b+c, y=a+c, z=a+b$. *It's easy to show that:* $x+y>z$ *and* $y+z>x$ *and* $x+z>y$ *then* x,y,z *are sides of a triangle, suppose that it's semiperimeter* (s) *and it's circumradius* (R) *and its inradius* (r) *then* $\sum x=2\sum a, \sum a=\frac{\sum x}{2}=s$, *and*

$a=\frac{y+z-x}{2}$, $b=\frac{x+z-y}{2}$, $c=\frac{x+y-z}{2}$ $\therefore$ *(2) is equivalent to* $\sum\frac{1}{s-x}\ge\frac{8s^2}{3xyz}$

$$\Leftrightarrow\frac{\sum(s-x)(s-y)}{\prod(s-x)}\ge\frac{8s^2}{3\cdot 4Rrs}\Leftrightarrow\frac{r(4R+r)}{r^2s}\ge\frac{8s^2}{3\cdot 4Rrs}$$

[where we've used the well-known identity $\sum(s-x)(s-y)=r(4R+r)$*]*

$\Leftrightarrow 4R+r\ge\frac{2s^2}{3R}$. *But using Doucet's inequality* $4R+r\ge\sqrt{3}s$

$4R+r\ge\sqrt{3}s\overset{MITRINOVIC}{\ge}\frac{2s^2}{3R}\Leftrightarrow\frac{3sR}{2}\ge s\Rightarrow$ *true because it is Mitrinovic's inequality.*

Equality holds if $a=b=c$.

Solution 2 by Boris Colakovic-Belgrade-Serbie

We use the well – known inequality $(a+b)(b+c)(c+a)\ge$

$$\ge\frac{8}{9}(a+b+c)(ab+bc+ca)$$

$$RHS\le\frac{3\cdot 8(a+b+c)^3}{8(a+b+c)(ab+bc+ca)}=\frac{3(a+b+c)^2}{ab+bc+ca}\le\frac{(a+b+c)(ab+bc+ca)}{abc}\Rightarrow$$

$\Rightarrow(ab+bc+ca)^2\ge 3(a+b+c)abc$ *true because*

$$(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)\ge$$

$$\ge ab\cdot bc+bc\cdot ca+ab\cdot ca+2abc(a+b+c)=3abc(a+b+c)$$

Solution 3 by Michael Sterghiou-Greece

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc}\frac{1}{a}\right) \geq \frac{8(a+b+c)^3}{3\prod_{cyc}(a+b)} \quad \textbf{(1)}$$

Let $\left(\sum_{cyc} a, \sum_{cyc} ab, abc\right) = (p, q, r)$

Then observe that $9\prod_{cyc}(a+b) \geq 8p \cdot q$ *(equivalent to* $\sum_{cyc} a(b-c)^2 \geq 0$*)*

so (1) can became the stronger inequality.

$p \cdot \frac{q}{r} \geq \frac{3\cdot 8p^3}{8pq}$ *or* $q^2 \geq 3pr$ *which holds. Done!*

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\left(\sum a\right)\left(\sum\frac{1}{a}\right) \overset{(1)}{\geq} \frac{8(\sum a)^3}{3\prod(a+b)}$$

Let $\sum a^2b + \sum ab^2 = p$

$$\textbf{(1)} \Leftrightarrow \frac{(\sum a)(\sum ab)}{abc} \geq \frac{8\sum a^3 + 24\prod(a+b)}{3\prod(a+b)} \Leftrightarrow \frac{p+3abc}{abc} - 8 \geq \frac{8\sum a^3}{3(2abc+p)}$$

$$\Leftrightarrow \frac{3(p-5abc)(p+2abc)}{abc} \geq 8\sum a^3 \Leftrightarrow \frac{3p^2}{abc} - 9p - 30abc \overset{(2)}{\geq} 8\sum a^3$$

Now, $\frac{3p^2}{abc} = \frac{3(\sum a^2b + \sum ab)^2}{abc} = \frac{3(\sum a^4b^2 + \sum a^2b^4 + 2abcp + 2\sum a^2b\cdot\sum ab^2)}{abc}$

$$= 3\left(\frac{\sum a^4b^2 + \sum c^2a^4}{abc}\right) + 6p + \frac{6\sum a^2b\cdot\sum ab^2}{abc}$$

$$= 3\sum\left\{a^3\left(\frac{b}{c}\right) + a^3\left(\frac{c}{b}\right)\right\} + 6p + \frac{6\{\sum a^3b^3 + 3a^2b^2c^2 + abc(\sum a^3)\}}{abc}$$

$$\overset{A-G}{\geq} 3\cdot 2\sum a^3 + 6p + 6\sum a^3 + \frac{6}{abc}\left(\sum a^3b^3 + 3a^2b^2c^2\right)$$

$$\overset{Schur}{\geq} 12\sum a^3 + 6p + \frac{6}{abc}\left(\sum a^2b^2c^2\cdot bc + \sum ab\cdot b^2c^2\right)$$

$$= 12\sum a^3 + 6p + \frac{6}{abc}\cdot abcp = 12\sum a^3 + 12p$$

$$\Rightarrow \frac{3p^2}{abc} - 9p - 30abc \geq 12\sum a^3 + 3p - 30abc$$

$$= 8\sum a^3 + \left\{4\sum a^3 + 3\left(\sum a^2b + \sum ab^2\right)\right\}$$

$$\overset{A-G}{\geq} 8\sum a^3 + (12abc + 18abc) - 30abc = 8\sum a^3$$

$\Rightarrow$ (2) $\Rightarrow$ (1) is true (Proved)

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $, b, c > 0$ *, we have the following: Because* $\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\ge a+b+c$

$$\Rightarrow 2(a+b+c)+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\ge 3(a+b+c)$$

$$\Rightarrow \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(ab+bc+ca)\ge 3(a+b+c)$$

$$\Rightarrow 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{8(ab+bc+ca)(a+b+c)}{9}\right)\ge 8(a+b+c)^2$$

$$\Rightarrow 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b)(b+c)(c+a)\ge 8(a+b+c)^2$$

$$\Rightarrow 3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b)(b+c)(c+a)\ge 8(a+b+c)^3$$

$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{8(a+b+c)^3}{3(a+b)(b+c)(c+a)}$ *ok. Therefore, it is true.*

Solution 6 by Anant Bansal-India

By $GM\ge HM$: $\left(\frac{a+b+c}{a}\right)\left(\frac{a+b+c}{b}\right)\left(\frac{a+b+c}{c}\right)\ge 3^3=27$ *(i)*

By $AM\ge GM$: $x=\prod_{cyc}\left(\frac{\sum_{cyc}a}{a+b}\right)=\left(1+\frac{c}{a+b}\right)\left(1+\frac{b}{a+c}\right)\left(1+\frac{a}{b+c}\right)\le\left(\frac{4}{3}\right)^3$

$$\frac{512}{81}\times\frac{8}{3}=\frac{8}{3}x$$

$$7>\frac{8}{3}x \quad \textit{(ii)}$$

From (i) and (ii): $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{8(a+b+c)^3}{3(a+b)(b+c)(c+a)}$

SP.228. Find the positive real numbers (x,y) such that:

$$\begin{cases}\dfrac{x^2}{y}+\dfrac{y^2}{x}=2\sqrt[4]{\dfrac{x^4+y^4}{2}}\\ x^2y^2-y^3+1=\sqrt{2x^2-2y+1}\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\begin{cases} \frac{x^2}{y}+\frac{y^2}{x}=2\sqrt[4]{\frac{x^4+y^4}{2}} & (1) \\ x^2y^2-y^3+1=\sqrt{2x^2-2y+1} & (2) \end{cases}, x,y>0$$

From (1), $x\neq 0, y\neq 0$ $D=]0,+\infty[$

$$\frac{x^2}{y}+\frac{y^2}{x}=2x\sqrt[4]{\frac{1}{2}\left(1+\frac{y^4}{x^4}\right)} \Rightarrow \frac{x}{y}\cdot x+\frac{y}{x}\cdot y=2x\sqrt[4]{\frac{1}{2}\left(1+\frac{y^4}{x^4}\right)}; x>0$$

$\frac{x}{y}+\frac{y}{x}\cdot\frac{y}{x}=2\sqrt[4]{\frac{1}{2}\left(1+\frac{y^4}{x^4}\right)}$. ***Suppose*** $t=\frac{y}{x}, t>0$, ***then:***

$$\frac{1}{t}+t^2=2\sqrt[4]{\frac{1}{2}(1+t^4)} \quad (*)\ D=]0,+\infty[$$

Let be the function: $f(t)=\frac{1}{t}+t^2-2\sqrt[4]{\frac{1}{2}(1+t^4)}, D=]0,+\infty[$

f ***is Derivative in*** D ***and:***

$$\lim_{t\to 0}[f(t)]=+\infty, \lim_{x\to+\infty}(f(x))=+\infty$$

$$\left(\lim_{t\to+\infty}[f(t)]=\lim_{t\to+\infty}\left[\underbrace{\frac{1}{t}}_{0}+\underbrace{t^2}_{+\infty}\left[1-2\cdot\underbrace{\frac{1}{t}}_{0}\sqrt[4]{\underbrace{\frac{1}{2}\left(\frac{1}{t^4}+1\right)}_{0}}\right]\right]=+\infty\right)$$

$$f'(t)=?, f(t)=\frac{1}{t}+t^2-2\left(\frac{1}{2}(1+t^4)\right)^{\frac{1}{4}} \Rightarrow$$

$$f'(t)=-\frac{1}{t^2}+2t-2\cdot\frac{1}{4}\left(\frac{1}{2}(1+t^4)\right)^{-\frac{3}{4}}(2t^3)$$

$f'(t)=-\frac{1}{t^2}+2t-\frac{t^3}{\sqrt[4]{\left(\frac{1}{2}(1+t^4)\right)^3}}, f'(t)=0\Rightarrow t=1$ ***in the interval*** $]0,+\infty[$

t	0 $\quad\quad\quad\quad$ 1 $\quad\quad\quad\quad$ $+$
$f'(t)$	$- - - - - - - - - - 0 + + + + + + + + +$
$f(t)$	$+\infty$ $\quad\quad\quad\quad$ 0 $\quad\quad\quad\quad$ $+\infty$

So: $t=1$ ***is solve to the equation (*). So:***

$\frac{y}{x} = 1 \Rightarrow y = x$. ***Substituted in equation (2):***

$$x^4 - x^3 + 1 = \sqrt{2x^2 - 2x + 1}$$

$$(x^4 - x^3)^2 + 2(x^4 - x^3) + 1 = 2x^2 - 2x + 1$$

$$\left(x^3(x-1)\right)^2 + 2\left(x^3(x-1)\right) = 2x(x-1)$$

$$(x-1)[x^6(x-1) + 2x^3 - 2x] = 0$$

$$(x-1)[x^6(x-1) + 2x^3 - 2x] = 0$$

$$x(x-1)\left[x^5(x-1) + 2x^2 - 2\right] = 0$$

$$x(x-1)\left[x^5(x-1) + 2(x-1)(x+1)\right] = 0$$

$$x(x-1)^2[x^2 + 2x + 1] = 0$$

$x = 0$ ***impossible***

$$x = 1 \Rightarrow y = 1$$

$x^2 + 2x + 1 = 0$ ***impossible***

$x^2 + 2x + 1 > 0$ ***impossible***

So: system $\begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 2\sqrt[4]{\frac{x^4+y^4}{2}} \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} \end{cases}, x, y > 0$

Have only: $(x, y) = (1, 1)$

$$S = \{(1, 1)\}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{x^4 + y^4}{2}} \le x^2 - xy + y^2 \Leftrightarrow 2(x^2 - xy + y^2)^2 \ge x^4 + y^4$$

$$\Leftrightarrow 2(x^4 + y^4 + 2x^2y^2) - 4xy(x^2 + y^2) + 2x^2y^2 \ge x^4 + y^4$$

$$\Leftrightarrow (x^2 + y^2)^2 - 4xy(x^2 + y^2) + 4x^2y^2 \ge 0 \Leftrightarrow (x^2 + y^2 - 2xy)^2 \ge 0 \rightarrow \textit{true}$$

$\therefore 2\sqrt[4]{\frac{x^4+y^4}{2}} \le 2\sqrt{x^2 - xy + y^2} \Rightarrow \frac{x^2}{y} + \frac{y^2}{x} \le 2\sqrt{x^2 - xy + y^2}$ ***(by first equation)***

$$\Rightarrow \frac{(x+y)(x^2 - xy + y^2)}{xy} \le 2\sqrt{x^2 - xy + y^2} \Rightarrow (x+y)\sqrt{x^2 - xy + y^2} \overset{(1)}{\le} 2xy$$

But A-G$\Rightarrow (x+y)\sqrt{x^2 - xy + y^2} \overset{(2)}{\ge} 2\sqrt{xy}\sqrt{xy} = 2xy$

$\therefore$ ***(1), (2)*** $\Rightarrow (x+y)\sqrt{x^2-xy+y^2} = 2xy$

and (2) suggests equality occurs when $x = y$

$\therefore x = y$ ***($\because$ equality occurs)***

Putting $x = y$ ***in*** $x^2y^2 - y^3 + 1 = \sqrt{2x^2-2y+1}$***, we get:***

$$x^4 - x^3 + 1 \overset{(3)}{=} \sqrt{2x^2-2x+1}$$

Let $x^2 - x = p$***. Then (3)*** $\Rightarrow x^2p + 1 = \sqrt{2p+1} \Rightarrow x^4p^2 + 1 + 2x^2p = 2p + 1$

$$\Rightarrow p(x^4p + 2x^2 - 2) = 0$$

If $p = 0, x(x-1) = 0 \Rightarrow x = 0, 1$ ***and both values satisfy (3), but*** $x, y \neq 0$

If $x^4p + 2x^2 - 2 = 0$***, then:*** $x^4(x^2-x) + 2(1+x)(x-1) = 0$

$$\Rightarrow x^5(x-1) + 2(x+1)(x-1) = 0 \Rightarrow (x-1)\left(x^5+2x+2\right) = 0$$

$$\Rightarrow x = 1 \ \left(\because x^5 + 2x + 2 > 2 \neq 0 \ as \ x > 0\right)$$

$\therefore$ ***combining all cases, all possible pairs of*** (x, y) ***satisfying given system is:***

$$\begin{pmatrix} x = 1 \\ y = 1 \end{pmatrix} \textit{(answer)}$$

Solution 3 by Orlando Irahola Ortega-Bolivia

$$\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = 2\sqrt[4]{\dfrac{x^4+y^4}{2}} & (1) \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2-2y+1} & (2) \end{cases} \quad \{x \wedge y \neq 0\}$$

$$\textbf{(2)} \rightarrow x^3 + y^3 = 2xy\sqrt[4]{\frac{x^4+y^4}{2}} \Rightarrow x^6 + y^6 + 2x^3y^3 = 4x^2y^2\sqrt{\frac{x^4+y^4}{2}} \Rightarrow$$

$$\Rightarrow \frac{x^3}{y^3} + \frac{y^3}{x^3} + 2 = \sqrt[4]{\frac{1}{2}\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right)}$$

Sea: $t = \frac{x}{y} + \frac{y}{x} \Rightarrow t^3 - 3t + 2 = 2\sqrt{2t^2-4} \Rightarrow t(t^2-3) = 2\left(\sqrt{2t^2-4} - 1\right)$

$$\Rightarrow t^2(t^2-3)^2 = 4\left(\sqrt{2t^2-4} - 1\right)^2 \Rightarrow \left(\sqrt{2t^2-4}^2 + 4\right)\left(\sqrt{2t^2-4}^2 - 2\right)^2 =$$

$$= 32\left(\sqrt{2t^2-4} - 1\right)^2; \textit{sea: } m = \sqrt{2t^2-4}$$

$$\Rightarrow (m^2+4)(m^2-2)^2 = 32(m-1)^2 \Rightarrow m^6 - 44m^2 + 64m - 16 = 0$$

$$(m-2)\left(m^5 + 2m^4 + 4m^3 + 8m^2 - 28m + 8\right) = 0$$

$m-2=0 \Rightarrow m_1=2 \Rightarrow \sqrt{2t^2-4}=2 \Rightarrow t=\pm 2$ ***pero:*** $t=\frac{x}{y}+\frac{y}{x} \Rightarrow \frac{t}{2} \geq \sqrt{\frac{x}{y}\cdot\frac{y}{x}} \Rightarrow t \geq 2$

$x \wedge y \in \mathbb{R}$

Then: $t=2 \Rightarrow \frac{x}{y}+\frac{y}{x}=2 \Rightarrow (x-y)^2=0 \Rightarrow x=y$ ***(3)***

Por altro lado: $m^5+2m^4+4m^3+8m^2-28m+8=0=f(m)$***: Condition:***

$m \geq 0 \rightarrow \sqrt{2t^2-4}=|m|$

Si: $\left.\begin{matrix} m=0 \Rightarrow f(0)=8>0 \\ m=1 \Rightarrow f(1)=-5<0 \end{matrix}\right\}$ $\exists$ ***variation of signs*** $\Rightarrow \exists$ ***al menos una solucion en*** $[0,1]$

$m>1 \Rightarrow f(m>1)>0 \Rightarrow \nexists$ ***solution for*** m

Conclusion: como existe al menos una solucion para m ***en*** $[0,1]$***. Pero:*** $t \geq 2$

$\Rightarrow$ ***para que*** $t \geq 2 \Rightarrow m \geq 2$ ***para*** $(x;y) \in \mathbb{R}$***, como*** $m \in [0,1] \Rightarrow (x,y) \notin \mathbb{R}$

(2)$\rightarrow y^2(2x^2-2y)+2=2\sqrt{2x^2-2y+1} \Rightarrow y^2\sqrt{2x^2-2y+1}^2+2-y^2=$

$=2\sqrt{2x^2-2y+1}$

Sea: $t=\sqrt{2x^2-2y+1} \Rightarrow t^2y^2+2-y^2=2t \Rightarrow (t-1)(ty^2+y^2-2)=0 \Rightarrow$

$t=1$ ***(*)***

$ty^2=2-y^2$ ***(**)***

(*)$\rightarrow t=1 \Rightarrow \sqrt{tx^2-2y+1}=1 \Rightarrow y=x^2$ ***(4)***

(**)$\rightarrow ty^2=2-y^2 \Rightarrow t^2y^4=y^4-4y^2+4 \Rightarrow y^4(2x^2-2y+1)=y^4-4y^2+4$

$\Rightarrow x^2y^4+2y^2=y^5+2$ ***(5)***

Ahora resolvemos: $\begin{cases} y=x \\ y=x^2 \end{cases} \Rightarrow x^2-x=0 \Rightarrow \begin{matrix} x=y=0 \\ x=y=1 \end{matrix} (x \wedge y \neq 0) \wedge \begin{cases} y=x \\ x^2y^4+2y^2=y^5+2 \end{cases}$

$\Rightarrow (x-1)(x^5+2x+2)=0$

$\Rightarrow x=1 \wedge x^5+2x+2=0; \forall x \geq 0; \exists$ ***solucion:*** $\forall x \in \mathbb{R}|-1 \leq x<0\ \exists x<0; y<0.$

Pero: (2)$\rightarrow 2x^2-2y+1 \geq$

$2x^2+1 \geq 2y \rightarrow y \geq 0$

Como $y<0 \Rightarrow \nexists x \wedge y$ ***que satisfaga el Sistema.***

Unica solucion: $x=y=1.$

SP.229. Let a, b, c be the lengths of a triangle such that $a + b + c = 3$. Prove that:

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} + 3 \cdot \sqrt[2018]{abc} \geq 2(ab + bc + ca)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Michael Sterghiou-Greece

$$\left(\sum_{cyc} \frac{1}{b+c-a}\right) + 3\sqrt[2018]{abc} \geq 2\sum_{cyc} ab \quad \textbf{(1)}$$

WLOG, let $c \leq b \leq a$ (c) and let $\left(\sum_{cyc} a, \sum_{cyc} ab, abc\right) = (p, q, r)$. $p = 3$ and $r \leq 1$. As

$a > b - c, a > c - b$ so $a > |b - c| \to a^2 > (b - c)^2$ **(2)**

Cyclic application of (2) and addition yields $2q \geq \sum_{cyc} a^2 = 9 - 2q$, so $q > \frac{9}{4}$

[Note: This is general without the condition ©]

Assume $c \leq \frac{1}{12}$. Then $|a - b| < c \leq 12$ and $\max(a + c - b, b + c - a) \leq \frac{1}{6}$ are one of $\frac{1}{a+c-b}, \frac{1}{b+c-a}$ is > 6. So, as $3 \cdot r^{\frac{1}{2018}} > 0$ and $2q \leq 6$ because $q \leq 3$ we are done!

Therefore, let $a \geq b \geq c \geq \frac{1}{12}$. In this case $r > \left(\frac{1}{12}\right)^3$. LHS (1) $\overset{BCS}{\geq} \frac{3^2}{\sum_{cyc}(b+c-a)} = 3$ so, it suffices to show that (3) $3 + 3r^{\frac{1}{2018}} - 2q \geq 0$ for $\left(\frac{1}{12}\right)^3 < r \leq 1, \frac{9}{4} < q \leq 3$. From the 3rd degree Schur inequality we have: $q \leq \frac{9+3r}{4}$, so (3) reduces to:

$f(r) = 2r^{\frac{1}{2018}} - r - 1 \geq 0$ with $r > \left(\frac{1}{12}\right)^3$ and $r \leq 1$. $f'(r) = \frac{1}{1009} \cdot \frac{1}{r^{\frac{2017}{2018}}} - 1$ with only one real root $r_0 = \left(\frac{1}{1009}\right)^{\frac{2018}{2017}}$. It is easy to show that $f\left(\frac{1}{12^3}\right) > 0, f(r_0) > 0$ and $f(1) = 0$ which means that $f(1) = 0$ is minimum for $f(r)$ on $\left(\frac{1}{12^3}, 1\right]$ and hence $f(r) \geq 0$. Equality for $a = b = c = 1$. Done!

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[2018]{abc} = \sqrt[2018]{abc\underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{2015\,times}} \overset{G-H}{\geq} \frac{2018}{\sum \frac{1}{a} + 2015} = \frac{2018}{\frac{\sum ab}{4Rrs} + 2015} = \frac{2018}{\frac{\sum ab}{2Rr(3)} + 2015}$$

$$\left(\because 2s=\sum a=3\right)=\frac{12108Rr}{\sum ab+12090Rr}\Rightarrow 3\sqrt[2018]{abc}\overset{(1)}{\geq}\frac{36324Rr}{\sum ab+12090Rr}$$

Now, $\sum\frac{1}{b+c-a}=\frac{1}{2}\sum\frac{1}{s-a}=\frac{\sum(s-b)(s-c)}{2r^2s}=\frac{\sum(s^2-s(b+c)+bc)}{2r^2s}$

$$=\frac{3s^2-4s^2+s^2+4Rr+r^2}{2r^2s}=\frac{4R+r}{r(2s)}\overset{(2)}{=}\frac{4R+r}{3r}\left(\because 2s=\sum a=3\right)$$

Again, $2\sum ab=\left(\frac{2}{3}\right)(3\sum ab)\leq\frac{2}{3}(\sum a)^2=\left(\frac{2}{3}\right)9=6\Rightarrow 2\sum ab\overset{(3)}{\leq}6$

(1), (2) $\Rightarrow LHS\overset{(4)}{\geq}\frac{4R+r}{3r}+\frac{36324Rr}{\sum ab+12090Rr}$

(4), (3) $\Rightarrow$ *it suffices to prove:* $\frac{4R+r}{3r}+\frac{36324Rr}{\sum ab+12090Rr}\geq 6\Leftrightarrow\frac{4R-17r}{3r}+\frac{36324Rr}{\sum ab+12090Rr}\geq 0$

$$\Leftrightarrow(4R-17r)\left(\sum ab+12090Rr\right)+108972Rr^2\geq 0$$

$$\Leftrightarrow(4R-17r)(s^2+12094Rr+r^2)+108972Rr^2\geq 0$$

$$\Leftrightarrow(4R-8r)s^2+(4R-17r)(12094Rr+r^2)+108972Rr^2\overset{(5)}{\geq}9rs^2$$

Now, LHS *of (5)* $\underset{(i)}{\overset{Gerretsen}{\geq}}(4R-8r)(16Rr-5r^2)+(4R-17r)(12094Rr+r^2)+$

$+108972Rr^2$ *and, RHS of (5)* $\underset{(ii)}{\overset{Gerretsen}{\leq}}9r(4R^2+4Rr+3r^2)$

(i), (ii) $\Rightarrow$ *in order to prove (5), it suffices to prove:*

$$(4R-8r)(16Rr-5r^2)+(4R-17r)(12094Rr+r^2)+108972Rr^2\geq$$

$$\geq 9r(4R^2+4Rr+3r^2)\Leftrightarrow 24202R^2-48403Rr-2r^2\geq 0$$

$\Leftrightarrow(R-2r)(24202R+r)\geq 0\rightarrow$ *true* $\because R\overset{Euler}{\geq}2r\Rightarrow$ *(5)* $\Rightarrow$ *proposed inequality is true (Proved)*

SP.230. Let a,b,c be positive real numbers such that $a+b+c=3$. Find the minimum value of:

$$T=\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+3\cdot\left(\frac{1}{abc+b^3+c^3}+\frac{1}{abc+c^3+a^3}+\frac{1}{abc+a^3+b^3}\right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Ruangkhaw Chaokha-Chiangrai-Thailand

Let $p = a + b + c = 3, q = abc + bc + ca, r = abc$

$$\because (ab - bc)^2 + (bc - ca)^2 + (ca - ab)^2 \geq 0$$

$(ab + bc + ca)^2 \geq 3abc(a + b + c) \Leftrightarrow q^2 \geq 9r \to$ ***(1) holds at*** $a = b = c = 1$

$\because (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0 \Leftrightarrow (a + b + c)^2 \geq 3(ab + bc + ca) \Leftrightarrow 3 \geq q \to$

(2) hold at $a = b = c = 1.$

Lemma $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \overset{??}{\geq} \frac{2(a^3+b^3+c^3)+3abc}{3}$ ***holds at*** $a = b = c = 1.$

Proof. $3(a^2 + b^2 + c^2) \geq 2abc(a^3 + b^3 + c^3) + 3a^2b^2c^2$

$$3(p^2 - 2q) \geq 2r(p^3 - 3pq + 3r) + 3r^2 \Leftrightarrow 3(9 - 2q) \geq 2r(27 - 9q + 3r) + 3r^2$$

$$0 \geq 9r^2 + 18(3 - q)r + 6q - 27 \Leftrightarrow 0 \geq \frac{q^4}{9} + 18(3 - q)\frac{q^2}{9} + 6q - 27 \overset{(1),(2)}{\geq}$$

$$\geq 9r^2 + 18(3 - q)r + 6q - 27$$

$0 \geq q^4 + 18(3 - q)q^2 + 9(6q - 27) \Leftrightarrow 0 \geq (q - 3)^2(q^2 - 12q - 27)$ ***True!!***

$\because$ ***(2);*** $0 < q \leq 3 \Leftrightarrow -6 < q - 6 \leq -3 \Leftrightarrow 9 \leq (q - 6)^2 < 36 \Leftrightarrow q^2 - 12q - 27 < -27 < 0$

Holds at $q = 3 \Leftrightarrow a = b = c = 1.$

$$T \geq \frac{2(a^3 + b^3 + c^3) + 3abc}{3} + 3\left(\frac{1}{abc + b^3 + c^3} + \frac{1}{a^3 + abc + c^3} + \frac{1}{a^3 + b^3 + abc}\right)$$

$$= \left(\frac{abc + b^3 + c^3}{3} + \frac{3}{abc + b^3 + c^3}\right) + \left(\frac{a^3 + abc + c^3}{3} + \frac{3}{a^3 + abc + c^3}\right) +$$

$$+\left(\frac{a^3 + b^3 + abc}{3} + \frac{3}{a^3 + b^3 + abc}\right) \overset{AM-GM}{\geq} 2 + 2 + 2 = 6$$

Holds at $\frac{abc+b^3+c^3}{3} = \frac{a^3+abc+c^3}{3} = \frac{a^3+b^3+abc}{3} = 1 \Leftrightarrow a = b = c = 1$

SP.231. In ΔABC the following relationship holds:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 4\sqrt{3}F + 2\log\left(a^{ab^2} \cdot b^{bc^2} \cdot c^{ca^2}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

We must show: $a^2b^2 + b^2c^2 + c^2a^2 \geq 4\sqrt{3}F + 2ab^2\ln a + 2bc^2\ln b + 2ca^2\ln c$ ***(1)***

Now we have $2\ln x \leq \frac{x^2-1}{x} \Leftrightarrow 2\ln x \leq x - \frac{1}{x}$ ***(2)***

(demonstration with derivative)

From (2)$\Rightarrow 2\ln a \le a - \frac{1}{a} \Rightarrow 2ab^2\ln a \le a^2b^2 - b^2$ *and similarly (3)*

From (1)+(3) we must show:

$$a^2b^2 + b^2c^2 + c^2a^2 \ge 4\sqrt{3}F + a^2b^2 + b^2c^2 + c^2a^2 - (a^2 + b^2 + c^2) \Leftrightarrow$$

$$\Leftrightarrow a^2 + b^2 + c^2 \ge 4\sqrt{3}F \text{ which its true in any } \Delta ABC$$

Observation: equality for $a = b = c = 1$.

SP.232. If $a, b \ge 1$ then:

$$2^{\frac{2}{a+b}} \cdot 3^{\frac{a+b}{2}} + 2^{\sqrt{ab}} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} + 2^{\frac{1}{\sqrt{ab}}} \cdot 3^{\sqrt{ab}} \ge 24$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{Let } x \ge 1 \Rightarrow 2^x \cdot 3^{\frac{1}{x}} + 2^{\frac{1}{x}} \cdot 3^x \ge 2\sqrt{2^{x+\frac{1}{x}} \cdot 3^{x+\frac{1}{x}}} \ge 2\sqrt{2^2 \cdot 3^2}$$

$$\Rightarrow 2^x \cdot 3^{\frac{1}{x}} + 2^{\frac{1}{x}} \cdot 3^x \ge 12$$

$$\left.\begin{array}{l} x = \dfrac{a+b}{2} \Rightarrow 2^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} + 2^{\frac{2}{a+b}} \cdot 3^{\frac{a+b}{2}} \ge 12 \\ x = \sqrt{ab} \Rightarrow 2^{\sqrt{ab}} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2^{\frac{1}{\sqrt{ab}}} \cdot 3^{\sqrt{ab}} \ge 12 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 2^{\frac{2}{a+b}} \cdot 3^{\frac{a+b}{2}} + 2^{\sqrt{ab}} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} + 2^{\frac{1}{\sqrt{ab}}} \cdot 3^{\sqrt{ab}} \ge 24$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b \ge 1$, *we have:* $2^{\frac{2}{a+b}}3^{\frac{a+b}{2}} + 2^{\sqrt{ab}}3^{\frac{1}{\sqrt{ab}}} + 2^{\frac{a+b}{2}}3^{\frac{2}{a+b}} + 2^{\frac{1}{\sqrt{ab}}} - 3^{\sqrt{ab}}$

$$\ge 4\sqrt[4]{\left(2^{\frac{2}{a+b}} \cdot 2^{\frac{a+b}{2}} \cdot 2^{\sqrt{ab}} \cdot 2^{\frac{1}{\sqrt{ab}}}\right)\left(3^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} \cdot 3^{\sqrt{ab}}3^{\frac{1}{\sqrt{ab}}}\right)}$$

$$= 4\sqrt[4]{2^{\frac{2}{a+b}+\frac{a+b}{2}+\sqrt{ab}+\frac{1}{\sqrt{ab}}} \cdot 3^{\frac{a+b}{2}+\frac{2}{a+b}+\sqrt{ab}+\frac{1}{\sqrt{ab}}}} \ge 4\sqrt[4]{2^4 3^4} = 2 \times 3 \times 4 = 24 \text{ ok}$$

$$\text{Because } \frac{a}{a+b} + \frac{a+b}{2} + \sqrt{ab} + \frac{1}{\sqrt{ab}} \ge 4\sqrt[4]{\left(\frac{2}{a+b}\right)\left(\frac{a+b}{2}\right)\left(\sqrt{ab}\right)\left(\frac{1}{\sqrt{ab}}\right)} = 4 \times 1 = 4$$

SP.233. If $A \in M_3(\mathbb{R}); Tr(A^2) = 0; \det = 1$ then:

$$\det(A^2 + A + I_3) \geq (Tr\ A)^3$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

$$\text{If } Tr(A^2) = 0 \text{ and } \det A = 1$$

$$P_A(x) = x^3 - Tr\ A\, x^2 + \frac{(Tr\ A)^2}{2} x - 1$$

$$\det(A^2 + A + I_3) = \det(A - \varepsilon I_3)(A - \bar{\varepsilon} I_3) =$$

$$= P_A(\varepsilon) \cdot P_A(\bar{\varepsilon}) = \left(\varepsilon^3 - Tr\ A\, \varepsilon^2 + \frac{(Tr\ A)^2}{2}\varepsilon - 1\right) \cdot \left(\bar{\varepsilon}^3 - Tr\ A \bar{\varepsilon}^2 + \frac{(Tr\ A)^2}{2}\bar{\varepsilon} - 1\right) =$$

$$= (Tr\ A)^2 \varepsilon \cdot \bar{\varepsilon}\left(\varepsilon - \frac{Tr\ A}{2}\right)\left(\bar{\varepsilon} - \frac{Tr\ A}{2}\right) = (Tr\ A)^2 \cdot \left(\varepsilon\bar{\varepsilon} - \varepsilon\frac{Tr\ A}{2} - \bar{\varepsilon}\frac{Tr\ A}{2} + \frac{(Tr\ A)^2}{4}\right)$$

$$= (Tr\ A)^2 \cdot \left(1 - \frac{Tr\ A}{2}(\varepsilon + \bar{\varepsilon}) + \frac{(Tr\ A)^2}{4}\right) = (Tr\ A)^2 \cdot \left(1 + \frac{Tr\ A}{2} + \frac{(Tr\ A)^2}{4}\right) \geq (Tr\ A)^3$$

$$1 + \frac{Tr\ A}{2} + \frac{(Tr\ A)^2}{4} \geq Tr\ A$$

$$4 + 2\, Tr\ A + (Tr\ A)^2 \geq 4 Tr\ A$$

$$(Tr\ A)^2 - 2 Tr\ A + 4 \geq 0$$

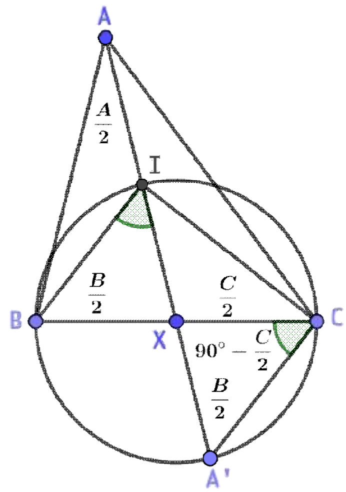
$$(Tr\ A - 1)^2 + 3 \geq 0$$

SP.234. In ΔABC, I – incentre; A' - is the intersection between AI and circumcircle of ΔBIC; B' - is the intersection between BI and circumcircle of ΔAIC; C' - is the intersection between CI and circumcircle of ΔAIB. Prove that:

$$\frac{IA}{IA'} + \frac{IB}{IB'} + \frac{IC}{IC'} \geq 2\left(1 - \frac{r}{R}\right)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AA' intersect BC at X. In ΔABX, $\angle AXB = 180^\circ - \left(B + \frac{A}{2}\right)$ using ΔBIX, $\angle BIX =$

$$= 180^\circ - \left(\frac{B}{2} + 180^\circ - \left(B + \frac{A}{2}\right)\right) = \frac{A+B}{2} = 90^\circ - \frac{C}{2} \Rightarrow$$

$$\Rightarrow \angle BCA' = \angle BIX = 90^\circ - \frac{C}{2}$$

$\therefore \angle ICA' = 90^\circ$. Also, $\angle IA'C = \angle IBC = \frac{B}{2}$

Using $\Delta IA'C$, $\sin\frac{B}{2} = \frac{IC}{IA'} = \frac{r}{IA'\sin\frac{C}{2}} \Rightarrow \frac{r}{4R\sin\frac{A}{2}} = \frac{r}{IA'} \Rightarrow IA' = 4R\sin\frac{A}{2} \Rightarrow$

$\Rightarrow \frac{IA}{IA'} \overset{(a)}{=} \frac{r}{4R\sin^2\frac{A}{2}}$. Similarly, $\frac{IB}{IB'} \overset{(b)}{=} \frac{r}{4R\sin^2\frac{B}{2}}$ and $\frac{IC}{IC'} \overset{(c)}{=} \frac{r}{4R\sin^2\frac{C}{2}}$

(a)+(b)+(c)$\Rightarrow LHS = \frac{r}{4R}\sum\csc^2\frac{A}{2} = \frac{r}{4R}\sum\frac{bc(s-a)}{r^2 s} =$

$$= \frac{s(s^2+4Rr+r^2)-12Rrs}{4Rrs} = \frac{s^2-8Rr+r^2}{4Rr} \overset{Gerretsen}{\geq} \frac{8Rr-4r^2}{4Rr} =$$

$$= 2 - \frac{r}{R} = 2\left(1 - \frac{r}{2R}\right)$$

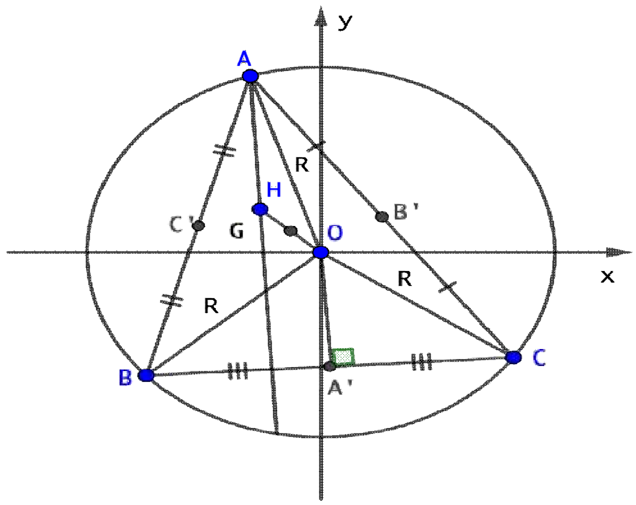
SP.235. Let be $A(z_1)$; $B(z_1)$; $C(z_3)$; $z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$;

$|z_1| = |z_2| = |z_3|$; $AB = c$; $BC = a$; $CA = b$.

If $(b+c)z_B z_C + (c+a)z_C z_A + (a+b)z_A z_B = 0$ then $AB = BC = CA$.

Proposed by Marian Ursărescu – Romania

Solution by Khaled Abd Imouti-Damascus-Syria



Let be $A(z_1), B(z_2), C(z_3), z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$

$$|z_1| = |z_2| = |z_3|, AB = c, BC = a, CA = b$$

If $(b + c) \cdot z_B \cdot z_C + (c + a) \cdot z_C \cdot z_A + (a + b) \cdot z_A \cdot z_B = 0$

Then $AB = BC = CA.$

$|z_A| = |z_B| = |z_C| = R$ **(**R **radius of circle)**

$$z_A \cdot \overline{z_A} = R^2, z_B \cdot \overline{z_B} = R^2, z_C \cdot \overline{z_C} = R^2$$

$$z_A = \frac{R^2}{\overline{z_A}}, z_B = \frac{R^2}{\overline{z_B}}, z_C = \frac{R^2}{\overline{z_C}}$$

$$(b + c) \cdot \left(\frac{R^2}{\overline{z_B}} \cdot \frac{R^2}{\overline{z_C}}\right) + (c + a)\left(\frac{R^2}{\overline{z_C}} \cdot \frac{R^2}{\overline{z_A}}\right) + (a + b)\left(\frac{R^2}{\overline{z_A}} \cdot \frac{R^2}{\overline{z_B}}\right) = 0$$

$$(b + c) \cdot \left(\frac{1}{\overline{z_B} \cdot \overline{z_C}}\right) + (c + d)\left(\frac{1}{\overline{z_C} \cdot \overline{z_A}}\right) + (a + b)\left(\frac{1}{\overline{z_A} \cdot \overline{z_C}}\right) = 0$$

$$\frac{(b + c)}{z_B \cdot z_C} + \frac{(c + a)}{z_C \cdot z_A} + \frac{a + b}{z_A \cdot z_B} = 0 \ \times (z_A \cdot z_B \cdot z_C \neq 0)$$

$$(b + c)z_A + (c + a)z_B + (a + b) \cdot z_C = 0$$

$$(b + c + a - a)z_A + (c + a + b - b)z_B + (a + b + c - c)z_C = 0$$

$$2p(z_A + z_B + z_C) = a \cdot z_A + b \cdot z_B + c \cdot z_C$$

$$6p \cdot z_G = a \cdot z_A + b \cdot z_B + c \cdot z_C$$

$$z_G = \frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{6p}$$

Suppose H is orthocenter:

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \quad (*)$$

$$\overrightarrow{OH} = \overrightarrow{OA} + \left(\overrightarrow{OB} + \overrightarrow{OC}\right) \Rightarrow \overrightarrow{ON} - \overrightarrow{OA} = 2\overrightarrow{OA'}$$

So: $\overrightarrow{AH} = 2 \cdot \overrightarrow{OA'}, (AH) \parallel (OA')$

but $OA' \perp BC \Rightarrow AH \perp BC$

in a similar way, $BH \perp AC$ ***and*** $CH \perp AB$

so, H ***is orthocenter in triangle*** ΔABC

from (*): $\overrightarrow{OH} = 3\overrightarrow{OG}$, ***so:***

O, H, G ***is collinear.***

not: $z_H = 3 \cdot z_G$

$$z_G - z_H = \frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{6p} - \frac{z_A + z_B + z_C}{1}$$

$$z_G - z_H = \frac{(a - 6p)z_A + (6 - 6p)z_B + (c - 6p)z_C}{6p}$$

$$-z_G + z_H = \frac{(6p - a)z_A + (6p - b)z_B + (6p - c)z_C}{6p}$$

$$6p(z_H - z_G) = (6p - a)z_A + (6p - b)z_B + (6p - c)z_C$$

$$\frac{6p(z_H - z_G)}{16p} = \frac{(6p - a)z_A + (6p - b)z_B + (6p - c)z_C}{16p}$$

$$\frac{3}{9}(z_H - z_G) = \frac{1}{16p}[(6p - a)z_A + (6p - b)z_B + (6p - c)z_C]$$

but: $(6p - a)z_A + (6p - b)z_B + (6p - c)z_C = 6p(z_A + z_B + z_C) - (az_A + bz_B + cz_C)$

$$(6p - a)z_A + (6p - b)z_B + (6p - c)z_C$$
$$= 18p \cdot \left(\frac{z_A + z_B + z_C}{3}\right) - (a \cdot z_A + b \cdot z_B + c \cdot z_C)$$

$$(6p - a) \cdot z_A + (6p - b)z_B + (6p - c)z_C = 18p \cdot z_G - (a \cdot z_A + b \cdot z_B + c \cdot z_C)$$

$$\frac{(6p - a)z_A + (6p - b)z_B + (6p - c)z_C}{16p} = \frac{18p \cdot z_G}{16p} - \frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{16p}$$

$$= \frac{9}{4}z_G - \frac{1}{8}\left(\frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{2p}\right)$$

$$= \frac{9}{4}z_G - \frac{1}{8} \cdot z_H$$

So: $\frac{3}{8}(z_H - z_G) = \frac{9}{4}z_G - \frac{1}{8}z_H$

$$\frac{3}{8}z_H + \frac{1}{8}z_H = \frac{9}{4}z_G + \frac{3}{8}z_G$$

$$\frac{1}{2}z_H = \frac{21}{8}z_G \Rightarrow z_H = \frac{21}{4}z_G$$

and: $z_H = 3 \cdot z_G$

$$\frac{21}{4}z_G - 3z_G = 0$$

$$\frac{9}{4}z_G = 0 \Rightarrow z_G = 0$$

$$G \equiv O$$

So ΔABC is equilateral triangle.

SP.236. In ΔABC the following relationship holds:

$$m_a + m_b + m_c \leq 3R^2\left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow x^2 + y^2 + z^2 \geq xy + yz + zx \Rightarrow x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{2} - \frac{(x^2+y^2+z^2)}{2}$$

$\Rightarrow 3(x^2 + y^2 + z^2) \geq (x+y+z)^2 \Leftrightarrow$ ***put*** $x = m_a, y = m_b, z = m_c$

$$\Rightarrow 3(m_a^2 + m_b^2 + m_c^2) \geq (m_a + m_b + m_c)^2 \Rightarrow \frac{9}{4}(a^2 + b^2 + c^2) \geq (m_a + m_b + m_c)^2$$

$$\Rightarrow m_a + m_b + m_c \leq \frac{3}{2}\sqrt{a^2 + b^2 + c^2} \overset{Leibnitz}{\leq} \frac{3}{2}\sqrt{9R^2} = \frac{9R}{2}$$

$\Leftrightarrow$ ***Need to show*** $\Leftrightarrow \frac{9R}{2} \leq 3R^2\left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2}\right) \Rightarrow \Omega = \frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \geq \frac{3}{2R}$

$$\Leftrightarrow \Omega \overset{AM-GM}{\geq} 3\left(\frac{r_a r_b r_c}{a^2b^2c^2}\right)^{\frac{1}{3}} = 3\left(\frac{s^2r}{16R^2\Delta^2}\right)^{\frac{1}{3}} = 3\left(\frac{s^2r}{16R^2r^2s^2}\right)^{\frac{1}{3}} = 3\left(\frac{1}{16R^2r}\right)^{\frac{1}{3}}$$

$$\Leftrightarrow 3\left(\frac{1}{16R^2r}\right)^{\frac{1}{3}} \geq \frac{3}{2R} \Rightarrow \frac{1}{16R^2r} \geq \frac{1}{8R^3} \Rightarrow \frac{R}{r} \geq 2 \Rightarrow R \geq 2r \Rightarrow \textit{(Euler * true)}$$

$$\Leftrightarrow m_a + m_b + m_c \leq 3R^2\left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2}\right) \quad \textit{(Proved)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{r_a}{a^2} = \sum \frac{\left(\frac{1}{a}\right)^2}{\frac{1}{r_a}} \overset{Bergstrom}{\geq} \frac{\left(\sum \frac{1}{a}\right)^2}{\sum \frac{1}{r_a}} = \frac{\left(\frac{\sum ab}{4Rrs}\right)^2}{\frac{1}{r}} = \frac{r(\sum ab)^2}{16R^2r^2s^2} = \frac{(s^2+4Rr+r^2)^2}{16R^2rs^2}$$

$$\Rightarrow 3R^2\left(\sum \frac{r_a}{a^2}\right) \geq \frac{3[s^4 + 2s^2(4Rr+r^2) + (4Rr+r^2)^2]}{16rs^2} \overset{?}{\geq} 4R + r$$

$$\Leftrightarrow 3s^4 + 6s^2(4Rr+r^2) + 3(4Rr+r^2)^2 \underset{(1)}{\overset{?}{\geq}} 16s^2(4Rr+r^2)$$

Now, LHS of (1) $\overset{Gerretsen}{\geq} 3s^2(16Rr - 5r^2) + 6s^2(4Rr + r^2) + 3(4Rr + r^2)^2 \overset{?}{\geq}$

$$\geq 16s^2(4Rr + r^2) \Leftrightarrow s^2(8Rr - 16r^2) + 3(4Rr + r^2)^2 \underset{(2)}{\overset{?}{\geq}} 9r^2s^2$$

Now, LHS of (2) $\underset{(i)}{\overset{Gerretsen}{\geq}} (16Rr - 5r^2)(8Rr - 16r^2) + 3(4Rr + r^2)^2$ *and*

RHS of (2) $\underset{(ii)}{\overset{Gerretsen}{\leq}} 9r^2(4R^2 + 4Rr + 3r^2)$

(i), (ii) $\Rightarrow$ ***in order to prove (2), it suffices to show:***

$$(16Rr - 5r^2)(8Rr - 16r^2) + 3(4Rr + r^2)^2 \geq 9r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 5R^2 - 11Rr + 2r^2 \geq 0 \Leftrightarrow (R - 2r)(5R - r) \geq 0$$

$\rightarrow$ ***true*** $\because R \overset{Euler}{\geq} 2r \Rightarrow$ ***(2)*** $\Rightarrow$ ***(1) is true.*** $\Rightarrow 3R^2\left(\sum \frac{r_a}{a^2}\right) \geq 4R + r \overset{Bager}{\geq} \sum m_a$

$$\therefore \sum m_a \leq 3R^2\left(\sum \frac{r_a}{a^2}\right) \quad \textit{(Proved)}$$

Solution 3 by Bogdan Fustei-Romania

We know that: $m_a + m_b + m_c \leq r_a + r_b + r_c = 4R + r \leq \frac{9}{2}R$

$\left(4R + r \leq \frac{9}{2}R \Rightarrow 8R + 2r \leq 9R \Rightarrow 2r \leq R\right.$ ***– Euler's inequality)***

$a^2 = 2R\frac{h_bh_c}{h_a}$ *(and the analogs)* $\Rightarrow \frac{r_a}{a^2} = \frac{h_ar_a}{2Rh_bh_c}$ *(and the analogs)*

$$3R^2\left(\frac{r_a}{a^2}+\frac{r_b}{b^2}+\frac{r_c}{c^2}\right) = 3R^2\cdot\frac{1}{2R}\left(\frac{h_ar_a}{h_bh_c}+\frac{h_br_b}{h_ah_c}+\frac{h_cr_c}{h_ah_b}\right) =$$

$= \frac{3}{2}R\cdot\left(\frac{h_ar_a}{h_bh_c}+\frac{h_br_b}{h_ah_c}+\frac{h_cr_c}{h_ah_b}\right)$. *We will prove that:*

Using the inequality between the arithmetic mean and the geometric mean:

$$\frac{h_ar_a}{h_bh_c}+\frac{h_br_b}{h_ah_c}+\frac{h_cr_c}{h_ah_b} \geq 3\cdot\sqrt[3]{\frac{h_ar_ah_br_bh_cr_c}{h_ah_bh_bh_ch_ah_c}} = 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}$$

$$h_a = \frac{2S}{a}; h_b = \frac{2b}{b}; h_c = \frac{2b}{c}; abc = 4RS; S^2 = rr_ar_br_c$$

$$h_ah_bh_c = \frac{4S\cdot 2S^2}{4RS} = \frac{2S^2}{R} = \frac{2r}{R}\cdot r_ar_br_c \Leftrightarrow h_ah_bh_c = \frac{2r}{R}r_ar_br_c$$

$\Rightarrow \frac{R}{2r} = \frac{r_ar_br_c}{h_ah_bh_c}; R \geq 2r$ *(Euler's inequality)* $\Rightarrow \frac{r_ar_br_c}{h_ah_bh_c} \geq 1$

So, we have: $\frac{h_ar_a}{h_bh_c}+\frac{h_br_b}{h_ah_c}+\frac{h_cr_c}{h_ah_b} \geq 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}} = 3\sqrt[3]{\frac{R}{2r}} \geq 3$

So, $3R^2\left(\frac{r_a}{a^2}+\frac{r_b}{b^2}+\frac{r_c}{c^2}\right) \geq \frac{3}{2}R\cdot 3 = \frac{9}{2}R \geq 4R + r \geq \sum m_a$

SP.237. In ΔABC the following relationship holds:

$$3r(m_a+m_b+m_c) \leq Rs\left(\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

We have: $\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c} = \frac{s^2+(4R+r)^2}{4Rs}$ *(1)*

From Doucet's inequality: $(4R+r)^2 \geq 3s^2$ *(2)*

From (1)+(2)$\Rightarrow \frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c} \geq \frac{s}{R} \Rightarrow$ *we must show:* $3r(m_a+m_b+m_c) \leq s^2$ *(3)*

But $\frac{m_a}{h_a} \leq \frac{R}{2r} \Rightarrow m_a \leq \frac{R}{2r}h_a \Rightarrow m_a+m_b+m_c \leq \frac{R}{2r}(h_a+h_b+h_c)$ *(4)*

From (3)+(4) we must show: $\frac{3}{2}R(h_a+h_b+h_c) \leq s^2$ *(5)*

But $h_a + h_b + h_c = \frac{s^2+r^2+4Rr}{2R}$ *(6)*

From (5)+(6) we must show: $\frac{3}{4}(s^2 + r^2 + 4Rr) \le s^2 \Leftrightarrow 3s^2 + 3r^2 + 12Rr \le 4s^2 \Leftrightarrow$

$s^2 \ge 12Rr + 3r^2$ *(7)*

From Gerretsen's inequality: $s^2 \ge 16Rr - 5r^2$ *(8)*

From (7)+(8) we must show: $16Rr - 5r^2 \ge 12Rr + 3r^2$

$\Leftrightarrow 4Rr \ge 8r^2 \Leftrightarrow R \ge 2r$, *true (Euler)*

Solution 2 by Bogdan Fustei-Romania

In ΔABC *we have the following relationship:*

$$3r(m_a + m_b + m_c) \le Rs\left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c}\right)$$

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{r_a h_a}{2S} + \frac{r_b h_b}{2S} + \frac{r_c h_c}{2S} = \frac{r_a h_a + r_b h_b + r_c h_c}{2S}$$

$2S = h_a \cdot a = h_b \cdot b = h_c \cdot c$, $h_a = \left(1 + \frac{b+c}{a}\right) \cdot r$ *(and the analogs)*

$\frac{b+c}{a} = \frac{r_a+h_a}{r_a}$ *(and the analogs);* $r_a h_a = \left(1 + \frac{b+c}{a}\right) r_a r$

$r_a h_a = \left(1 + \frac{r_a+h_a}{r_a}\right) r_a r = (2r_a + h_a) r$ *(and the analogs)*

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{[2(r_a + r_b + r_c) + h_a + h_b + h_c] \cdot r}{2S}; S = s \cdot r$$

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{[2(r_a + r_b + r_c) + h_a + h_b + h_c]}{2s} \Bigg| \cdot Rs$$

$$Rs\left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c}\right) = \frac{R}{2}[(2r_a + 2r_b + 2r_c) + h_a + h_b + h_c]$$

The inequality from enunciation becomes:

$$3r(m_a + m_b + m_c) \le \frac{R}{2}[2(r_a + r_b + r_c) + h_a + h_b + h_c]$$

$$6r(m_a + m_b + m_c) \le 2R(r_a + r_b + r_c) + R(h_a + h_b + h_c)$$

$\frac{R}{2r} \ge \frac{m_a}{h_a}$ *(and the analogs)* $\Rightarrow R \cdot h_a \ge 2rm_a \Rightarrow R(h_a + h_b + h_c) \ge 2r\sum m_a$

We will prove that $4R(m_a + m_b + m_c) \le 2R(r_a + r_b + r_c)$

$m_a + m_b + m_c \le r_a + r_b + r_c = 4R + r$ *– true.*

$4r \le 2R \Rightarrow 2r \le R$ *(Euler's inequality)*

So, $4r(m_a+m_b+m_c)\le 2R(r_a+r_b+r_c)$ *is true.*

$2r(m_a+m_b+m_c)\le R(h_a+h_b+h_c)$ *is true.*

Finally, the inequality from enunciation is proved.

Solution 3 by Soumava Chakraborty-Kolkata-India

Firstly, $\sum ar_a=\sum 4R\sin\frac{A}{2}\cos\frac{A}{2}s\tan\frac{A}{2}=2Rs\sum 2\sin^2\frac{A}{2}=2Rs\sum(1-\cos A)$

$$=2Rs\left(3-1-\frac{r}{R}\right)\overset{(1)}{=}2s(2R-r)$$

Now, $Rs\sum\frac{r_a}{a}=Rs\sum\frac{ra^2}{ar_a}\overset{Bergstrom}{\ge}\frac{Rs(4R+r)^2}{\sum ar_a}$

$$\overset{by\ (1)}{=}\frac{Rs(4R+r)^2}{2s(2R-r)}\overset{Bager}{\ge}\frac{R(4R+r)(\sum m_a)}{2(2R-r)}\overset{?}{\ge}3r\sum m_a\Leftrightarrow 4R^2+Rr\overset{?}{\ge}12Rr-6r^2$$

$$\Leftrightarrow 4R^2-11Rr+6r^2\overset{?}{\ge}0\Leftrightarrow(R-2r)(4R-3r)\overset{?}{\ge}0$$

$$\to \textit{true} \because R\overset{Euler}{\ge}2r\therefore 3r\sum m_a\le Rs\sum\frac{r_a}{a}\quad \textit{(proved)}$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

We have: $r_a+r_b+r_c=4R+r$

$$m_a+m_b+m_c\le r_a+r_b+r_c;\ ar_a+br_b+cr_c=$$

$$=s\left(a\tan\left(\frac{A}{2}\right)+b\tan\left(\frac{B}{2}\right)+c\tan\left(\frac{C}{2}\right)\right)=$$

$=2s(2R-r)$. *Using Schwarz's inequality, we have:*

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}=\frac{r_a^2}{ar_a}+\frac{r_b^2}{br_b}+\frac{r_c^2}{cr_c}\ge\frac{(r_a+r_b+r_c)^2}{ar_a+br_b+cr_c}=\frac{(4R+r)^2}{2s(2R-r)}$$

$$\to RHS\ge\frac{R(4R+r)^2}{2(2R-r)}$$

We must show that: $\frac{R(4R+r)^2}{2(2R-r)}\ge 3r(4R+r)\ge LHS$

$$R(4R+r)\ge 6r(2R-r)$$

$$4R^2-11Rr+6r^2\ge 0$$

$$(4R-3r)(R-2r)\ge 0$$

It is true because: $R\ge 2r\to 4R-3r\ge 8r-3r=5r>0$ *(proved)*

SP.238. Let a, b, c be positive real numbers such that

$$\min\{b+c, c+a, a+b\} > \max\{a, b, c\}. \text{ Prove that:}$$

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ ***and*** $\min\{a+b, b+c, c+a\} > \max\{a, b, c\}$, ***we have:***

$a+b > c, b+c > a, c+a > b$ ***consider because*** $2(a^2+b^2+c^2)(ab+bc+ca) \geq$

$$\geq 3((ab)^2+(bc)^2+(ca)^2+a^2bc+ab^2c+abc^2) \Rightarrow$$

$$\Rightarrow 4\left((a^2+b^2+c^2)(ab+bc+ca)\right)^2 \geq$$

$$\geq 9((ab)^2+(bc)^2+(ca)^2+a^2bc+ab^2c+abc^2)^2$$

$$\Rightarrow 4(a^2+b^2+c^2)^3(ab+bc+ca) \geq$$

$$\geq 9((ab)^2+(bc)^2+(ca)^2+a^2bc+ab^2c+abc^2)^2$$

$$\Rightarrow \frac{4(a^2+b^2+c^2)^4}{((ab)^2+(bc)^2+(ca)^2+a^2bc+ab^2c+abc^2)^2} \geq \frac{9(a^2+b^2+c^2)}{(ab+bc+ca)}$$

$$\Rightarrow \frac{2(a^2+b^2+c^2)^2}{(ab)^2+(bc)^2+(ca)^2+a^2bc+ab^2c+abc^2} \geq 3\sqrt{\frac{(a^2+b^2+c^2)}{(ab+bc+ca)}}$$

$$\Rightarrow \frac{4(a^2+b^2+c^2)^2}{2(ab)^2+(bc)^2+(ca)^2+a^2bc+ab^2c+abc^2} \geq 3\sqrt{\frac{(a^2+b^2+c^2)}{(ab+bc+ca)}}$$

$$\Rightarrow 4\left(\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2}\right) \geq 3\sqrt{\frac{(a^2+b^2+c^2)}{(ab+bc+ca)}}$$

$$\Rightarrow \left(\frac{2a}{b+c}\right)^2 + \left(\frac{2b}{c+a}\right)^2 + \left(\frac{2c}{a+b}\right)^2 \geq 3\sqrt{\frac{(a^2+b^2+c^2)}{(ab+bc+ca)}}$$

$$\Rightarrow \left(\sqrt{\frac{a}{b+c-a}}\right)^2 + \left(\sqrt{\frac{b}{c+a-b}}\right)^2 + \left(\sqrt{\frac{c}{a+b-c}}\right)^2 \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\Rightarrow \frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$ ***ok. Therefore, it is to be true.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\min\{b+c,c+a,a+b\}\geq\max\{a,b,c\}\Rightarrow b+c>a,c+a>b,a+b>c$$

$\Rightarrow a,b,c$ ***are sides of a triangle.***

$$\sum\frac{a}{b+c-a}=\frac{1}{2}\sum\frac{a}{s-a}=\frac{1}{2}\sum\frac{a-s+s}{s-a}=\frac{1}{2}\left(-3+\frac{s\sum(s-b)(s-c)}{sr^2}\right)$$

$$=\frac{1}{2}\left(-3+\frac{\sum s(s^2-s(b+c)+bc)}{r^2}\right)=\frac{1}{2}\left(-3+\frac{3s^2-4s^2+s^2+4Rr+r^2}{r^2}\right)$$

$$=\frac{1}{2}\left(-3+\frac{4R+r}{r}\right)=\frac{2R-r}{r}\geq 3\sqrt{\frac{\sum a^2}{\sum ab}}\Leftrightarrow\left(\frac{2R-r}{r}\right)^2\geq\frac{18(s^2-4Rr-r^2)}{s^2+4Rr+r^2}$$

$$\Leftrightarrow(2R-r)^2(s^2+4Rr+r^2)\geq 18r^2(s^2-4Rr-r^2)$$

$$\Leftrightarrow(4R^2-4Rr-8r^2)s^2+(2R-r)^2(4Rr+r^2)+18r^2(4Rr+r^2)\overset{(1)}{\geq}9r^2s^2$$

$$\because 4R^2-4Rr-8r^2=4(R+r)(R-2r)\overset{Euler}{\geq}0$$ $\therefore$ ***LHS of (1)*** $\underset{(a)}{\overset{Gerretsen}{\geq}}$

$\geq(4R^2-4Rr-8r^2)(16Rr-5r^2)+(2R-r)^2(4R+r^2)+18r^2(4Rr+r^2)$ ***and,***

RHS of (1) $\underset{(a)}{\overset{Gerretsen}{\leq}} 9r^2(4R^2+4Rr+3r^2)$

(a), (b) $\Rightarrow$ ***in order to prove (1), it suffices to prove:***

$$(4R^2-4Rr-8r^2)(16Rr-5r^2)+(2R-r)^2(4Rr+r^2)+18r^2(4Rr+r^2)\geq$$

$$\geq 9r^2(4R^2+4Rr+3r^2)\Leftrightarrow 20t^3-33t^2-18t+8\geq 0\left(t=\frac{R}{r}\right)$$

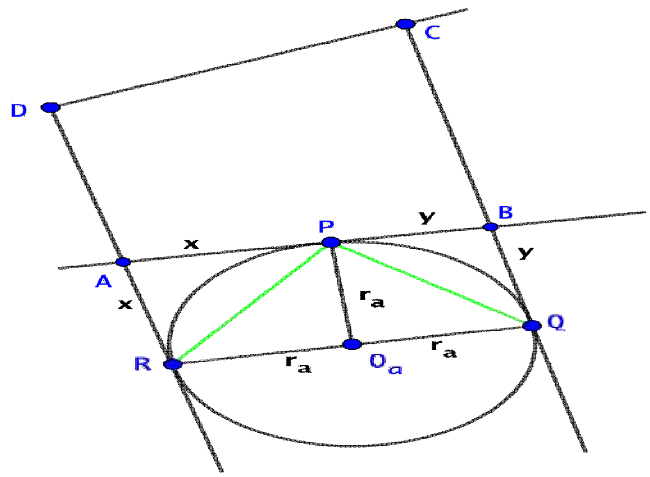
$\Leftrightarrow(t-2)(20t^2+7(t-2)+10)\geq 0\rightarrow$ ***true*** $\because t\overset{Euler}{\geq}2\Rightarrow$ ***(1)*** $\Rightarrow$ ***proposed inequality is true (Proved)***

SP.239 If a,b,c,d are sides in a cyclic quadrilateral, r_a,r_b,r_c,r_d – exradii, s – semiperimeter then:

$$\frac{a}{r_a^2}+\frac{b}{r_b^2}+\frac{c}{r_c^2}+\frac{d}{r_d^2}\geq\frac{32}{s}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India



Let $AB = a, BC = b, CD = c, DA = d$

$$\because \angle O_aPA + \angle O_aRA = 90^\circ + 90^\circ = 180^\circ$$

$$\therefore \angle PO_aR = 180^\circ - \angle PAR$$

($\because$ APO_aR ***is a cyclic quadrilateral)***

$$= 180^\circ - (180^\circ - A) = A$$

using cosine law on ΔPAR ***and*** ΔPO_aR***,***

$$x^2 + x^2 - 2x^2\cos(180^\circ - A) = r_a^2 + r_a^2 - 2r_a^2\cos A \Rightarrow 2x^2 \cdot 2\cos^2\frac{A}{2} = 2r_a^2 \cdot 2\sin^2\frac{A}{2}$$

$$\Rightarrow x \overset{(1)}{=} r_a\tan\tfrac{A}{2}\text{. Similarly, } \angle PO_aQ = B \text{ and } \therefore y \overset{(2)}{=} r_a\tan\tfrac{B}{2}$$

$$(1)+(2) \Rightarrow r_a\left(\tan\tfrac{A}{2} + \tan\tfrac{B}{2}\right) = x + y = a \Rightarrow \frac{1}{r_a} \overset{(a)}{=} \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a}$$

$$\text{Similarly, } \frac{1}{r_b} \overset{(b)}{=} \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b}, \frac{1}{r_c} \overset{(c)}{=} \frac{\tan\frac{C}{2}+\tan\frac{D}{2}}{c} \text{ and } \frac{1}{r_d} \overset{(d)}{=} \frac{\tan\frac{D}{2}+\tan\frac{A}{2}}{d}$$

(a), (b), (c), (d)$\Rightarrow$ ***LHS***

$$= \frac{\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)^2}{a} + \frac{\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)^2}{b} + \frac{\left(\tan\frac{C}{2} + \tan\frac{D}{2}\right)^2}{c} + \frac{\left(\tan\frac{D}{2} + \tan\frac{A}{2}\right)^2}{d}$$

$$\overset{A-G}{\geq} 4\left(\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{a} + \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{b} + \frac{\tan\frac{C}{2}\tan\frac{D}{2}}{c} + \frac{\tan\frac{D}{2}\tan\frac{A}{2}}{d}\right)$$

$$\overset{A-G}{\geq} 16\sqrt[4]{\frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}\tan^2\frac{D}{2}}{abcd}}$$

$$= 16\sqrt[4]{\frac{(s-a)(s-d)}{(s-b)(s-c)}\cdot\frac{(s-b)(s-a)}{(s-c)(s-d)}\cdot\frac{(s-c)(s-b)}{(s-a)(s-d)}\cdot\frac{(s-d)(s-c)}{(s-a)(s-b)}\cdot\frac{1}{abcd}}$$

$$= 16\sqrt[4]{\frac{1}{abcd}} \overset{?}{\geq} \frac{32}{s} \Leftrightarrow s^4 \overset{?}{\geq} 16abcd \Leftrightarrow (a+b+c+d)^4 \overset{?}{\geq} 256abcd$$

$$\Leftrightarrow a+b+c+d \overset{?}{\geq} 4\sqrt[4]{abcd} \rightarrow \textit{true by A-G (Hence proved)}$$

SP.240 If a, b, c, d are sides in a cyclic quadrilateral, r_a, r_b, r_c, r_d – exradii, s – semiperimeter then:

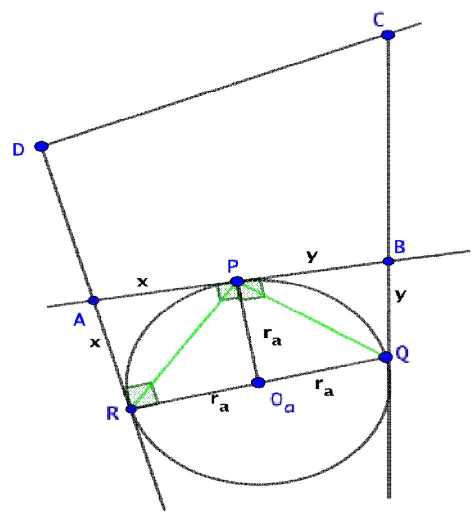
$$\frac{r_a^2}{a^3}+\frac{r_b^2}{b^3}+\frac{r_c^2}{c^3}+\frac{r_d^2}{d^3} \geq \frac{2}{s}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{r_a^2}{a^3} \geq \frac{2}{s} = \frac{4}{\sum a} \Leftrightarrow \left(\sum a\right)\left(\sum\frac{r_a^2}{a^3}\right) \overset{(i)}{\geq} 4$$

Now, $\left(\sum a\right)\left(\sum\frac{r_a^2}{a^3}\right) \overset{CBS}{\geq} \left(\sum\frac{r_a}{a}\right)^2 \overset{?}{\geq} 4 \Leftrightarrow \sum\frac{r_a}{a} \underset{(ii)}{\overset{?}{\geq}} 2$



Let $AB = a, BC = b, CD = c, DA = d$

$\because \angle O_aPA + \angle O_aRA = 90^\circ + 90^\circ = 180^\circ$

($\therefore APO_aR$ is a cyclic quadrilateral)

$\therefore \angle PO_aR = 180^\circ - (180^\circ - A) = A$

using cosine law on ΔPAR and ΔPO_aR,

$$x^2 + x^2 - 2x^2\cos(180^\circ - A) = r_a^2 + r_a^2 - 2r_a^2\cos A \Rightarrow 2x^2 \cdot 2\cos^2\frac{A}{2} = 2r_a^2 \cdot 2\sin^2\frac{A}{2}$$

$\Rightarrow x \overset{(1)}{=} r_a \tan\frac{A}{2}$. *Similarly,* $\angle PO_aQ = B$ *and* $\therefore y \overset{(2)}{=} r_a \tan\frac{B}{2}$

(1)+(2)$\Rightarrow r_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = x + y = a \Rightarrow \frac{r_a}{a} = \frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}} = \frac{1}{\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}+\sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}}}$

$$= \frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(s-a)(s-d) + (s-a)(s-b)} = \frac{\Delta}{(s-a)(a+c)} \Rightarrow \frac{r_a}{a} \overset{(a)}{=} \frac{\Delta}{(s-a)(a+c)}$$

Now, $\frac{r_b}{b} = \frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}$ *(similarly)*

$$= \frac{1}{\sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}} + \sqrt{\frac{(s-c)(s-b)}{(s-a)(s-d)}}} \overset{(b)}{=} \frac{\Delta}{(s-b)(b+d)}$$

Again, $\frac{r_c}{c} = \frac{1}{\tan\frac{C}{2}+\tan\frac{D}{2}} = \frac{1}{\sqrt{\frac{(s-c)(s-b)}{(s-a)(s-d)}}+\sqrt{\frac{(s-d)(s-c)}{(s-a)(s-b)}}} \overset{(c)}{=} \frac{\Delta}{(s-c)(a+c)}$

Also, $\frac{r_d}{d} = \frac{1}{\tan\frac{D}{2}+\tan\frac{A}{2}} = \frac{1}{\sqrt{\frac{(s-d)(s-c)}{(s-a)(s-b)}}+\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}} \overset{(d)}{=} \frac{\Delta}{(s-d)(b+d)}$

(a)+(b)+(c)+(d)$\Rightarrow$

$$\sum\frac{r_a}{a} = \frac{\Delta}{a+c}\left(\frac{1}{s-a} + \frac{1}{s-c}\right) + \frac{\Delta}{b+d}\left(\frac{1}{s-b} + \frac{1}{s-d}\right) =$$

$$= \frac{\Delta(b+d)}{(s-a)(s-c)(a+c)} + \frac{\Delta(a+c)}{(s-b)(s-d)(b+d)}$$

$$\overset{A-G}{\geq} 2\sqrt{\frac{\Delta^2}{(s-a)(s-b)(s-c)(s-d)}} = 2\sqrt{\frac{\Delta^2}{\Delta^2}} = 2$$

$\Rightarrow$ *(ii)* $\Rightarrow$*(i)* $\Rightarrow$ *given inequality is true (Proved)*

UP.226. Find all positive real numbers (x, y, z) such that:

$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ x^3y + y^3z + z^3x = \dfrac{2x}{y^2 + z^2} + \dfrac{2y}{z^2 + x^2} + \dfrac{2z}{x^2 + y^2} \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Ruangkhaw Chaoka-Chiangrai-Thailand

$$x^2 + y^2 + z^2 = 3 \quad (a)$$

$$x^3y + y^3z + z^3x = \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \quad (b)$$

Lemma. $\frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \geq 3$ ***(1) (hold at*** $x = y = z = 1$***)***

Proof. $\frac{2x}{y^2+z^2} = \frac{2x}{3-x^2} \overset{??}{\geq} x^2 \Leftrightarrow x^4 - 3x^2 + 2x = x(x-1)^2(x+2) \geq 0$ ***true***

(hold at $x = 1$***)***

Similarly: $\frac{2y}{z^2+x^2} \geq y^2$ ***(holds at*** $y = 1$***) and*** $\frac{2z}{x^2+y^2} \geq z^2$ ***(holds at*** $z = 1$***)***

$\therefore \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \geq x^2 + y^2 + z^2 = 3$ ***(holds at*** $x = y = z = 1$***)***

(Vasile Cîrtoaje, 1992) $(x^2 + y^2 + z^2)^2 \geq 3(x^3y + y^3z + z^3x)$

(holds at $x^2 - xy + yz = y^2 - yz + zx = z^2 - zx + xy$)

$$\frac{1}{2}\left\{\begin{matrix} [(x^2 - xy + yz) - (y^2 - yz + zx)]^2 + [(y^2 - yz + zx) - (z^2 - zx + xy)]^2 + \\ +[(z^2 - zx + xy) - (x^2 - xy + yz]^2 \end{matrix}\right\} \geq 0 \quad (2)$$

(1), (2): $\frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \geq 3 = \frac{(x^2+y^2+z^2)^2}{3} \geq (x^3y + y^3z + z^3x)$

(holds at $x = y = z = 1$***)***

$\therefore$ ***(b) is the hold point of inequality at*** $x = y = z = 1$

UP.227. If $x_n = \sum_{k=1}^{n} \sqrt[k+1]{1 + \frac{1}{k}}; n \in \mathbb{N}; n \geq 1$**, then find:**

$$\Omega = \lim_{n\to\infty}\left(\log\left(\sum_{k=1}^{n} x_k^2\right) - 3H_n\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}\left(\ln\left(\sum_{k=1}^{n} x_k^2\right) - 3H_n\right) =$$

$$= \lim_{n\to\infty} \ln\left(\sum_{k=1}^{n} x_k^2\right) - 3\ln n = 3\lim_{n\to\infty}(H_n - \ln n) \quad (1)$$

$$\lim_{n\to\infty}(H_n - \ln n) = \gamma \quad (2)$$

$$\lim_{n\to\infty} \ln\left(\sum_{k=1}^{n} x_k^2\right) - \ln n^3 = \lim_{n\to\infty} \ln\left(\frac{\sum_{k=1}^{n} x_k^2}{n^3}\right) = \ln\left(\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n} x_k^2}{n^3}\right)\right) \quad (3)$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n} x_k^2}{n^3} \overset{C.S.}{=} \lim_{n\to\infty}\frac{x_{n+1}^2}{(n+1)^3 - n^3} = \lim_{n\to\infty}\frac{x_{n+1}^2}{3n^2+3n+1} = \lim_{n\to\infty}\left(\frac{x_{n+1}}{n+1}\right)^2 \cdot \frac{(n+1)^2}{3n^2+3n+1}$$

$$= \frac{1}{3}\left(\lim_{n\to\infty}\frac{x_{n+1}}{n+1}\right)^2 = \frac{1}{3}\left(\lim_{n\to\infty}\frac{x_n}{n}\right)^2 \quad (4)$$

$$\lim_{n\to\infty}\frac{x_n}{n} \overset{C.S.}{=} \lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1-n} = \lim_{n\to\infty}\sqrt[n+2]{1+\frac{1}{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{\frac{1}{n+2}} = 1 \quad (5)$$

From (1)+(2)+(3)+(4)+(5)$\Rightarrow \Omega = \ln\frac{1}{3} - 3\gamma = -(\ln 3 + 3\gamma)$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\left(\ln\left(\sum_{k=1}^{n} x_k^2\right) - 3(H_n - \ln(n) + \ln(n))\right) = -3\gamma + \lim_{n\to\infty}\ln\left(\frac{\sum_{k=1}^{n} x_k^2}{n^3}\right) \overset{Stolz\ Cesaro}{=}$$

$$= -3\gamma + \lim_{n\to\infty}\ln\left(\frac{x_{n+1}^2}{3n^2+3n+1}\right) = -3\gamma + 2\lim_{n\to\infty}\ln\left(\frac{x_{n+1}}{\sqrt{3n^2+3n+1}}\right) \overset{Stolz\ Cesaro}{=}$$

$$= -3\gamma + 2\lim_{n\to\infty}\ln\left(\frac{\sqrt[n+3]{1+\frac{1}{n+1}}}{\frac{6n+6}{\sqrt{3n^2+9n+7}+\sqrt{3n^2+3n+1}}}\right) =$$

$$= -3\gamma + 2\lim_{n\to\infty}\ln\left(\frac{1}{\frac{n\left(6+\frac{6}{n}\right)}{n\left(\sqrt{3+\frac{9}{n}+\frac{7}{n^2}}+\sqrt{3+\frac{3}{n}+\frac{1}{n^2}}\right)}}\right) =$$

$$= -3\gamma + 2\ln\left(\frac{1}{\sqrt{3}}\right) = -3\gamma - \ln 3 \Rightarrow \Omega = -3\gamma - \ln(3)$$

Solution 3 by Ali Jaffal-Lebanon

By Bernoulli's inequality we have:

$(1+n)^{\alpha} \le 1+\alpha n$ *for* $x \ge -1$ *and* $0 \le \alpha \le 1$, *so,*

$$\left(1+\frac{1}{k}\right)^{\frac{1}{k+1}} \le 1+\frac{1}{k(k+1)} = 1+\frac{1}{k}-\frac{1}{k+1}$$

then $1 \le \left(1+\frac{1}{k}\right)^{\frac{1}{k+1}} \le 1+\frac{1}{k}-\frac{1}{k+1}$ *then* $n \le x_n \le n+1-\frac{1}{n+1}$

so, $n \le x_n \le n+1$ *and* $n^2 \le x_n^2 \le (n+1)^2$

then $\sum_{k=1}^{k=n} k^2 = \sum_{k=1}^{k=n} x_k^2 \le \sum_{k=1}^{k=n} (k+1)^2$

$$\frac{n(n+1)(2n+1)}{6} \le \sum_{k=1}^{k=n} x_k^2 \le \frac{n(n+1)(2n+1)}{6} - 14(n+1)^2$$

We have: $\lim_{n\to+\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$. *Then* $\lim_{n\to\infty} n^{\frac{1}{3}} \sum_{k=1}^{k=n} x_k^2 = \frac{1}{3}$

by $\Omega_n = \log\left(\sum_{k=1}^{k=n} x_k^2\right) - 3H_n = \log\left(\sum_{k=1}^{k=n} x_k^2\right) - 3\ln n - 3\gamma + \varphi'(n)$

Since $H_n = \ln n + \gamma + \varphi(n) = \log\left(n^{\frac{1}{3}\sum_{k=1}^{k=n} x_k^2}\right) - 3\gamma + \varphi'(n)$

$$\lim_{n\to+\infty} \Omega_n = \log\left(\frac{1}{3}\right) - 3\gamma$$

UP.228. If $f:\mathbb{R}\to(0,\infty)$, f continuous; $a,b\in\mathbb{R}$; $a\le b$ then:

$$\int_a^b\int_a^b \left(\frac{\left(f^2(x)+f(y)\right)\left(f^2(y)+f(x)\right)}{\left(1+f(x)\right)\left(1+f(y)\right)}\right) dx\,dy \ge \left(\int_a^b f(x)\,dx\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$f:\mathbb{R}\to(0,\infty)$ *and* f *is continuous. Hence* $f(x)>0$ *for all* $x\in\mathbb{R}$

$\left(f^2(x)+f(y)\right)\left(1+\frac{1}{f(y)}\right) \overset{CAUCHY\ SCHWARZ}{\ge} \left(1+f(x)\right)^2$ *where* $x,y\in\mathbb{R}$

$\Rightarrow \frac{f^2(x)+f(y)}{1+f(y)} \ge f(y)\left(\frac{1+f(x)}{1+f(y)}\right)^2$ *similarly,* $\frac{f^2(y)+f(x)}{1+f(x)} \ge f(x)\left(\frac{1+f(y)}{1+f(x)}\right)^2$

$$\therefore \int_a^b\int_a^b \frac{\left(f^2(x)+f(y)\right)\left(f^2(y)+f(x)\right)}{(1+f(x))(1+f(y))}dx\,dy \ge \int_a^b\int_a^b f(y)\left(\frac{1+f(x)}{1+f(y)}\right)^2\cdot f(x)\left(\frac{1+f(y)}{1+f(x)}\right)^2 dx\,dy$$

$$=\left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right)=\left(\int_a^b f(x)dx\right)^2$$

(proved)

Solution 2 by Nassim Nicholas Taleb-USA

$g=\frac{\left(f(x)^2+f(y)\right)\left(f(x)+f(y)^2\right)}{(f(x)+1)(f(y)+1)}$. ***Rewrite, with*** $f(x)=v, f(y)=w$***:***

$$g=\frac{(v^2+w)(v+w^2)}{(1+v)(1+w)}, v,w>0$$

We have $g\ge vw$***, with identity for*** $v=w$

So, $\int_a^b\int_a^b\left(\frac{\left(f(x)^2+f(y)\right)\left(f(x)+f(y)^2\right)}{(f(x)+1)(f(y)+1)}\right)dy\,dx\ge\int_a^b\int_a^b f(x)\,f(y)\,dx\,dy=$

$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right)=\left(\int_a^b f(x)\,dx\right)^2$ ***as required.***

Note:

$$\frac{(v^2+w)(v+w^2)}{(1+v)(1+w)}-vw=\frac{(v-w)^2(v+w)}{(1+v)(1+w)}\ge 0$$

UP.229. Calculate the limit:

$$\Omega=\lim_{x\to\infty}\left(x^2\int_x^{x+\frac{2}{x}}\arctan\left(\frac{1}{t}\right)dt\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Florentin Vișescu-Romania

$$\exists\alpha\in\left(x;x+\frac{2}{x}\right)ax$$

$$\int_x^{x+\frac{2}{x}} \arctan\frac{1}{t}dt = \left(x+\frac{2}{x}-x\right)\arctan\alpha = \frac{2}{x}\arctan\alpha$$

$$0 < x < \alpha < x+\frac{2}{x} \qquad \frac{1}{x} > \frac{1}{\alpha} > \frac{x}{x^2+2}$$

$$\arctan\frac{1}{x} > \arctan\frac{1}{\alpha} > \arctan\frac{x}{x^2+2}$$

$$2x\arctan\frac{1}{x} > \underbrace{2x\arctan\frac{1}{\alpha}}_{x^2\int_x^{x+\frac{2}{x}}\arctan\frac{1}{t}dt} > 2x\arctan\frac{x}{x^2+2}$$

$$\lim_{n\to\infty} x^2\int_x^{x+\frac{2}{x}} \arctan\frac{1}{t}dt = 2$$

$$\Omega = 2$$

Solution 2 by Ali Jaffal-Lebanon

Let $I(x) = x^2\int_x^{x+\frac{2}{x}}\arctan\left(\frac{1}{t}\right)dt, x > 0$

We have: $x < t < x+\frac{2}{x}$ *then:*

$$x < t < \frac{x^2+2}{x}$$

$$\frac{x}{x^2+2} < \frac{1}{t} < \frac{1}{x}$$

But $\varphi(t) = \arctan t$ *is increasing on* $]-\infty,\infty[$

So, $\arctan\left(\frac{x}{x^2+2}\right) < \arctan\left(\frac{1}{t}\right) < \arctan\left(\frac{1}{x}\right)$

then $2x\arctan\left(\frac{x}{x^2+2}\right) < I(x) < 2x\arctan\left(\frac{1}{x}\right)$

we have $\lim_{u\to 0}\frac{\arctan(u)}{u} = 1$

so, $\lim_{x\to+\infty} 2x\arctan\left(\frac{1}{x}\right) = \lim_{x\to\infty} 2\frac{\arctan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to 0} 2\frac{\arctan(u)}{u} = 2$

and $\lim_{x\to+\infty} 2x\arctan\left(\frac{x}{x^2+2}\right) = \lim_{u\to 0^+}\frac{2}{u}\arctan\left(\frac{u}{2u^2+1}\right)$

$$= \lim_{u\to 0}\frac{2}{u}\times\frac{u}{2u^2+1}\times\frac{\arctan\left(\frac{u}{2u+1}\right)}{\frac{2a}{2u^2+1}} = 2\times 1 = 2$$

Then by sandwich theorem: $\lim_{n\to\infty} I(x) = \Omega = 2$

UP.230. If $a, b > 1$ then:

$$\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}}\cdot\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}}\geq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$. *We are to prove:* $H^{G-A}\cdot G^{A-H}\cdot A^{H-G}\geq 1$

$$\Leftrightarrow \frac{H^G}{H^A}\cdot\frac{G^A}{G^H}\cdot\frac{A^H}{A^G}\geq 1 \Leftrightarrow \left(\frac{H}{A}\right)^G\left(\frac{G}{H}\right)^A\left(\frac{A}{G}\right)^H\geq 1$$

$$\Leftrightarrow \left(\frac{H}{A}\right)^G\left(\frac{A}{G}\right)^A\left(\frac{A}{G}\right)^H\geq 1\ \left(\because \frac{G}{H}=\frac{A}{G}\ as\ G^2 = AH\right)$$

$$\Leftrightarrow \left(\frac{H}{A}\right)^G\left(\frac{A}{\sqrt{AH}}\right)^{A+H}\geq 1\ \ (\because G=\sqrt{AH}) \Leftrightarrow \left(\frac{H}{A}\right)^G\left(\sqrt{\frac{A}{H}}\right)^{A+H}\overset{(1)}{\geq} 1$$

Now, $\left(\sqrt{\frac{A}{H}}\right)^{A+H}\overset{(2)}{\geq}\left(\sqrt{\frac{A}{H}}\right)^{2G}\Leftrightarrow (A+H)\log\sqrt{\frac{A}{H}}\geq 2G\log\sqrt{\frac{A}{H}}$

$$\Leftrightarrow \left(\log\sqrt{\frac{A}{H}}\right)(A+H-2G)\geq 0 \to \textit{true} \because \log\sqrt{\frac{A}{H}}\geq 0\ \ (as\ \sqrt{\frac{A}{H}}\geq 1)$$

and $A+H-2G \overset{A-G}{\geq} 2\sqrt{AH}-2G = 2G-2G = 0$

$\therefore \left(\frac{H}{A}\right)^G\left(\sqrt{\frac{A}{H}}\right)^{A+H}\overset{by\ (2)}{\geq}\left(\frac{H}{A}\right)^G\left(\sqrt{\frac{A}{H}}\right)^{2G} = \left(\frac{H}{A}\right)^G\left(\frac{A}{H}\right)^G = 1 \Rightarrow$ ***(1)*** $\Rightarrow$ ***given inequality is true***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = \ln x$ *for all* $x > 1, f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2} < 0$ *for all* $x > 1$ *hence* f *is a concave function and* $\frac{a+b}{2}\geq\sqrt{ab}\geq\frac{2ab}{a+b}$

$$\frac{f(\sqrt{ab})-f\left(\frac{2ab}{a+b}\right)}{\sqrt{ab}-\frac{2ab}{a+b}}\geq\frac{f\left(\frac{a+b}{2}\right)-f\left(\frac{2ab}{a+b}\right)}{\frac{a+b}{2}-\frac{2ab}{a+b}}$$

$$\Rightarrow\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)f(\sqrt{ab})+f\left(\frac{2ab}{a+b}\right)\left(\frac{2ab}{a+b}-\frac{a+b}{2}\right)\geq$$

$$\geq\left(\sqrt{ab}-\frac{2ab}{a+b}\right)f\left(\frac{a+b}{2}\right)-\left(\sqrt{ab}-\frac{2ab}{a+b}\right)f\left(\frac{2ab}{a+b}\right)$$

$$\Rightarrow\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)f(\sqrt{ab})+\left(\sqrt{ab}-\frac{a+b}{2}\right)f\left(\frac{2ab}{a+b}\right)+\left(\frac{2ab}{a+b}-\sqrt{ab}\right)f\left(\frac{a+b}{2}\right)\geq 0$$

$$\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\ln(\sqrt{ab})+\left(\sqrt{ab}-\frac{a+b}{2}\right)\ln\left(\frac{2ab}{a+b}\right)+\left(\frac{2ab}{a+b}-\sqrt{ab}\right)\ln\left(\frac{a+b}{2}\right)\geq 0$$

$$\Rightarrow\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}\frac{a+b}{2}}\cdot\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}}\geq 1\quad \textit{(Proved)}$$

Solution 3 by Marian Ursărescu Romania

Let $\frac{2ab}{a+b}=x,\sqrt{ab}=y,\frac{a+b}{2}=z\Rightarrow x<y<z$. *We must show:* $x^{y-z}\cdot y^{z-x}\cdot z^{x-y}\geq 1\Leftrightarrow$

$$\ln(x^{y-z}\cdot y^{z-x}\cdot z^{x-y})\geq 0\Leftrightarrow(y-z)\ln x+(z-x)\ln y+(x-y)\ln z\geq 0\quad (1)$$

Theorem: Let $f:I\to\mathbb{R},I\subset\mathbb{R},I$ *– internal* f *its concave* $\Leftrightarrow\forall x,y,z\in I$ *with:* $x<y<z$

we have: $\frac{f(x)-f(y)}{x-y}\geq\frac{f(y)-f(z)}{y-z}$. *In our case* $f(x)=\ln x$

From theorem $\Rightarrow(y-z)f(x)+(z-x)f(y)+(x-y)f(z)\geq 0$

Let $f(x)=\ln x\Rightarrow$ ***(1)*** *it's true.*

UP.231. Prove that for any acute triangle ABC the following inequality holds:

$$\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\tan C}+\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\geq 3\sqrt[4]{3}+\frac{3}{\sqrt[4]{3}}$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

Let $\sin A=x,\sin B=y,\sin C=z$

$\cos A=a,\cos B=b,\cos C=c$

We have:

1. $x+y+z\le\frac{3\sqrt{3}}{2}$

2. $a+b+c\le\frac{3}{2}$

3. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\ge 3\sqrt{3}$

4. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\ge\sqrt{3}$

Consider $\frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}{3}\le x+y+z\le\frac{3\sqrt{3}}{2}\Rightarrow\sqrt{x}+\sqrt{y}+\sqrt{z}\le\frac{3\sqrt[4]{3}}{\sqrt{2}}$

and $\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{3}\le a+b+c\le\frac{3}{2}\Rightarrow\sqrt{a}+\sqrt{b}+\sqrt{c}\le\frac{3}{\sqrt{2}}$

by using a method above, we have: $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}\ge 3\sqrt[4]{3}$ ***and*** $\sqrt{\frac{a}{x}}+\sqrt{\frac{b}{y}}+\sqrt{\frac{c}{z}}\ge\frac{3}{\sqrt[4]{3}}$

Hence $\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\tan C}+\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\ge 3\sqrt[4]{3}+\frac{3}{\sqrt[4]{3}}$

Therefore, it is true.

Solution 2 by Tran Hong-Dong Thap-Vietnam

Let $f(x)=\sqrt{\tan x}+\sqrt{\cot x},\left(0<x<\frac{\pi}{2}\right)$

$$\to f'(x)=\frac{1}{2}\csc x\sec x\left(\sqrt{\tan x}-\sqrt{\cot x}\right)$$

$$\to f''(x)=\frac{1}{4}\left(\sqrt{\cot x}\left[(\csc x\sec x)^2+\frac{2\cos 2x}{(\cos x\sin x)^2}\right]+\sqrt{\tan x}\left[(\csc x\sec x)^2-\frac{2\cos 2x}{(\cos x\sin x)^2}\right]\right)$$

$$=\frac{1}{4}\left((\csc x\sec x)^2\left[\sqrt{\cot x}+\sqrt{\tan x}\right]+\frac{2\cos 2x}{(\cos x\sin x)^2}\left[\sqrt{\cot x}-\sqrt{\tan x}\right]\right)$$

$$=\frac{1}{4}\left((\csc x\sec x)^2\left[\sqrt{\cot x}+\sqrt{\tan x}\right]+\frac{2\cos 2x}{(\cos x\sin x)^2}\cdot\frac{\cot x-\tan x}{\sqrt{\cot x}+\sqrt{\tan x}}\right)$$

$$=\frac{1}{4}\left((\csc x\sec x)^2\left[\sqrt{\cot x}+\sqrt{\tan x}\right]+\frac{2\cos^2 2x}{(\cos x\sin x)^3}\cdot\frac{1}{\sqrt{\cot x}+\sqrt{\tan x}}\right)>0$$

(with $0<x<\frac{\pi}{2}$***)***

Using Jensen's inequality with $0<A,B,C<\frac{\pi}{2}$ ***we have:***

$$f(A)+f(B)+f(C)\ge 3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)$$

$$=3\left(\sqrt{\tan\frac{\pi}{3}}+\sqrt{\cot\frac{\pi}{3}}\right)=3\left(\frac{1}{\sqrt[4]{3}}+\sqrt[4]{3}\right)=RHS$$

Proved. Equality if and only if $A = B = C = \frac{\pi}{3}$.

UP.232. If $a, b, c > 0$; $a + b + c = 3$ **then:**

$$\left(1+\frac{1}{a}\right)^{a^2+2ac}\cdot\left(1+\frac{1}{b}\right)^{b^2+2ba}\cdot\left(1+\frac{1}{c}\right)^{c^2+2cb}\geq 512$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\left.\begin{matrix}\left(1+\frac{1}{a}\right)^{a^2+2ac}=\left[\left(1+\frac{1}{a}\right)^{a}\right]^{1+2c}\\ \left(1+\frac{1}{a}\right)^{a}\geq 2\ (Bernoulli)\end{matrix}\right\}\Rightarrow\left(1+\frac{1}{a}\right)^{a^2+2ac}\geq 2^{1+2c}\Rightarrow$$

$$\left(1+\frac{1}{a}\right)^{a^2+2ac}\cdot\left(1+\frac{1}{b}\right)^{b^2+2ab}\cdot\left(1+\frac{1}{c}\right)^{c^2+2bc}\geq 2^{1+2c}\cdot 2^{1+2a}\cdot 2^{1+2b}\Rightarrow$$

$$\Rightarrow\left(1+\frac{1}{a}\right)^{a^2+2ac}\cdot\left(1+\frac{1}{b}\right)^{b^2+2ab}\cdot\left(1+\frac{1}{c}\right)^{c^2+2bc}\geq 2^{3+2(a+b+c)}=2^9=512$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ *and* $a + b + c = 3$, *we get as follows:*

1. $a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2 = 3^2 = 9$

2. $abc \leq 1 \Rightarrow (abc)^2 \leq abc \leq 1$

Hence $a^{(a^2+2ac)}b^{(b^2+2ba)}c^{(c^2+2cb)} \geq a^{2(a^2+2ac)}b^{2(b^2+2ba)}c^{2(c^2+2cb)}$

$$\Rightarrow a^{\frac{a^2+2ac}{2}}b^{\frac{b^2+2bc}{2}}c^{\frac{c^2+2cb}{2}}\geq a^{(a^2+2ac)}b^{(b^2+2ba)}c^{(c^2+2cb)}$$

$$\Rightarrow 2^9\left[a^{\frac{(a^2+2ac)}{2}}b^{\frac{(b^2+2ba)}{2}}c^{\frac{(c^2+2cb)}{2}}\right]\geq 512a^{(a^2+2ac)}b^{(b^2+2ba)}c^{(c^2+2cb)}$$

$$\Rightarrow 2^{[(a^2+2ac)+(b^2+2ba)+(c^2+2cb)]}a^{\frac{(a^2+2ac)}{2}}b^{\frac{(b^2+2ba)}{2}}c^{\frac{(c^2+2cb)}{2}}\geq$$

$$\geq 512a^{(a^2+2ac)}b^{(b^2+2ba)}c^{(c^2+2cb)}$$

$$\Rightarrow (a+1)^{(a^2+2ac)}\cdot(b+1)^{(b^2+2ba)}\cdot(c+1)^{(c^2+2cb)}\geq 512a^{(a^2+2ac)}b^{(b^2+2ba)}c^{(c^2+2cb)}$$

$$\Rightarrow\left(\frac{a+1}{a}\right)^{(a^2+2ac)}\cdot\left(\frac{b+1}{b}\right)^{(b^2+2ba)}\cdot\left(\frac{c+1}{c}\right)^{(c^2+2cb)}\geq 512$$

$$\Rightarrow\left(1+\frac{1}{a}\right)^{(a^2+2ac)}\cdot\left(1+\frac{1}{b}\right)^{(b^2+2ba)}\cdot\left(1+\frac{1}{c}\right)^{(c^2+2cb)}\ge 512\ ok$$

Therefore, it is true.

UP.233. If $m,p\in\mathbb{N}-\{0\};m\ge p$ then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((2n+1)!!)^m}-\sqrt[n]{((2n-1)!!)^m}}{n^{m-p}\left(\sqrt[n+1]{((n+1)!)^p}-\sqrt[n]{(n!)^p}\right)}\right)$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{((2n-1)!)^m}\left(\frac{\sqrt[n+1]{(2n+1)!!^m}}{\sqrt[n]{(2n-1)!!^m}}-1\right)}{n^{m-p}\sqrt[n]{n!^p}\left(\frac{\sqrt[n+1]{(n+1)!^p}}{\sqrt[n]{n!^p}}-1\right)}=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^m\cdot\left(\frac{n}{\sqrt[n]{n!}}\right)^p\cdot\frac{n\left(\left(\frac{\sqrt[n]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m-1\right)}{n\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p-1\right)}\quad \textit{(1)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{2}{e}\quad \textit{(2)}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=e\quad \textit{(3)}$$

$$\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m-1\right)=\lim_{n\to\infty}\frac{n\left(e^{\ln\left(\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m\right)}-1\right)}{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m}\cdot\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m$$

$$= \lim_{n\to\infty} m \ln^{n+1} \frac{\sqrt{(2n+1)!!}^{\,n}}{(2n-1)!!} = m \ln\left(\lim_{n\to\infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right) =$$

$$= m \ln\left(\lim_{n\to\infty}\left(\frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right)\right) \overset{(2)}{=} m \ln\left(2 \cdot \frac{e}{2}\right) = m \quad \textbf{(4)}$$

$$\lim_{n\to\infty} n\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p - 1\right) = \lim_{n\to\infty} n \frac{\left(e^{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p} - 1\right)}{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p} \cdot \ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p$$

$$= \lim_{n\to\infty} p \ln \frac{\sqrt[n+1]{(n+1)!}^{\,n}}{n!} = p \ln\left(\lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}}\right)$$

$$= p \ln\left(\lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}}\right) \overset{(3)}{=} p \ln e = p \quad \textbf{(5)}$$

From (1)+(2)+(3)+(4)+(5)$\Rightarrow \Omega = \left(\frac{2}{e}\right)^m \cdot e^p \cdot m \cdot p = 2^m \cdot m \cdot p \cdot e^{p-m}$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(2n-1)!!}\right)^m}{\left(\sqrt[n]{n!}\right)^p} \cdot \frac{\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m - 1}{n^{m-p}\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p - 1\right)} =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^m \cdot \left(\frac{n}{\sqrt[n]{n}}\right)^p \cdot \frac{\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m - 1}{\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p - 1} \quad \textbf{(1)}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} = \frac{1}{e} \cdot \lim_{n\to\infty} \frac{2n+1}{n+1} = \frac{2}{e}$$

(2)

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{e} \cdot \lim_{n\to\infty} \frac{n+1}{n+1} = \frac{1}{e} \Rightarrow \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} =$$

e ***(3)***

$$\overset{(1);(2);(3)}{\Rightarrow}\Omega=\left(\frac{2}{e}\right)^m\cdot e^p\cdot\lim_{n\to\infty}\frac{\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m-1}{\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}-1}\cdot\frac{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1}{\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{p-1}}\cdot$$

$$\cdot\frac{\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}-1}{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1}=mp\left(\frac{2}{e}\right)^m\cdot e^p\cdot\lim_{n\to\infty}\frac{e^{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}-1}{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}\cdot$$

$$\cdot\frac{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)}{e^{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)}-1}\cdot\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)\cdot\frac{1}{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)}=$$

$$=mp\left(\frac{2}{e}\right)^m\cdot e^p\cdot 1\cdot 1\cdot\lim_{n\to\infty}\frac{n\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}{n\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)}=$$

$$=mp\left(\frac{2}{e}\right)^m\cdot e^p\cdot\lim_{n\to\infty}\frac{\ln\left(\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)}{\ln\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)}=$$

$$=mp\left(\frac{2}{e}\right)^m\cdot e^p\cdot\lim_{n\to\infty}\frac{\ln\left(\frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}\right)}{\ln\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)}=mp\left(\frac{2}{e}\right)^m\cdot e^p\cdot\lim_{n\to\infty}\frac{\ln\left(\sqrt[n+1]{\frac{(2n+1)^{n+1}}{(2n+1)!!}}\right)}{\ln\left(\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}}\right)}$$

$$=mp\left(\frac{2}{e}\right)^m e^p\lim_{n\to\infty}\ln\left(\frac{(2n+3)^{n+2}}{(2n+3)!!}\cdot\frac{(2n+1)!!}{(2n+1)^{n+1}}\right)\cdot\frac{1}{\ln\left(\frac{(n+2)^{n+2}}{(n+2)!}\cdot\frac{(n+1)!}{(n+1)^{n+1}}\right)}=$$

$$=mp\cdot 2^m\cdot e^{p-m}\ln(e)\cdot\frac{1}{\ln(e)}=mp2^me^{p-m}\Rightarrow\Omega=mp2^me^{p-m}$$

UP.234. If $m,p\geq 0$ then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{m+p+1}-\left(\sqrt[n]{n!}\right)^{m+p+1}}{n^m\cdot\left(\sqrt[n]{(2n-1)!!}\right)^p}\right)$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution 1 by Marian Ursărescu – Romania

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{n!}^{m+p+1}\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}-1\right)}{n^m\sqrt[n]{(2n-1)!!}^p}=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^{m+p+1}\cdot\left(\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^p\cdot n\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}-1\right)\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}\quad (2)$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}$$

$$=\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^n=\frac{e}{2}\quad (3)$$

$$\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}-1\right)=$$

$$=\lim_{n\to\infty}\frac{\left(e^{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}}-1\right)}{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}}\cdot\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}=$$

$$=\lim_{n\to\infty}(m+p+1)\cdot\ln\frac{\sqrt[n+1]{(n+1)!}^n}{n!}=(m+p+1)\ln\left(\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}^n}{n!}\right)$$

$$= (m+p+1)\ln\left(\lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right) =$$

$$= (m+p+1)\ln\left(\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right) = (m+p+1)\ln\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)$$

$$\overset{(2)}{=} (m+p+1)\ln e = m+p+1 \quad (4)$$

From (1)+(2)+(3)+(4): $\Omega = \left(\frac{1}{e}\right)^{m+p+1}\cdot\left(\frac{e}{2}\right)^p\cdot(m+p+1) = \frac{(m+p+1)}{2^p\cdot e^{m+1}}$

Solution 2 by Remus Florin Stanca – Romania

$$\Omega = \lim_{n\to\infty}\frac{\left(\sqrt[n]{n!}\right)^{m+p+1}\cdot\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}-1\right)}{n^m\left(\sqrt[n]{(2n-1)!!}\right)^p} =$$

$$= \lim_{n\to\infty}\left(\sqrt[n]{n!}\right)^{m+p+1}\cdot\frac{1}{n^m}\cdot\left(\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^p\cdot\frac{1}{n^p}\cdot\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}-1\right) =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^{m+p+1}\cdot\left(\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^p\cdot n\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1}-1\right) \quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \frac{1}{e}$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n} = \frac{e}{2}$$

$$\overset{(1)}{\Rightarrow} \Omega = \frac{m+p+1}{e^{p+m+1}}\cdot\left(\frac{e}{2}\right)^p\cdot\lim_{n\to\infty}\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n =$$

$$= \frac{m+p+1}{e^{m+1}}\cdot\frac{1}{2^p}\cdot\lim_{n\to\infty}\left((n+1)\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right) = \frac{m+p+1}{2^p\cdot e^{m+1}} \Rightarrow \Omega = \frac{m+p+1}{2^p\cdot e^{m+1}}$$

UP.235. If $a>0; r\ge 0; (b_n)_{n\ge 0}\subset(0,\infty); \lim_{n\to\infty}\frac{b_{n+1}}{n^r\cdot b_n}=b>0$ then find:

$$\Omega=\lim_{n\to\infty}\left(\left(\left(\sqrt[n+1]{b_{n+1}}\right)^a-\left(\sqrt[n]{b_n}\right)^a\right)\cdot n^{1-ar}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu – Romania

$$\Omega=\lim_{n\to\infty}\frac{n}{n^{ar}}\cdot\left(\sqrt[n]{b_n}\right)^a\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}^a-1\right)=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n^r}\right)^a\cdot n\left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^a-1\right)\quad\textit{(1)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^r}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{(n^n)^r}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{(n+1)r}}\cdot\frac{n^{nr}}{b_n}=$$

$$=\lim_{n\to\infty}\frac{b_{n+1}}{n^r\cdot b_n}\cdot\frac{n^r}{(n+1)^r}\cdot\left(\frac{n}{n+1}\right)^{nr}=b\cdot 1\cdot\left(\frac{1}{e}\right)^r=\frac{b}{e^r}\quad\textit{(2)}$$

$$\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^a-1\right)=\lim_{n\to\infty}n\frac{\left[e^{\ln\left(\frac{\sqrt[n+1]{b_n+1}}{\sqrt[n]{b_n}}\right)^a}-1\right]}{\ln\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^a}\cdot\ln\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^a$$

$$=\lim_{n\to\infty}na\cdot\ln\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}=a\lim_{n\to\infty}\ln\frac{\sqrt[n+1]{b_{n+1}}^n}{b_n}=$$

$$=a\cdot\ln\left(\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{1}{\sqrt[n+1]{b_{n+1}}}\right)=a\ln\left(\lim_{n\to\infty}\frac{b_{n+1}}{n^r\cdot b_n}\cdot\frac{n^r}{\sqrt[n+1]{b_{n+1}}}\right)$$

$$=a\ln\left(b\lim_{n\to\infty}\frac{(n+1)^r}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{n^r}{(n+1)^r}\right)\overset{(2)}{=}a\ln\left(b\frac{e^r}{b}\right)=a\ln e^r=ar\ \textit{(3)}$$

From (1)+(2)+(3) $\Rightarrow\ \Omega=\frac{b^a}{e^{ar}}\cdot ar=\frac{arb^a}{e^{ar}}$

Solution 2 by Remus Florin Stanca – Romania

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{b_n}\right)^a\left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^a-1\right)n^{1-ar}=a\lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n^r}\right)^a n\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1\right)\quad\textit{(1)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^r}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^{rn}}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{r(n+1)}}\cdot\frac{n^{nr}}{b_n}=\left(\frac{1}{e}\right)^r\cdot b\Rightarrow$$

$$\Rightarrow\Omega=a\cdot\left(\frac{1}{e}\right)^{ra}b^a\lim_{n\to\infty}\ln\left(\frac{b_{n+1}}{n^rb_n}\cdot\frac{n^r}{\sqrt[n+1]{b_{n+1}}}\right)=$$

$$=ar\left(\frac{1}{e}\right)^{ra}b^a=\frac{arb^a}{e^{ar}}\Rightarrow\Omega=\frac{arb^a}{e^{ar}}$$

UP.236. If $f,g:(0,\infty)\to(0,\infty)$ are such that exists:

$$\lim_{x\to\infty}\frac{f(x+1)}{x\cdot f(x)}=a>0;\lim_{x\to\infty}\frac{g(x+1)}{x\cdot g(x)}=b>0;\lim_{x\to\infty}\frac{f(x)^{\frac{1}{x}}}{x},\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}$$

then find:

$$\Omega=\lim_{x\to\infty}\left((f(x))^{\frac{2}{x}}\cdot\left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(x+1)^2}-\frac{(g(x))^{\frac{1}{x}}}{x^2}\right)\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu-Romania

We have (from Heine): $\lim_{n\to\infty}\frac{a_{n+1}}{n^{a_n}}=a_1,\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b$

and we must find: $\Omega=\lim_{n\to\infty}\left[\sqrt[n]{a_n^2}\left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2}-\frac{\sqrt[n]{b_n}}{n^2}\right)\right]$ *(1)*

$$\Omega=\lim_{n\to\infty}\sqrt[n]{a_n^2}\cdot\frac{\sqrt[n]{b_n}}{n^2}\left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2}\cdot\frac{n^2}{\sqrt[n]{b_n}}-1\right)=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{n}\cdot n\left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2}\cdot\frac{n^2}{\sqrt[n]{b_n}}-1\right)\quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C.D.}{=}\lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}\right)$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}\cdot\frac{n}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{a}{e}\quad (3)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}\overset{C.D.}{=}\lim_{n\to\infty}\left(\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}\right)=$$

$$=\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\cdot\frac{n}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{b}{e}\quad (4)$$

$$\lim_{n\to\infty}n\left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2}\cdot\frac{n^2}{\sqrt[n]{b_n}}-1\right)=$$

$$=\lim_{n\to\infty}n\frac{\left(e^{\ln\left(\frac{\sqrt[n]{b_{n+1}}}{\sqrt[n]{b_n}}\right)}-1\right)}{\ln\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)}\cdot\ln\left(\frac{\sqrt[n]{b_{n+1}}}{\sqrt[n]{b_n}}\right)=$$

$$=\lim_{n\to\infty}n\ln\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)=\lim_{n\to\infty}\ln\frac{\sqrt[n+1]{b_{n+1}}^{\,n}}{b_n}=$$

$$=\ln\left(\lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n+1]{b_{n+1}}}\right)=\ln\left(\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\right)=$$

$$\overset{(4)}{=}\ln\left(b\cdot\frac{e}{b}\right)=\ln e=1\quad (5)$$

From (1)+(2)+(3)+(4)+(5)⇒

$$\Omega=\left(\frac{a}{e}\right)^2\frac{b}{e}\cdot 1=\frac{a^2b}{e^3}$$

Solution 2 by Remus Florin Stanca-Romania

We know that there are two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a>0 \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b>0$$

$$\Omega=\lim_{n\to\infty}\left((a_n)^{\frac{2}{n}}\cdot\left(\frac{b_{n+1}^{\frac{1}{n+1}}}{(n+1)^2}-\frac{b_n^{\frac{1}{n}}}{n^2}\right)\right)=\lim_{n\to\infty}(a_n)^{\frac{2}{n}}\cdot\frac{b_n^{\frac{1}{n}}}{n^2}\left(\frac{b_{n+1}^{\frac{1}{n+1}}}{b_n^{\frac{1}{n}}}\cdot\left(\frac{n}{n+1}\right)^2-1\right)=$$

$$=\lim_{n\to\infty}\left(\frac{a_n^{\frac{1}{n}}}{n}\right)^2\cdot\frac{b_n^{\frac{1}{n}}}{n}\cdot n\cdot\lim_{n\to\infty}\ln\left(\frac{b_{n+1}^{\frac{1}{n+1}}}{b_n^{\frac{1}{n}}}\left(\frac{n}{n+1}\right)^2\right)\ (1)$$

$$\lim_{n\to\infty}\left(\frac{a_n^{\frac{1}{n}}}{n}\right)=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\frac{a}{e}\ \textit{and}$$

$$\lim_{n\to\infty}\frac{b_n^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\left(\frac{b_n}{n^n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}=\frac{b}{e}\Rightarrow$$

$$\Rightarrow\Omega=\frac{a^2b}{e^2}\ln\left(\frac{1}{e^2}\cdot\lim_{n\to\infty}\left(\frac{b_{n+1}}{nb_n}\cdot\frac{n+1}{b_{n+1}^{\frac{1}{n+1}}}\right)\right)=\frac{a^2b}{e^2}\ln\left(\frac{1}{e^2}\cdot b\cdot\frac{e}{b}\right)=-\frac{a^2b}{e^2}\Rightarrow$$

$$\Rightarrow\Omega=-\frac{a^2b}{e^2}$$

We also notice that if $h(x)=\ln\big(f(x)\big)-x\ln(x)-x$ ***and*** $h_1(x)=\ln\big(g(x)\big)-x\ln(x)-x$ ***were continuous and differentiable functions on*** $(0;+\infty)$ ***then we would apply Lagrange's theorem on*** $(x;x+1)$***:***

$$\frac{h(x+1)-h(x)}{x+1-x}=h'(c_x)\ \textit{and}\ \frac{h_1(x+1)-h_1(x)}{x+1-x}=h_1'(c_x')\Leftrightarrow$$

$$\Leftrightarrow\ln\big(f(x+1)-(x+1)\ln(x+1)-x-1-\ln\big(f(x)\big)\big)+x\ln(x)+x=$$

$$=\frac{f'(c_x)}{f(c_x)}-\ln(c_x)-2$$

Where $c_x\in(x,x+1)\Rightarrow\lim_{x\to\infty}\left(\ln\left(\frac{f(x+1)}{f(x)}\right)-2-\ln(x)\right)=\lim_{x\to\infty}\left(\frac{f'(x)}{f(x)}-\ln(x)-2\right)$

(1)

$$\lim_{x\to\infty}\frac{f(x+1)}{xf(x)}=e^{\lim_{x\to\infty}\left(\ln\left(\frac{f(x+1)}{f(x)}-\ln(x)\right)\right)}=a\Rightarrow\lim_{x\to\infty}\ln\left(\frac{f(x+1)}{f(x)}\right)-\ln(x)=\ln(a)$$

$$\overset{(1)}{\Rightarrow}\lim_{x\to\infty}\frac{f'(x)}{f(x)}-\ln(x)=\ln(a)\quad\textbf{\textit{(2)}}$$

$$\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}=\lim_{x\to\infty}e^{\frac{\ln\left(\frac{f(x)}{x^x}\right)}{x}}\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}}\lim_{x\to\infty}e^{\frac{f'(x)}{f(x)}-\ln(x)-1}\overset{(2)}{\Rightarrow}\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}=e^{\ln(a)-1}=\frac{a}{e}$$

and

in the same way $\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}=\frac{b}{e}$ ***so,***

$$\Omega=\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}\cdot\left(\frac{(f(x))^{\frac{1}{x}}}{x}\right)^2x\ln\left(\frac{g(x+1)^{\frac{1}{x+1}}}{g(x)^{\frac{1}{x}}}\cdot\left(\frac{x}{x+1}\right)^2\right)=\frac{ba^2}{e^2}\cdot$$

$$\cdot\ln\left(\frac{1}{e^2}\lim_{x\to\infty}\left(\frac{g(x+1)}{g(x)}\cdot\frac{1}{g(x+1)^{\frac{1}{x+1}}}\right)\right)=\frac{ba^2}{e^2}\ln\left(\frac{1}{e^2}\cdot b\cdot\frac{e}{b}\right)=-\frac{ba^2}{e^2}$$

$$\Rightarrow\Omega=-\frac{ba^2}{e^2}$$

UP.237. Let be $f:(0,\infty)\to(0,\infty)$ a continuous function; $(a_n)_{n\ge1}\subset(0,\infty)$; $(b_n)_{n\ge1}\subset(0,\infty)$; $\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a>0$; $\lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^2}=b>0$. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\int_{\sqrt[n]{n!\cdot a_n}}^{\sqrt[n+1]{(n+1)!\cdot a_{n+1}}}f\left(\frac{x}{\sqrt[n]{b_n}}\right)dx\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania

Solution by Remus Florin Stanca – Romania

f = continuous function $\Rightarrow\exists\zeta\in\left[\sqrt[n]{n!\,a_n}\text{ and }\sqrt[n+1]{(n+1)!\,a_{n+1}}\right]$ such that:

$$\int_{\sqrt[n]{n!a_n}}^{\sqrt[n+1]{(n+1)!a_{n+1}}}f\left(\frac{x}{\sqrt[n]{b_n}}\right)dx=\left(\sqrt[n+1]{(n+1)!\,a_{n+1}}-\sqrt[n]{n!\,a_n}\right)f\left(\frac{\zeta}{\sqrt[n]{b_n}}\right)\Rightarrow$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{(n+1)!\,a_{n+1}}-\sqrt[n]{n!\,a_n}\right)f\left(\frac{\zeta}{\sqrt[n]{b_n}}\right)\ \textit{(a)}$$

$$\zeta\in\left[\sqrt[n]{n!\,a_n};\ \sqrt[n+1]{(n+1)!\,a_{n+1}}\right]\Rightarrow\sqrt[n]{n!\,a_n}\le\zeta\le\sqrt[n+1]{(n+1)!\,a_{n+1}}\Rightarrow$$

$$\Rightarrow\frac{\sqrt[n]{n!a_n}}{\sqrt[n]{b_n}}\le\frac{\zeta}{\sqrt[n]{b_n}}\le\frac{\sqrt[n+1]{(n+1)!a_{n+1}}}{\sqrt[n]{b_n}}\ \textit{(1)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!\,a_n}}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot\frac{n^2}{\sqrt[n]{b_n}}\cdot\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{b_n}}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}=$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)\cdot\lim_{n\to\infty}\left(\frac{(n+1)^{2n+2}}{b_{n+1}}\cdot\frac{b_n}{n^{2n}}\right)\cdot\lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}\right)=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\left(e^2\cdot\frac{(n+1)^2}{b_{n+1}}b_n\right)\lim_{n\to\infty}\left(\frac{1}{e}\cdot\frac{a_{n+1}}{na_n}\right)=\frac{1}{e}e^2\frac{1}{b}\cdot\frac{1}{e}a=\frac{a}{b}\ \textit{(2)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!\,a_{n+1}}}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{(n+1)^2}{\sqrt[n]{b_n}}=$$

$$=\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}\lim_{n\to\infty}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)^{n+1}}}\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{b_n}}=$$

$$=\lim_{n\to\infty}\left(\frac{(n+2)!}{(n+2)^{n+2}}\cdot\frac{(n+1)^{n+1}}{(n+1)!}\right)\cdot\lim_{n\to\infty}\left(\frac{a_{n+2}}{(n+2)^{n+2}}\cdot\frac{(n+1)^{n+1}}{a_{n+1}}\right)\cdot$$

$$\cdot\lim_{n\to\infty}\left(\frac{(n+1)^{2n+2}}{b_{n+1}}\cdot\frac{b_n}{n^{2n}}\right)=\frac{1}{e}\cdot\left(\frac{1}{e}\lim_{n\to\infty}\frac{a_{n+2}}{(n+1)a_{n+1}}\right)\left(e^2\lim_{n\to\infty}\frac{nb_n}{b_{n+1}}\right)=\frac{1}{e^2}ae^2\frac{1}{b}=\frac{a}{b}\ \textbf{(3)}$$

$$\overset{(1);(2);(3)}{\Rightarrow}\lim_{n\to\infty}\frac{\zeta}{\sqrt[n]{b_n}}=\frac{a}{b},\ f=\text{continuous}\Rightarrow\lim_{n\to\infty}f\left(\frac{\zeta}{\sqrt[n]{b_n}}\right)=f\left(\frac{a}{b}\right)\overset{(a)}{\Rightarrow}$$

$$\Rightarrow\Omega=f\left(\frac{a}{b}\right)\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{n!\,a_n}\left(\frac{\sqrt[n+1]{(n+1)!\,a_{n+1}}}{\sqrt[n]{n!\,a_n}}-1\right)=$$

$$=f\left(\frac{a}{b}\right)\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{a_n}}{n}n\left(\frac{e^{\ln\left(\frac{\sqrt[n+1]{(n+1)!a_{n+1}}}{\sqrt[n]{n!a_n}}\right)}-1}{\ln\left(\frac{\sqrt[n+1]{(n+1)!\,a_{n+1}}}{\sqrt[n]{n!\,a_n}}\right)}\right)\cdot\ln\left(\frac{\sqrt[n+1]{(n+1)!\,a_{n+1}}}{\sqrt[n]{n!\,a_n}}\right)=$$

$$=\frac{1}{e}\cdot f\left(\frac{a}{b}\right)a\cdot 1\cdot\lim_{n\to\infty}\ln\left(\left(\frac{\sqrt[n+1]{(n+1)!\,a_{n+1}}}{\sqrt[n]{n!\,a_n}}\right)^n\right)$$

$$=\frac{a}{e}f\left(\frac{a}{b}\right)\lim_{n\to\infty}\ln\left(\frac{(n+1)!\,a_{n+1}}{n!\cdot a_n}\cdot\frac{1}{\sqrt[n]{(n+1)!\,a_{n+1}}}\right)=$$

$$=\frac{a}{e}f\left(\frac{a}{b}\right)\lim_{n\to\infty}\ln\left(\frac{a_{n+1}}{na_n}\cdot\frac{n^2}{\sqrt[n]{b_n}}\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{(n+1)!\,a_{n+1}}}\right)=\frac{a}{e}f\left(\frac{a}{b}\right)\ln\left(a\frac{e^2}{b}\cdot\frac{b}{a}\right)=$$

$$=\frac{2a}{e}f\left(\frac{a}{b}\right)\Rightarrow\Omega=\frac{2a}{e}f\left(\frac{a}{b}\right)$$

UP.238. If $m\ge 0$ then find:

$$\Omega=\lim_{x\to\infty}\left(\left((x+1)^m\cdot\Gamma(x+2)\right)^{\frac{1}{x+1}}-\left(x^m\cdot\Gamma(x+1)\right)^{\frac{1}{x}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

(another approach by sequence)

Because $\Gamma(n+1)=n!$ ***we must calculate (from Heine):***

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^m(n+1)!}-\sqrt[n]{n^m n!}\right)=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{n^m n!}}{n}\cdot n\left(\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}-1\right)\quad \textbf{(1)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n^m\cdot n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^m n!}{n^m}}\overset{C.S.}{=}\lim_{n\to\infty}\frac{(n+1)^m(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n^m-n!}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^m\cdot\left(\frac{n}{n+1}\right)^n=\frac{1}{e}\quad \textbf{(2)}$$

$$\lim_{n\to\infty}n\left(\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}\right)=$$

$$=\lim_{n\to\infty}\frac{n\left(e^{\ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m\cdot n!}}}-1\right)}{\ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}}\cdot\ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}$$

$$=\lim_{n\to\infty}\ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}^{\,n}}{n^m n!}=\ln\left(\lim_{n\to\infty}\frac{(n+1)^m(n+1)!}{n^m n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)^m(n+1)!}}\right)=$$

$$=\ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^m\cdot\frac{n+1}{\sqrt[n+1]{(n+1)^m(n+1)!}}\right)\overset{(2)}{=}\ln e=1$$ ***(3) From (1)+(2)+(3)***$\Rightarrow\Omega=\frac{1}{e}$

Solution 2 by Remus Florin Stanca-Romania

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{x\Gamma(x)}\overset{n\in\mathbb{N}}{=}\lim_{n\to\infty}\frac{n!}{n\cdot(n-1)!}=1\quad \textbf{(1)}$$

$\Gamma(x)$ ***is continuous and differentiable so the function*** $f(x)=\ln(\Gamma(x))-x\ln x$ ***is differentiable and continuous*** $\Rightarrow$

$$\Rightarrow\frac{\ln(\Gamma(x+1))-(x+1)\ln(x+1)-\ln(\Gamma(x))+x\ln(x)}{x+1-x}=\frac{\Gamma'(c_x)}{\Gamma(c_x)}-1-\ln c_x,$$

$c_x\in(x;x+1)$ ***(Lagrange)***$\Rightarrow\ln(\Gamma(x+1))-\ln(\Gamma(x))-x\ln\left(\frac{x+1}{x}\right)-\ln(x+1)=$

$$=\frac{\Gamma'(c_x)}{\Gamma(c_x)}-1-\ln(c_x)\Rightarrow\lim_{x\to\infty}\left(\ln\left(\frac{\Gamma(x+1)}{\Gamma(x)}\right)-\ln(x+1)\right)-1=$$

$$= \lim_{x\to\infty}\frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln(c_x) \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\left(\ln\left(\frac{\Gamma(x+1)}{\Gamma(x)}\right) - \ln(x)\right) - 1 = \lim_{x\to\infty}\left(\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1\right) \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\ln\left(\frac{\Gamma(x+1)}{x\Gamma(x)}\right) - 1 = \lim_{x\to\infty}\left(\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1\right)$$

$$\overset{(1)}{\Rightarrow} \lim_{x\to\infty}\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1 = -1 \quad (2)$$

$$\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{x\to\infty}\left(\frac{\Gamma(x+1)}{x^x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln(\Gamma(x+1)) - x\ln(x)}{x}} =$$

$$\underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{x\to\infty} e^{\frac{\Gamma'(x)}{\Gamma(x)} - 1 - \ln(x)} \overset{(2)}{=} e^{-1} = \frac{1}{e} \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1} = \frac{1}{e} \quad (3)$$

$$\Gamma = \lim_{x\to\infty}(x^m\Gamma(x+1))^{\frac{1}{x}}\left(\frac{((x+1)^m\Gamma(x+2))^{\frac{1}{x+1}}}{(x^m\Gamma(x+1))^{\frac{1}{x}}} - 1\right) =$$

$$= \lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}x\ln\left(\frac{((x+1)^m\Gamma(x+2))^{\frac{1}{x+1}}}{(x^m\Gamma(x+1))^{\frac{1}{x}}}\right) =$$

$$= \frac{1}{e}\lim_{x\to\infty}\ln\left(\frac{(x+1)^m}{x^m}\cdot\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{((x+1)^m\Gamma(x+2))^{\frac{1}{x+1}}}\right) =$$

$$= \frac{1}{e}\lim_{x\to\infty}\ln\left(\frac{\Gamma(x+2)}{(x+1)\Gamma(x+1)}\cdot\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right) = \frac{1}{e}\ln(e) = \frac{1}{e} \Rightarrow \Omega = \frac{1}{e}$$

Solution 3 by Rohan Shinde-India

Using Lemma 1 of Stolz Cesaro theorem that for two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of real numbers, if $0 < b_1 < b_2 < \cdots < b_n < \cdots$ and $\lim_{n\to\infty} b_n = \infty$ then if $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \Omega$ then also $\lim_{n\to\infty}\frac{a_n}{b_n} = \Omega$

Here if $a_n = (n^m\Gamma(n+1))^{\frac{1}{n}}$ and $b_n = n$ then

$$\lim_{n\to\infty}\left\{\left((n+1)^m\Gamma(n+2)\right)^{\frac{1}{n+1}}-\left(n^m\Gamma(n+1)\right)^{\frac{1}{n}}\right\}=\Omega$$

$$\Rightarrow \lim_{n\to\infty}\frac{\left(n^m\Gamma(n+1)\right)^{\frac{1}{n}}}{n}=\Omega \Rightarrow \lim_{n\to\infty}\left(n^m\cdot\frac{\Gamma(n+1)}{n^n}\right)^{\frac{1}{n}}$$

Using Stirling's approximation for Gamma function,

$$\Omega=\lim_{n\to\infty}\left(n^m\times\frac{\sqrt{2\pi n}}{n^n}\times\left(\frac{n}{e}\right)^n\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left(n^{\frac{m}{n}}\times\left(\sqrt{2\pi n}\right)^{\frac{1}{n}}\times\frac{1}{e}\right)$$

But

$$\lim_{n\to\infty}n^{\frac{m}{n}}=\lim_{n\to\infty}e^{\frac{m\ln n}{n}}=e^{\lim_{n\to\infty}\frac{m\ln n}{n}}=e^0=1$$

and $\lim_{n\to\infty}\left(\sqrt{2\pi n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}e^{\frac{\ln(2\pi n)}{2n}}=e^{\lim_{n\to\infty}\left(\frac{\ln(2\pi n)}{2n}\right)}=e^0=1$

$$\Rightarrow \Omega=\frac{1}{e}$$

UP.239. If $0<a\le b<\frac{\pi}{2}$ then:

$$\frac{1}{2}\int_a^b\int_a^b(1+\tan x)(1+\tan y)(1+\tan x\tan y)\,dxdy\ge(\tan b-\tan a)^2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\frac{2}{\cos^2 x\cos^2 y}=2\cdot\frac{1}{\cos^2 x}\cdot\frac{1}{\cos^2 y}=2\cdot\frac{\sin^2 x+\cos^2 x}{\cos^2 x}\cdot\frac{\sin^2 y+\cos^2 y}{\cos^2 y}=$$

$$=2(\tan^2 x+1)(\tan^2 y+1)=$$

$$=(\tan^2 x+1)(\tan^2 y+1)+(\tan^2 x+1)(\tan^2 y+1)\ge$$

$$\overset{QM-AM}{\ge}2\left(\frac{\tan x+1}{2}\right)^2(\tan^2 y+1)+(\tan^2 x+1)\cdot 2\left(\frac{\tan y+1}{2}\right)^2=$$

$$=\frac{1}{2}\left[(\tan x+1)^2(\tan^2 y+1)+(\tan^2 x+1)(\tan y+1)^2\right]=$$

$$=\frac{1}{2}\left[(\tan^2 x+2\tan x+1)(\tan^2 y+1)+(\tan^2 x+1)(\tan^2 y+2\tan y+1)\right]=$$

$$=\frac{1}{2}\left(\begin{matrix}\tan^2 x\tan^2 y+\tan^2 x+2\tan x\tan^2 y+2\tan x+\\+\tan^2 y+1+\tan^2 x\tan^2 y+2\tan^2 x\tan y+\tan^2 x+\tan^2 y+2\tan y+1\end{matrix}\right)=$$

$$=\tan^2 x\tan^2 y+\tan x\tan^2 y+\tan^2 x\tan y+\tan x+\tan y+1+\tan^2 x+\tan^2 y\ge$$

$$\overset{AM-GM}{\ge}\tan^2 x\tan^2 y+\tan x\tan^2 y+\tan^2 x\tan y+\tan x+\tan y+1+2\tan x\tan y=$$

$$=\tan^2 x\tan^2 y+\tan x\tan^2 y+\tan^2 x\tan y+\tan x\tan y+\tan x\tan y+\tan x+$$

$$+\tan y+1=\tan x\tan y(\tan x\tan y+\tan x+\tan y+1)+$$

$$+(\tan x\tan y+\tan x+\tan y+1)=$$

$$=(\tan x\tan y+\tan x+\tan y+1)(\tan x\tan y+1)=$$

$$=[\tan x(\tan y+1)+(\tan y+1)](\tan x\tan y+1)=$$

$$=(\tan x+1)(\tan y+1)(\tan x\tan y+1)$$

$$\int_a^b\int_a^b(1+\tan x)(1+\tan y)(1+\tan x\tan y)dxdy\ge$$

$$\ge\int_a^b\int_a^b\frac{2}{\cos^2 x\cos^2 y}dxdy=2\left(\int_a^b\frac{1}{\cos^2 x}dx\right)\left(\int_a^b\frac{1}{\cos^2 y}dy\right)$$

$$\frac{1}{2}\int_a^b\int_a^b(1+\tan x)(1+\tan y)(1+\tan x\tan y)dxdy\ge$$

$$\ge\left(\tan x\Big|_a^b\right)\cdot\left(\tan y\Big|_a^b\right)=(\tan b-\tan a)^2$$

Equality holds for $a=b$.

UP.240. Let a,b,c be positive real numbers such that $a+b+c=3$. Find the minimum value of:

$$T=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Michael Sterghiou-Greece

$$T=\left(\sum_{cyc}\frac{a}{b}\right)+\sum_{cyc}\frac{1}{a^3+b^3+abc}\quad\textbf{(1)}$$

We will prove that $T \geq 4$. We use the following lemma: With the problem's conditions it holds that: $G\left(\sum_{cyc}\frac{a}{b}\right) + 3 \geq 7 \cdot \sum_{cyc} a^2$ (L). To prove this, we homogenize the inequality by multiplying by $3abc$ (and taking into account that $\sum_{cyc} a = 3$) as follows:

(L) $\to 2\left(\sum_{cyc} a\right)^2 \cdot \left(\sum ab^2\right) + abc \cdot \left(\sum_{cyc} a\right)^2 - 21abc\sum_{cyc} a^2 \geq 0$ or the equivalent $\sum_{cyc} a^4c + \sum_{cyc} a^3b^2 + 2\sum_{cyc} a^2b^3 + 4abc \cdot \sum_{cyc} ab - 8abc\sum_{cyc} a^2 \geq 0$. The last reduces to: $\sum_{cyc} a\,(a-b)^2(b-2c)^2 \geq 0$ which is true. Now, as $\sum_{cyc} a^2 \geq 3$ be adding this to (L) we get $\left(\sum_{cyc}\frac{a}{b}\right) \geq \sum_{cyc} a^2$ (2). Let $(p, q, r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)$ with $p = 3$ (1) becomes the stronger inequality $\left(\sum_{cyc} a^2\right) + \frac{9}{2\left(\sum_{cyc} a^3\right) + 3abc} \geq 4$ [by (2) and BCS] or $9 - 2q + \frac{1}{6-2q+r} - 4 \geq 0$ [Note that $\sum_{cyc} a^3 = p^3 - 3pq + 3r$] which reduces to $4q^2 - 22q + (5-2q)r + 31 \geq 0$ (3). This is either an increasing or decreasing function of r (depending on the sign of $5 - 2q$). In either case it suffices to hold when $r = \max$ or $r = \min$ which according to V. Cîrtoaje theorem happens (for any fixed q) when any two of a, b, c are equal. Let WLOG $b = c\left(< \frac{3}{2}\right) \Rightarrow a = 3 - 2b$. Now (3) becomes $-(b-1)^2(12b^3 - 54b^2 + 70b - 31) \geq 0$ for $0 < b < \frac{3}{2}$. This can be easily show to be true because $12b^3 - 54b^2 + 70b - 31 < 0$ for $b \in \left(0, \frac{3}{2}\right)$. Done!

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru