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SOLUTIONS

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# ***SOLUTIONS***

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JP.151. Let  $a, b$  be positive real numbers. Find the maximum of  $k$  such that inequality is true:

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{8k + 32}{(a + b)^4}$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

*Solution 1 by Michael Sterghiou-Greece*

$a, b > 0$ . Find the max of  $k$  so that the below inequality is true:

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{8k + 32}{(a + b)^4} \quad (1)$$

(1) homogeneous to  $a, b \rightarrow$  assume  $a + b = 1$ .

LHS (1)  $\stackrel{BCS}{\geq} \frac{k+4}{a^4 + b^4}$  which needs to be  $\geq \frac{9(k+4)}{1^4}$ , so it suffices that

$$(k + 4) \cdot \left[ \frac{1}{x^4 + (1-x)^4} - 8 \right] \geq 0 \quad (2) \text{ where } a = x, b = 1 - x$$

$$f(x) = \frac{1}{x^4 + (1-x)^4} - 8 \text{ which } f'(x) = -\frac{4[x^3 - (1-x)^3]}{[(1-x)^4 + x^4]^2} = 0$$

for  $x = \frac{1}{2}$  only and  $\min f(x) \rightarrow -7$  when  $x \rightarrow 0$  or  $x \rightarrow 1$  and  $\max f(x) = 0$  where

$x = \frac{1}{2}$ . Therefore  $f(x) \leq 0$ . But  $(k + 4)f(x) \geq 0 \rightarrow k \leq -4$  so that  $k + 4 \leq 0$ . Therefore

$\max k = -4$  for which (1) becomes true by BCS. Equality for  $a = b$ .

*Solution 2 by Khaled Abd Almuti-Damascus-Syria*

As you know:  $\sqrt{x + y} \cdot \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} \right) \geq 2\sqrt{2}; x = a^2, y = b^2$

$$\sqrt{a^4 + b^4} \cdot \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \geq 2\sqrt{2}$$

$$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2\sqrt{2}}{\sqrt{a^4 + b^4}}$$

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{2}{a^2 \cdot b^2} \geq \frac{8}{a^4 + b^4}$$

$$\frac{1}{a^4} + \frac{1}{b^4} - \frac{8}{a^4 + b^4} \geq \frac{-2}{a^2 \cdot b^2}$$

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$$\frac{1}{a^4} + \frac{1}{b^4} \geq \frac{8}{a^4 + b^4} - \frac{2}{a^2 \cdot b^2}$$

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{8}{a^4 + b^4} - \frac{2}{a^2 b^2} + \frac{k}{a^4 + b^4}$$

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{k+8}{a^4 + b^4} - \frac{2}{a^2 b^2} \quad (I) \text{ is true } \forall a, b > 0$$

$$\text{But: } \frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{8k+32}{(a+b)^4} \quad (II) \text{ from assumption}$$

$$8k + 32 = 0 \Rightarrow k = -4$$

$k$	$-\infty$	$-4$	$+\infty$
$8k + 32$	-----0+++++		

when  $k = -4$ :  $\frac{1}{a^4} + \frac{1}{b^4} - \frac{4}{a^4 + b^4} \geq \frac{4}{a^4 + b^4} - \frac{2}{a^2 \cdot b^2}$  is true  $\frac{1}{a^4} + \frac{1}{b^4} - \frac{4}{a^4 + b^4} \geq 0$

$$\text{So: for } k \leq -4: \frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{8k+32}{(a+b)^4}$$

**JP.152.** Let  $ABC$ ;  $h_a, h_b, h_c$  denote the lengths of altitudes from  $A, B, C$ ;  $l_a, l_b, l_c$  are the lengths of the symmetric divergence lines from  $A, B, C$ ;  $r_a, r_b, r_c$  are radii of the circle next to the corners  $A, B, C$ . Prove that:

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \geq 3$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\text{In any } \Delta ABC, \sum \frac{h_a r_a}{w_a^2} \geq 3$$

$$\begin{aligned} \because w_a^2 &= \frac{4bcs(s-a)}{(b+c)^2}, \therefore \sum \frac{h_a r_a}{w_a^2} = \sum \frac{2rs}{a} \cdot \frac{rs}{s-a} \cdot \frac{rs}{s-a} \cdot \frac{(b+c)^2}{4bcs(s-a)} = \\ &= \frac{2r^2 s^2}{4 \cdot 4Rrs^2} \sum \frac{(b+c)^2}{(s-a)^2} = \frac{r}{8R} \sum \frac{(s+s-a)^2}{(s-a)^2} \stackrel{(1)}{=} \frac{r}{8R} \sum \frac{s^2 + (s-a)^2 + 2s(s-a)}{(s-a)^2} \end{aligned}$$

$$\text{Now, } \sum (s-b)(s-c) = \sum \{s^2 - s(b+c) + bc\} =$$

$$3s^2 - 4s^2 + s^2 + 4Rr + r^2 \stackrel{(2)}{=} 4Rr + r^2$$

$$(1) \Rightarrow \sum \frac{h_a r_a}{w_a^2} = \frac{rs^2}{8Rr^4 s^2} \sum (s-b)^2 (s-c)^2 + \frac{3r}{8R} + \frac{2rs}{8R \cdot r^2 s} \sum (s-b)(s-c) =$$

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$$= \frac{1}{8Rr^3} \left[ \left\{ \sum (s-b)(s-c)^2 - 2r^2s \{ \sum (s-a) \} \right\} \right] + \frac{3r}{8R} + \frac{1}{4Rr} (4Rr + r^2) \text{ (by (2))}$$

$$= \frac{r^2(4Rr+r)^2 - 2r^2s^2}{8Rr^3} + \frac{3r}{8R} + \frac{4R+r}{4R} \text{ (by (2))} = \frac{(4R+r)^2 - 2s^2 + 3r^2 + 8Rr + 2r^2}{8Rr} \geq 0$$

$$\Leftrightarrow 16R^2 - 8Rr + 6r^2 \geq 2s^2 \Leftrightarrow s^2 \leq 8R^2 - 4Rr + 3r^2. \text{ Now, } s^2 \stackrel{\text{Gerretsen}}{\leq}$$

$$\leq 4R^2 + 4Rr + 3r^2 \stackrel{?}{\leq} 8R^2 - 4Rr + 3r^2 \Leftrightarrow 4R^2 \stackrel{?}{\geq} 8Rr \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (Euler)}$$

(Proved)

**Solution 2 by Marin Chirciu – Romania**

We prove the following lemma:

**Lemma:**

2) In  $\Delta ABC$ :

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

**Proof**

Using  $h_a = \frac{2S}{a}$ ,  $r_a = \frac{S}{s-a}$ ,  $l_a = \frac{2bc}{b+c} \cos \frac{A}{2}$ ,  $\cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}$ , we obtain:

$$\begin{aligned} \sum \frac{h_a r_a}{l_a^2} &= \sum \frac{\frac{2S}{a} \cdot \frac{S}{s-a}}{\left( \frac{2bc}{b+c} \cos \frac{A}{2} \right)^2} = \frac{2S^2}{4abcs} \sum \frac{(b+c)^2}{(s-a)^2} = \frac{r}{8R} \cdot \frac{2(8R^2 + 8Rr + 3r^2 - s^2)}{r^2} = \\ &= \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \end{aligned}$$

Back to the main problem:

Inequality that we have to prove can be written:

$$\frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \geq 3 \Leftrightarrow s^2 \leq 8R^2 - 4Rr + 3r^2, \text{ which follows from Gerretsen's inequality:}$$

$s^2 \leq 4R^2 + 4Rr + 3r^2$ . It remains to prove that:

$$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 4Rr + 3r^2 \Leftrightarrow 4R^2 \geq 8Rr \Leftrightarrow R \geq 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

**Remark.**

The inequality can be strengthened:

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3) In  $\Delta ABC$ :

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \geq \frac{R}{r} + 1$$

Proposed by Marin Chirciu – Romania

**Solution**

Using the Lemma and Gerretsen's inequality:  $s^2 \leq 4R^2 + 4Rr + 3r^2$  we obtain:

$$\begin{aligned} \sum \frac{h_a r_a}{l_a^2} &= \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \geq \frac{8R^2 + 8Rr + 3r^2 - 4R^2 - 4Rr - 3r^2}{4Rr} = \\ &= \frac{4R^2 + 4Rr}{4Rr} = \frac{R}{r} + 1 \end{aligned}$$

Equality holds if and only if the triangle is equilateral.

**Remark.**

Inequality 3) is stronger than inequality 1):

4) In  $\Delta ABC$ :

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \geq \frac{R}{r} + 1 \geq 3$$

**Solution**

See inequality 3) and  $\frac{R}{r} + 1 \geq 3 \Leftrightarrow R \geq 2r$  (Euler's inequality).

Equality holds if and only if the triangle is equilateral.

**Remark.**

Let's emphasize an inequality having an opposite sense:

5) In  $\Delta ABC$ :

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \leq 2 \left( \frac{R}{r} - \frac{r}{R} \right)$$

**Solution**

Using the Lemma and Gerretsen's inequality:  $s^2 \geq 16Rr - 5r^2$  we obtain:

$$\sum \frac{h_a r_a}{l_a^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \leq \frac{8R^2 + 8Rr + 3r^2 - 16Rr + 5r^2}{4Rr} =$$

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$$= \frac{8R^2 - 8Rr + 8r^2}{4Rr} = \frac{2(R^2 - Rr + r^2)}{Rr} \leq \frac{2(R^2 - r^2)}{Rr} = 2\left(\frac{R}{r} - \frac{r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

**Remark.**

We can write the double inequality:

6) In  $\Delta ABC$ :

$$\frac{R}{r} + 1 \leq \frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \leq 2\left(\frac{R}{r} - \frac{r}{R}\right)$$

Proposed by Marin Chirciu – Romania

**Solution**

See inequalities 3) and 5)

Equality holds if and only if the triangle is equilateral.

JP.153. In  $\Delta ABC$  the lengths  $BC, CA, AB$  are  $a, b, c$ . Let  $l_a, l_b, l_c$  be the lengths of the bisectors from the vertices  $A, B, C$  in triangle  $ABC$ . Prove that:

$$\frac{l_a l_b}{\sin \frac{C}{2}} + \frac{l_b l_c}{\sin \frac{A}{2}} + \frac{l_c l_a}{\sin \frac{B}{2}} \leq \frac{3}{2} \sqrt{3abc(a+b+c)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

**Solution 1 by Soumava Chakraborty-Kolkata-India**

Firstly,  $\prod(a+b) = 2abc + \sum ab(2s-c) = 2s(s^2 + 4Rr + r^2) - 4Rrs \stackrel{(1)}{=} 2s(s^2 + 2Rr + r^2)$

$$\begin{aligned} \text{Now, } \sum \frac{w_a w_b}{\sin \frac{C}{2}} &= \frac{\sum \frac{2bc}{b+c} \cos \frac{A}{2} \frac{2ca}{c+a} \cos \frac{B}{2} \sin \frac{A}{2} \sin \frac{B}{2}}{(\prod \sin \frac{A}{2})} = \frac{4R}{r} \sum \left(2 \sin \frac{A}{2} \cos \frac{A}{2}\right) \left(2 \sin \frac{B}{2} \cos \frac{B}{2}\right) \frac{abc^2(a+b)}{\prod(a+b)} \\ &= \frac{4R}{2rs(s^2 + 2Rr + r^2)} \sum \frac{ab}{4R^2} \cdot abc^2(a+b) \quad (\text{by (1)}) \end{aligned}$$

$$= \frac{4R \cdot 16R^2 r^2 s^2}{8R^2 rs(s^2 + 2Rr + r^2)} \sum (a+b) = \frac{64R^3 r^2 s^2 \cdot 4s}{8R^2 rs(s^2 + 2Rr + r^2)} = \frac{32Rrs^2}{s^2 + 2Rr + r^2} \leq$$

$$\leq \frac{3}{2} \sqrt{3abc(a+b+c)} \Leftrightarrow \frac{1024R^2 r^2 s^4}{(s^2 + 2Rr + r^2)^2} \leq \frac{9}{4} \cdot 3 \cdot 4Rrs(2s) \Leftrightarrow$$

$$\Leftrightarrow 27(s^2 + 2Rr + r^2)^2 \geq 512Rrs^2 \Leftrightarrow$$

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$$\Leftrightarrow 27s^4 + 27r^2(2R + r)^2 + 54s^2(2Rr + r^2) \stackrel{(2)}{\geq} 512Rrs^2$$

Now, LHS of (2)  $\stackrel{\text{Gerretsen}}{\geq} \{27(16Rr - 5r^2) + 54(2Rr + r^2)\}s^2 + 27r^2(2R + r)^2 \stackrel{?}{\geq} 512Rrs^2$

$$\Leftrightarrow s^2(28R - 81r) + 27r(2R + r)^2 \stackrel{?}{\geq} 0 \Leftrightarrow$$

$$\Leftrightarrow s^2(28R - 56r) + 27r(2R + r)^2 \stackrel{?}{\geq} 25rs^2 \quad (3)$$

Now, LHS of (3)  $\stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(28R - 56r) + 27r(2R + r)^2$  &

$$\text{RHS of (3)} \stackrel{\text{Gerretsen}}{\geq} 25r(4R^2 + 4Rr + 3r^2)$$

(a), (b)  $\Rightarrow$  in order to prove (3), it suffices to prove:

$$(16R - 5r)(28R - 56r) + 27(2R + r)^2 \geq 25(4R^2 + 4Rr + 3r^2) \Leftrightarrow$$

$$\Leftrightarrow 114R^2 - 257Rr + 58r^2 \geq 0 \Leftrightarrow (R - 2r)(114R - 29r) \geq 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r$$

(proved)

### Solution 2 and generalizations by Marin Chirciu – Romania

We prove the following lemma:

Lemma:

2) In  $\Delta ABC$ :

$$\frac{l_b l_c}{\sin \frac{A}{2}} + \frac{l_c l_a}{\sin \frac{B}{2}} + \frac{l_a l_b}{\sin \frac{C}{2}} = \frac{32s^2 Rr}{s^2 + r^2 + 2Rr}$$

Proof

Using  $l_a = \frac{2bc}{b+c} \cos \frac{A}{2}$ ,  $2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A$ ,  $\frac{a}{\sin A} = 2R$ , we obtain:

$$\begin{aligned} \sum \frac{l_b l_c}{\sin \frac{A}{2}} &= \sum \frac{\frac{2ac}{a+c} \cos \frac{B}{2} \cdot \frac{2ab}{a+b} \cos \frac{C}{2}}{\sin \frac{A}{2}} = \sum \frac{\frac{8a^2 bc}{(a+b)(a+c)} \cdot \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \\ &= 8abc \prod \cos \frac{A}{2} \sum \frac{a}{(a+b)(a+c) \sin A} = 32Rrs \cdot \frac{s}{4R} \cdot 2R \sum \frac{1}{(a+b)(a+c)} = \\ &= 16Rrs^2 \cdot \frac{2}{s^2 + r^2 + 2Rr} = \frac{32s^2 Rr}{s^2 + r^2 + 2Rr} \end{aligned}$$

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Let's get back to the main problem:

Using the Lemma, the inequality that we have to prove can be written:

$$\begin{aligned} \frac{32s^2Rr}{s^2+r^2+2Rr} &\leq \frac{3}{2}\sqrt{3abc(a+b+c)} \Leftrightarrow \left(\frac{32s^2Rr}{s^2+r^2+2Rr}\right)^2 \leq \frac{9}{4} \cdot 3abc(a+b+c) \Leftrightarrow \\ &\Leftrightarrow \left(\frac{32s^2Rr}{s^2+r^2+2Rr}\right)^2 \leq \frac{9}{4} \cdot 12Rrs \cdot 2s \Leftrightarrow \left(\frac{32s^2Rr}{s^2+r^2+2Rr}\right)^2 \leq 54Rrs^2 \Leftrightarrow \\ &\Leftrightarrow 512s^2Rr \leq 27(s^2+r^2+2Rr)^2 \Leftrightarrow s^2(27s^2+54r^2-404Rr) + 27r^2(2R+r)^2 \\ &\geq 0 \end{aligned}$$

We distinguish the following cases:

Case 1). If  $(27s^2+54r^2-404Rr) \geq 0$ , the inequality is obvious.

Case 2). If  $(27s^2+54r^2-404Rr) < 0$ , we write the inequality:

$27r^2(2R+r)^2 \geq s^2(404Rr-54r^2-27s^2)$ , which follows from Gerretsen's inequality:

$16Rr-5r^2 \leq s^2 \leq 4R^2+4Rr+3r^2$ . It remains to prove that:

$$\begin{aligned} 27r^2(2R+r)^2 &\geq (4R^2+4Rr+3r^2)(404Rr-54r^2-27(16Rr-5r^2)) \Leftrightarrow \\ &\Leftrightarrow 28R^3-26R^2r-33Rr^2-54r^3 \geq 0 \Leftrightarrow (R-2r)(28R^2+30Rr+27r^2) \geq 0, \end{aligned}$$

obviously from Euler's inequality. Equality holds if and only if the triangle is equilateral.

Remark.

We can prove that:

3) In  $\triangle ABC$ :

$$\frac{l_b l_c}{\sin \frac{A}{2}} + \frac{l_c l_a}{\sin \frac{B}{2}} + \frac{l_a l_b}{\sin \frac{C}{2}} \leq \frac{27R^2}{2}$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma, the inequality can be written:

$$\frac{32s^2Rr}{s^2+r^2+2Rr} \leq \frac{27R^2}{2} \Leftrightarrow s^2(27R-64r) + 27Rr(2R+r) \geq 0.$$

We distinguish the cases:

Case 1). If  $(27R-64r) \geq 0$ , inequality is obvious.

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Case 2). If  $(27R - 64r) < 0$ , inequality can be rewritten:

$27Rr(2R + r) \geq s^2(64r - 27R)$ , which follows from Gerretsen's inequality:

$s^2 \leq 4R^2 + 4Rr + 3r^2$ . It remains to prove that:

$$27Rr(2R + r) \geq 9(4R^2 + 4Rr + 3r^2)(64r - 27R) \Leftrightarrow 54R^3 - 47R^2r - 74Rr^2 - 96r^3 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R - 2r)(54R^2 + 61Rr + 48r^2) \geq 0, \text{ obviously Euler's inequality.}$$

Equality holds if and only if the triangle is equilateral.

Remark.

Let's highlight an inequality having an opposite sense.

4) In  $\Delta ABC$ :

$$\frac{l_b l_c}{\sin \frac{A}{2}} + \frac{l_c l_a}{\sin \frac{B}{2}} + \frac{l_a l_b}{\sin \frac{C}{2}} \geq 27Rr$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma, the inequality can be written:  $\frac{32s^2Rr}{s^2+r^2+2Rr} \geq 27Rr \Leftrightarrow$

$$\Leftrightarrow 5s^2 \geq 27r(2R + r), \text{ which follows from Gerretsen's inequality: } s^2 \geq 16Rr - 5r^2. \text{ It}$$

remains to prove that:

$$5(16Rr - 5r^2) \geq 27r(2R + r) \Leftrightarrow 26Rr \geq 52r^2 \Leftrightarrow R \geq 2r \text{ (Euler's inequality)}$$

Remark.

We write the double inequality:

5) In  $\Delta ABC$ :

$$27Rr \leq \frac{l_b l_c}{\sin \frac{A}{2}} + \frac{l_c l_a}{\sin \frac{B}{2}} + \frac{l_a l_b}{\sin \frac{C}{2}} \leq \frac{27R^2}{2}$$

Proposed by Marin Chirciu – Romania

Solution

See inequalities 3) and 4).

Equality holds if and only if the triangle is equilateral.

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**JP.154.** In triangle  $ABC$  with sides  $BC = a, CA = b, AB = c, r_a, r_b, r_c$  are exradii,  $h_a, h_b, h_c$  are distances from  $A, B, C$  to  $BC, CA, AB$ . Prove that:

$$\frac{r_a r_b}{\sin \frac{A}{2} \sin \frac{B}{2}} + \frac{r_b r_c}{\sin \frac{B}{2} \sin \frac{C}{2}} + \frac{r_c r_a}{\sin \frac{C}{2} \sin \frac{A}{2}} \geq \frac{h_a h_b}{\sin^2 \frac{C}{2}} + \frac{h_b h_c}{\sin^2 \frac{A}{2}} + \frac{h_c h_a}{\sin^2 \frac{B}{2}}$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sum \frac{r_a r_b}{\sin \frac{A}{2} \sin \frac{B}{2}} &= \sum \frac{s^2 \tan \frac{A}{2} \tan \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = s^2 \sum \frac{\cos \frac{C}{2}}{\left(\prod \cos \frac{A}{2}\right)} = s^2 \sum \frac{\cos \frac{C}{2}}{\left(\frac{s}{4R}\right)} = \\ &= 2Rs \sum \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{\cos \frac{A-B}{2}} \geq 2Rs \sum (\sin A + \sin B) \\ &\quad \left(\because 0 < \cos \frac{A-B}{2} \leq 1 \text{ as } -\frac{\pi}{2} < \frac{A-B}{2} < \frac{\pi}{2}\right) \\ &= 4Rs (\sum \sin A) = 4Rs \left(\frac{s}{R}\right) = 4s^2 \stackrel{(1)}{\Rightarrow} LHS \geq 4s^2 \\ \sum \frac{h_a h_b}{\sin^2 \frac{C}{2}} &= \sum \frac{4r^2 s^2}{ab} \cdot \frac{ab(s-c)}{(s-a)(s-b)(s-c)} = \frac{4r^2 s^2}{r^2 s} \sum (s-c) = 4s^2 \stackrel{\text{by (1)}}{\leq} LHS \text{ (Proved)} \end{aligned}$$

**JP.155.** Let  $a, b, c$  be positive real numbers such that:  $a + b + c = 3$ .

**Prove that:**

$$\frac{a^2}{b^4 c^3 \sqrt[3]{4(b^6 + 1)}} + \frac{b^2}{c^4 a^3 \sqrt[3]{4(b^6 + 1)}} + \frac{c^2}{a^4 b^3 \sqrt[3]{4(a^6 + 1)}} \geq \frac{3}{2}$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^2}{b^4 c^3 \sqrt[3]{4(b^6 + 1)}} + \frac{b^2}{c^4 a^3 \sqrt[3]{4(b^6 + 1)}} + \frac{c^2}{a^4 b^3 \sqrt[3]{4(a^6 + 1)}} \stackrel{(1)}{\geq} \frac{3}{2}$$

Firstly,  $4(a^6 + 1) \leq (3a^2 - 4a + 3)^3 \Leftrightarrow (a - 1)^4(23a^2 - 16a + 23) \geq 0 \Leftrightarrow$

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$$\Leftrightarrow (a-1)^4\{23(a-1)^2 + 30a\} \geq 0 \rightarrow \text{true}$$

$$\therefore \sqrt[3]{4(a^6+1)} \stackrel{(a)}{\leq} 3a^2 - 4a + 3. \text{ Similarly, } \sqrt[3]{4(b^6+1)} \stackrel{(b)}{\leq} 3b^2 - 4b + 3 \ \&$$

$$\sqrt[3]{4(c^6+1)} \stackrel{(c)}{\leq} 3c^2 - 4c + 3; \ (a), \ (b), \ (c) \Rightarrow \text{LHS of (1)}$$

$$\geq \frac{a^2}{b^4c(3b^2-4b+3)} + \frac{b^2}{c^4a(3c^2-4c+3)} + \frac{c^2}{a^4b(3a^2-4a+3)} =$$

$$= \frac{a^3}{ab^4c(3b^2-4b+3)} + \frac{b^2}{bc^4a(3c^2-4c+3)} + \frac{c^2}{a^4bc(3a^2-4a+3)} =$$

$$= \frac{1}{abc} \left\{ \frac{\left(\frac{a}{b}\right)^3}{3b^2-4b+3} + \frac{\left(\frac{b}{c}\right)^3}{3c^2-4c+3} + \frac{\left(\frac{c}{a}\right)^3}{3a^2-4a+3} \right\} \geq$$

$$\geq \frac{\left(\frac{a}{b}\right)^3}{3b^2-4b+3} + \frac{\left(\frac{b}{c}\right)^3}{3c^2-4c+3} + \frac{\left(\frac{c}{a}\right)^3}{3a^2-4a+3}$$

$$\left( \because \sum a = 3 \stackrel{A-G}{\geq} 3\sqrt[3]{abc} \Rightarrow abc \leq 1 \right)$$

$$\stackrel{\text{Holder}}{\geq} \frac{\left(\sum \frac{a}{b}\right)^3}{3(3\sum a^2 - 4\sum a + 9)} = \frac{\left(\sum \frac{a}{b}\right)^3}{3(3\sum a^2 - 4 \cdot 3 + 9)} \quad \left( \because \sum a = 3 \right)$$

$$= \frac{\left(\sum \frac{a}{b}\right)^3}{9\left(\sum a^2 - \left(\frac{\sum a}{3}\right)^2\right)} \quad \left( \because 1 = \frac{\sum a}{3} \right) = \frac{\left(\sum \frac{a}{b}\right)^3}{9\{9\sum a^2 - (\sum a)^2\}} = \frac{\left(\sum \frac{a}{b}\right)^3}{9\sum a^2 - (\sum a)^2}$$

$$\text{Now, } \sum \frac{a}{b} = \sum \frac{a^2}{ab} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum a)^2}{\sum ab} \Rightarrow \left(\sum \frac{a}{b}\right)^3 \stackrel{(3)}{\geq} \frac{(\sum a)^6}{(\sum ab)^3}$$

$$(2), (3) \Rightarrow \text{LHS of (1)} \geq \frac{(\sum a)^6}{(\sum ab)^3\{9\sum a^2 - (\sum a)^2\}} \stackrel{?}{\geq} \frac{3}{2}$$

$$\Leftrightarrow 2\left(\sum a\right)^6 \stackrel{?}{\geq} 3\left(\sum ab\right)^3\left\{9\sum a^2 - \left(\sum a\right)^2\right\} \Leftrightarrow$$

$$\Leftrightarrow \frac{2}{9}\left(\sum a\right)^8 \stackrel{?}{\geq} 3\left(\sum ab\right)^3\left\{9\sum a^2 - \left(\sum a\right)^2\right\} \quad \left( \because 9 = \left(\sum a\right)^2 \right)$$

$$\Leftrightarrow 2\left(\sum a^2 + 2\sum ab\right)^4 \stackrel{?}{\geq} 27\left(\sum ab\right)^3\left\{9\sum a^2 - \left(\sum a\right)^2\right\}$$

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$$\Leftrightarrow 2(x + 2y)^4 \stackrel{?}{\geq} 27y^3(9x - x - 2y) \quad \left( \begin{array}{l} x = \sum a^2, \\ y = \sum ab \end{array} \right)$$

$$\Leftrightarrow (x + 2y)^4 \stackrel{?}{\geq} 27y^3(4x - y) \Leftrightarrow t^4 + 8t^3 + 24t^2 - 76t + 43 \stackrel{?}{\geq} 0 \quad \left( t = \frac{x}{y} = \frac{\sum a^2}{\sum ab} \right) \Leftrightarrow$$

$$\Leftrightarrow (t - 1)^2(t^2 + 10t + 43) \stackrel{?}{\geq} 0 \rightarrow \text{true} \left( \because t = \frac{\sum a^2}{\sum ab} \geq 1 \right)$$

(Hence proved)

JP.156. Let  $ABC$  be a triangle having the area  $S$ . Let be  $A' \in (BC)$  such that the incircles of  $\Delta AA'B$ ,  $\Delta AA'C$  have the same radius. Analogous, we obtain the points

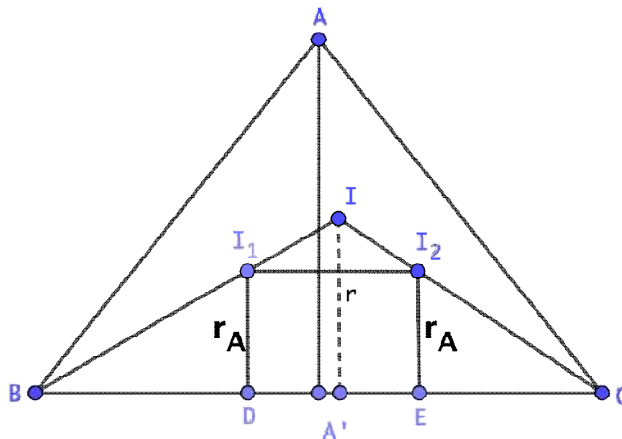
$B' \in (AC)$ ,  $C' \in (AB)$ . Prove that:

$$S = \frac{AA' \cdot BB' \cdot CC'}{s}$$

where  $s$  is the semiperimeter of  $\Delta ABC$ .

Proposed by Marian Ursărescu – Romania

Solution by proposer



Let be  $r_A$  the inradii in  $\Delta ABA'$ ,  $\Delta ACA'$

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$$\begin{aligned} S &= \sigma_{ABA'} + \sigma_{ACA'} = s_{ABA'} \cdot r_A + s_{ACA'} \cdot r_A = r_A(s_{ABA'} + s_{ACA'}) = \\ &= r_A(s + AA') \Rightarrow r_A(s + AA') = S \quad (1) \end{aligned}$$

Let be  $I_1, I_2$  the incenters in  $\Delta ABA', ACA'$  and  $I$  the incenter in  $\Delta ABC$

$$\begin{aligned} \Delta I I_1 I_2 \sim \Delta IBC &\Rightarrow \frac{I_1 I_1}{BC} = \frac{r - r_A}{r} \Rightarrow 1 - \frac{r_A}{r} = \frac{I_1 I_2}{BC} \Rightarrow \\ &\Rightarrow \frac{r_A}{r} = 1 - \frac{I_1 I_2}{a} \quad (2) \end{aligned}$$

Let  $D \wedge E$  be the intersection points with side  $BC$  of the projections of incenters.

$I_1 I_2 ED$  is a rectangle  $\Rightarrow$

$$I_1 I_2 = DE = DA' + A'E = s_{ABA'} - c + s_{ACA'} - b = s - b - c + AA' \quad (3)$$

$$\text{From (2)+(3)} \Rightarrow \frac{r_A}{r} = 1 - \frac{s-b-c+AA'}{a} = \frac{s-AA'}{a} \Rightarrow r_A = \frac{r}{a}(s - AA') \quad (4)$$

$$\text{From (1)+(4)} \Rightarrow \frac{r}{a}(s - AA')(s + AA') = S \Rightarrow s^2 - AA'^2 = as \Rightarrow$$

$$\begin{aligned} \Rightarrow AA'^2 = s^2 - sa \Rightarrow AA' &= \sqrt{s(s-a)} \text{ analogous } BB' = \sqrt{s(s-b)}, CC' = \sqrt{s(s-c)} \\ &\Rightarrow AA' \cdot BB' \cdot CC' \end{aligned}$$

**JP.157. Prove that in  $\Delta ABC$ , the following inequality holds:**

$$\sum_{cyc} \left( a \cot \frac{A}{2} \cdot \sin^2 B \cdot \sin^2 C \right) \geq \frac{81r^4}{2R^2(2R-r)}$$

*Proposed by Marian Ursărescu – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} LHS &= \sum \left( 4R \sin \frac{A}{2} \cos \frac{A}{2} \cdot \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} \cdot \frac{b^2 c^2}{16R^4} \right) = \frac{1}{4R^3} \sum b^2 c^2 \cdot \frac{s(s-a)}{bc} \\ &= \frac{s}{4R^3} \sum bc(s-a) \\ &= \frac{s}{4R^3} \{s(s^2 + 4Rr + r^2) - 12Rrs\} = \frac{s^2}{4R^3} (s^2 - 8Rr + r^2) \stackrel{\text{Gerretsen}}{\geq} \frac{s^2(8Rr - 4r^2)}{4R^3} \\ &\stackrel{s \geq 3\sqrt{3}r}{\geq} \frac{27r^2(8Rr - 4r^2)}{4R^3} \stackrel{?}{\geq} \frac{81r^4}{2R^2(R-r)} \\ &\Leftrightarrow (4R - 2r)(R - r) \stackrel{?}{\geq} 3Rr \Leftrightarrow 4R^2 - 9Rr + 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R - 2r)(4R - r) \stackrel{?}{\geq} 0 \end{aligned}$$

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true  $\because R \stackrel{\text{Euler}}{\geq} 2r$  (Proved)

JP.158. Let  $a, b, c > 0$ . Prove that:

$$\sum_{cyc} \frac{1}{a} + \sum_{cyc} \frac{a}{b^2 + c^2} \geq 3 \sum_{cyc} \frac{1}{b+c}$$

Proposed by Andrei Ştefan Mihalcea – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{1}{a} + \sum \frac{a^2}{ab^2 + ac^2} \stackrel{\text{Bergstrom}}{\geq} \frac{\sum ab}{abc} + \frac{(\sum a)^2}{\sum a^2 b + \sum ab^2} \stackrel{?}{\geq} 3 \sum \frac{1}{b+c}$$

$$\Leftrightarrow \frac{(\sum ab)(\sum a^2 b + \sum ab^2) + abc(\sum a)^2}{abc(\sum a^2 b + \sum ab^2)} \stackrel{?}{\geq} \frac{3((\sum a)^2 + \sum ab)}{(a+b)(b+c)(c+a)}$$

$$\Leftrightarrow \sum a^5 b^3 + \sum a^3 b^5 + abc(\sum a^4 b + \sum ab^4) + 2 \sum a^4 b^4$$

$$\stackrel{(1)}{\geq} 2abc(\sum a^3 b^2 + \sum a^2 b^3) + 2a^2 b^2 c^2 (\sum a^2)$$

(Simplifying & re-arranging)

$$\because \sum x^2 \geq \sum xy \because 2 \sum a^4 b^4 \stackrel{(a)}{\geq} 2a^2 b^2 c^2 (\sum a^2)$$

$$(x = a^2 b^2, y = b^2 c^2, z = c^2 a^2)$$

$$\text{Again, } abc(\sum a^4 b + \sum ab^4) = abc \cdot \sum ab(a^3 + b^3)$$

$$\stackrel{(b)}{\geq} abc \sum ab \cdot ab(a+b) = abc \sum a^2 b^2 (a+b) = abc(\sum a^3 b^2 + \sum a^2 b^3)$$

$$\text{Also, } \sum a^5 b^3 + \sum a^3 b^5 = \sum c^5 (a^3 + b^3) \geq \sum c^5 ab(a+b) = abc(\sum c^4 (a+b))$$

$$= abc(\sum a^4 b + \sum ab^4) \stackrel{\text{by (b)}}{\geq} abc(\sum a^3 b^2 + \sum a^2 b^3)$$

(a)+(b)+(c)  $\Rightarrow$  (1) is true (Proved)

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For  $a, b, c > 0$  we have identity inequality below.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} = \frac{2}{2\sqrt{ab}} + \frac{2}{2\sqrt{bc}} + \frac{2}{2\sqrt{ca}} \geq 2 \left( \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right)$$

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Consider  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a}{b^2+c^2} + \frac{b}{c^2+a^2} + \frac{c}{a^2+b^2} = (a^2 + b^2 + c^2) \left( \frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)} \right)$

**1ca2+b2**

$$\begin{aligned} &\geq (a^2 + b^2 + c^2) \frac{\left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 2 \left( \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \right) \right)}{2(a^2 + b^2 + c^2)} \\ &= \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 2 \left( \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \right) \right) \\ &\geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + 2 \left( \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \\ &= 3 \left( \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \text{ ok} \end{aligned}$$

Therefore, it is to be true.

**JP.159. Prove that in any  $\triangle ABC$  the following inequality holds:**

$$a^2 h_b h_c + b^2 h_a h_c + c^2 h_a h_b \leq 4(R + r)^4$$

*Proposed by Marian Ursărescu – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sum a^2 h_b h_c &= \sum \frac{a^2 \cdot ca \cdot ab}{4R^2} = \frac{4Rrs}{4R^2} \left( \sum a^3 \right) = \frac{rs}{R} \cdot 2s(s^2 - 6Rr - 3r^2) \leq 4(R + r)^4 \\ &\Leftrightarrow rs^4 - rs^2(6Rr + 3r^2) \stackrel{(1)}{\leq} 2R(R + r)^4 \end{aligned}$$

$$\text{Now, LHS of (1)} \stackrel{\text{Rouche}}{\leq} rs^2 \{ 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} - (6Rr + 3r^2) \}$$

$$\stackrel{\text{Rouche}}{\leq} r \{ 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \}$$

$$\left( 2R^2 + 4Rr - 4r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \right) =$$

$$= r \left[ (2R^2 + 10Rr - r^2)(2R^2 + 4Rr - 4r^2) \right.$$

$$\left. + 4R(R - 2r)^3 2(R - 2r)\sqrt{R^2 - 2Rr}(4R^2 + 14Rr - 5r^2) \right]$$

$$\stackrel{?}{\leq} 2R(R + r)^4 \Leftrightarrow (R - 2r)(R^4 + 2R^3r + 8R^2r^2 - 19Rr^3 + r^4) \stackrel{?}{\geq} (2)$$

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$$\geq r(R - 2r)\sqrt{R^2 - 2Rr}(4R^2 + 14Rr - 5r^2)$$

$$\because R - 2r \stackrel{\text{Euler}}{\geq} 0 \text{ \& } R^4 + 2R^3r + 8R^2r^2 - 19Rr^3 + r^4 \stackrel{\text{Euler}}{\geq}$$

$$\geq R^4 + R^3r + Rr \cdot 4r^2 + 8Rr^2 \cdot 2r - 19Rr^3 + r^4 > 0$$

$\therefore$  in order to prove (2), it suffices to prove:

$$(R^4 + 2R^3r + 8R^2r^2 - 19Rr^3 + r^4)^2 - R(R - 2r)r^2(4R^2 + 14Rr - 5r^2)^2 > 0$$

$$\Leftrightarrow t^8 + 4t^7 + 4t^6 - 86t^5 + 58t^5 + 152t^3 + 72t^2 + 12t + 1 > 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(t^6 + 8t^5 + 32t^2(t^2 - 4) + 98t^2 + 10t(t^2 - 4) + 32t + 160) + 684\} \\ + 729 > 0$$

$$\rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (2) \text{ is true} \Rightarrow (1) \text{ is true (Proved)}$$

**JP.160. Prove that for all non-negative real numbers  $x, y, z$  the following inequality holds:**

$$\frac{y + z}{(1 + x)^2} + \frac{z + x}{(1 + y)^2} + \frac{x + y}{(1 + z)^2} \geq \frac{x}{1 + yz} + \frac{y}{1 + zx} + \frac{z}{1 + xy}$$

*Proposed by Nguyen Viet Hung – Hanoi – Vietnam*

*Solution 1 by Soumava Chakraborty-Kolkata-India*

$$\text{Given inequality} \Leftrightarrow x\left(\frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} - \frac{1}{1+yz}\right) + y\left(\frac{1}{(1+z)^2} + \frac{1}{(1+x)^2} - \frac{1}{1+zx}\right) + \\ + z\left(\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} - \frac{1}{1+xy}\right) \stackrel{(a)}{\geq} 0. \text{ Now, } \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} - \frac{1}{1+yz} \geq 0 \Leftrightarrow \frac{(1+z)^2 + (1+y)^2}{(1+y)^2(1+z)^2} \geq \frac{1}{1+yz} \\ \Leftrightarrow (1 + yz)\{(1 + z)^2 + (1 + y)^2\} \geq (1 + y)^2(1 + z)^2 \quad (\because 1 + yz \geq 1 > 0) \\ \Leftrightarrow y^3z + yz^3 - y^2z^2 - 2yz + 1 \geq 0 \Leftrightarrow (y^3z + yz^3 - 2y^2z^2) + (y^2z^2 - 2yz + 1) \geq 0 \\ \Leftrightarrow yz(y - z)^2 + (yz - 1)^2 \geq 0 \rightarrow \text{true} \because yz(y - z)^2 \geq 0 \quad (\because yz \geq 0 \text{ as } y, z \geq 0) \\ \text{\& } (yz - 1)^2 \geq 0 \Rightarrow yz(y - z)^2 + (yz - 1)^2 \geq 0 \therefore \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} - \frac{1}{1+yz} \geq 0 \\ \Rightarrow x\left(\frac{1}{(1 + y)^2} + \frac{1}{(1 + z)^2} - \frac{1}{1 + yz}\right) \stackrel{(1)}{\geq} 0 \quad (\because x \geq 0)$$

$$\text{Similarly, } y\left(\frac{1}{(1+z)^2} + \frac{1}{(1+x)^2} - \frac{1}{1+zx}\right) \stackrel{(2)}{\geq} 0 \text{ \& } z\left(\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} - \frac{1}{1+xy}\right) \stackrel{(3)}{\geq} 0$$

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(1)+(2)+(3)  $\Rightarrow$  (a) is true (Proved)

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\text{For } a, b, c > 0, \text{ we have: } \frac{a}{(1+b)^2} + \frac{a}{(1+c)^2} \geq \frac{a}{(1+bc)}$$

$$\text{Because } 1 + b^3c + bc^3 \geq 2bc + b^2c^2$$

$$\text{Hence: } 1 + 2b + b^2 + 1 + 2c + c^2 + 2bc + 2bc^2 + 2b^2c + b^3c + bc^3$$

$$\geq 1 + 2b + b^2 + 2c + 4bc + 2bc^2 + 2b^2c + c^2 + b^2c^2$$

$$\Rightarrow [(1+b)^2 + (1+c)^2](1+bc) \geq (1+b)^2(1+c)^2 \Rightarrow \frac{a}{(1+b)^2} + \frac{a}{(1+c)^2} \geq \frac{a}{(1+bc)}$$

Therefore for  $x, y, z > 0$  we will get that:

$$\begin{aligned} & \frac{x+y}{(1+z)^2} + \frac{y+z}{(1+x)^2} + \frac{z+x}{(1+y)^2} \\ &= \frac{x}{(1+z)^2} + \frac{x}{(1+y)^2} + \frac{y}{(1+x)^2} + \frac{y}{(1+z)^2} + \frac{z}{(1+x)^2} + \frac{z}{(1+y)^2} \geq \\ & \geq \frac{x}{(1+yz)} + \frac{y}{(1+zx)} + \frac{z}{(1+xy)} \text{ ok} \end{aligned}$$

**JP.161. Let  $a, b, c, d$  be positive real numbers such that  $abcd \geq 1$ . Prove that:**

$$\frac{a^2 + b^2 + c^2 + 1}{a^3 + b^3 + c^3 + 1} + \frac{b^2 + c^2 + d^2 + 1}{b^3 + c^3 + d^3 + 1} + \frac{c^2 + d^2 + a^2 + 1}{c^3 + d^3 + a^3 + 1} + \frac{d^2 + a^2 + b^2 + 1}{d^3 + a^3 + b^3 + 1} \leq 4$$

**Proposed by Nguyen Viet Hung – Hanoi – Vietnam**

**Solution by Soumava Chakraborty-Kolkata-India**

$$\text{Firstly, } \forall x, y, z, w > 0, x^3 + y^3 + z^3 + w^3 \geq \frac{1}{4}(x^2 + y^2 + z^2 + w^2)(x + y + z + w)$$

$$\begin{aligned} & \Leftrightarrow \{(x^3 + y^3) - (x^3y + xy^2)\} + \{(x^3 + z^3) - (x^2z + xz^2)\} + \\ & + \{(x^3 + w^3) - (x^2w + xw^2)\} + \{(y^3 + z^3) - (y^2z + yz^2)\} + \\ & + \{(y^3 + w^3) - (y^2w + yw^2)\} + \{(z^3 + w^3) - (z^2w + zw^2)\} \geq 0 \rightarrow \text{true} \end{aligned}$$

$$\because \forall x, y \geq 0, x^3 + y^3 \geq xy(x + y) \text{ etc}$$

$$\Rightarrow x^3 + y^3 + z^3 + w^3 \stackrel{(a)}{\geq} \frac{1}{4}(x^2 + y^2 + z^2 + w^2)(x + y + z + w)$$

$$(a) \Rightarrow a^3 + b^3 + c^3 + 1 \stackrel{(i)}{\geq} \frac{1}{4}(a^2 + b^2 + c^2 + 1)(a + b + c + 1),$$

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$$b^3 + c^3 + d^3 + 1 \stackrel{(ii)}{\geq} \frac{1}{4} (b^2 + c^2 + d^2 + 1)(b + c + d + 1),$$

$$c^3 + d^3 + a^3 + 1 \stackrel{(iii)}{\geq} \frac{1}{4} (c^2 + d^2 + a^2 + 1)(c + d + a + 1),$$

$$d^3 + a^3 + b^3 + 1 \stackrel{(iv)}{\geq} \frac{1}{4} (d^2 + a^2 + b^2 + 1)(d + a + b + 1)$$

$$(i), (ii), (iii), (iv) \Rightarrow \text{LHS of (1)} \leq \frac{4}{a+b+c+1} + \frac{4}{b+c+d+1} + \frac{4}{c+d+a+1} + \frac{4}{d+a+b+1} \stackrel{?}{\leq} 4$$

$$\Leftrightarrow (b + c + d + 1)(c + d + a + 1)(d + a + b + 1) + \\ + (c + d + a + 1)(d + a + b + 1)(a + b + c + 1) \\ + (d + a + b + 1)(a + b + c + 1)(b + c + d + 1) +$$

$$+ (a + b + c + 1)(b + c + d + 1)(c + d + a + 1) \leq$$

$$\leq (a + b + c + 1)(b + c + d + 1)(c + d + a + 1)(d + a + b + 1)$$

$$\Leftrightarrow a^3(b + c + d) + b^3(a + c + d) + c^3(a + b + d) + d^3(a + b + c) +$$

$$+ 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) +$$

$$+ 4(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab$$

$$+ d^2ac + d^2bc) +$$

$$+ 9abcd \stackrel{(2)}{\geq} 3(a^2 + b^2 + c^2 + d^2) + 7(ab + ac + ad + bc + bd + cd) + \\ + 6(a + b + c + d) + 3$$

$$\text{Now, } a^3(b + c + d) + b^3(a + c + d) + c^3(a + b + d) + d^3(a + b + c) \geq$$

$$\stackrel{A-G}{\geq} 3 \left( a^3(\sqrt[3]{bcd}) + b^3(\sqrt[3]{cad}) + c^3(\sqrt[3]{bcd}) + d^3(\sqrt[3]{abc}) \right) \geq$$

$$\stackrel{abcd \geq 1}{\geq} 3 \left( a^3 \sqrt[3]{\frac{1}{a}} + b^3 \sqrt[3]{\frac{1}{b}} + c^3 \sqrt[3]{\frac{1}{c}} + d^3 \sqrt[3]{\frac{1}{d}} \right)$$

$$= 3 \left( a^2 \cdot a^{\frac{2}{3}} + b^2 \cdot b^{\frac{2}{3}} + c^2 \cdot c^{\frac{2}{3}} + d^2 \cdot d^{\frac{2}{3}} \right)$$

$$\stackrel{\text{Chebyshev}}{\geq} \frac{3}{4} (\sum a^2) \left( \sum a^{\frac{2}{3}} \right) (\because \text{WLOG, if we assume } a \geq b \geq c \geq d \text{ then, } a^{\frac{2}{3}} \geq b^{\frac{2}{3}} \geq c^{\frac{2}{3}} \geq d^{\frac{2}{3}})$$

$$\stackrel{A-G}{\geq} \frac{3}{4} \left( \sum a^2 \right) \left( 4 \sqrt[4]{(abcd)^{\frac{2}{3}}} \right) = 3 \left( \sum a^2 \right) (abcd)^{\frac{1}{6}} \geq 3 \sum a^2 (\because abcd \geq 1)$$

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$$\Rightarrow a^3(b+c+d) + b^3(a+c+d) + c^3(a+b+d) + d^3(a+b+c) \stackrel{(v)}{\geq} 3 \sum a^2$$

$$\begin{aligned} & \text{Also, } a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 + \\ & + 2(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab \\ & + d^2ac + d^2bc) + \\ & + 6abcd = (ab + ac + ad + bc + bd + cd)^2 \\ & \stackrel{A-G}{\geq} 6\sqrt[6]{a^3b^3c^3d^3} (ab + ac + ad + bc + bd + cd) \\ = & 6(abcd)^{\frac{1}{2}}(ab + ac + ad + bc + bd + cd) \stackrel{abcd \geq 1}{\geq} 6(ab + ac + ad + bc + bd + cd) \\ & \Rightarrow a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 + \\ & + 2(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab \\ & + d^2ac + d^2bc) + 6abcd \\ & \stackrel{(vi)}{\geq} 6(ab + ac + ad + bc + bd + cd) \end{aligned}$$

$$\begin{aligned} \text{Now, by A-G, } a^2b^2 + a^2c^2 & \geq 2a^2bc, a^2b^2 + a^2d^2 \geq 2a^2bd, \\ a^2c^2 + a^2d^2 & \geq 2a^2cd, b^2a^2 + b^2c^2 \geq 2b^2ac, b^2a^2 + b^2d^2 \geq 2b^2ad, \\ b^2c^2 + b^2d^2 & \geq 2b^2cd, c^2a^2 + c^2b^2 \geq 2c^2ab, c^2a^2 + c^2d^2 \geq 2c^2ad, \\ c^2b^2 + c^2d^2 & \geq 2c^2bd, d^2a^2 + d^2b^2 \geq 2d^2ab, d^2a^2 + d^2c^2 \geq 2d^2ac \text{ \& } \\ & d^2b^2 + d^2c^2 \geq 2d^2bc \end{aligned}$$

Adding the last 12 inequalities, we have

$$\begin{aligned} 4P & \stackrel{(b)}{\geq} 2Q, \text{ where } P = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 \text{ \& } \\ Q & = a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + \\ & + d^2ab + d^2ac + d^2bc \end{aligned}$$

$$\text{Again, } P \stackrel{A-G}{\geq} 6\sqrt[6]{a^6b^6c^6d^6} = 6abcd \Rightarrow P \stackrel{(b)}{\geq} 6abcd$$

$$(a)+(b) \Rightarrow 6P \geq 2Q + 6abcd + P = (ab + ac + ad + bc + bd + cd)^2$$

$$\begin{aligned} & \stackrel{A-G}{\geq} 6\sqrt[6]{a^3b^3c^3d^3}(ab + ac + ad + bc + bd + cd) \\ = & 6\sqrt[6]{abcd}(ab + ac + ad + bc + bd + cd) \stackrel{abcd \geq 1}{\geq} 6(ab + ac + ad + bc + bd + cd) \\ \Rightarrow & a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 \stackrel{(vii)}{\geq} ab + ac + ad + bc + bd + cd \end{aligned}$$

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Moreover,  $2Q = [a^2(bc + bd + cd) + b^2(ac + ad + cd) + c^2(ab + ad + bd) + d^2ab + ac + bc \cdot 2$

$$\begin{aligned} &\stackrel{A-G}{\geq} 6 \left( a^2 \left( \sqrt[3]{b^2 c^2 d^2} \right) + b^2 \left( \sqrt[3]{a^2 c^2 d^2} \right) + c^2 \left( \sqrt[3]{a^2 b^2 d^2} \right) + d^2 \left( \sqrt[3]{a^2 b^2 c^2} \right) \right) \\ &\stackrel{abcd \geq 1}{\geq} 6 \left( a^2 \cdot \sqrt[3]{\frac{1}{a^2}} + b^2 \cdot \sqrt[3]{\frac{1}{b^2}} + c^2 \cdot \sqrt[3]{\frac{1}{c^2}} + d^2 \cdot \sqrt[3]{\frac{1}{d^2}} \right) \\ &= 6 \left( a \cdot a^{\frac{1}{3}} + b \cdot b^{\frac{1}{3}} + c \cdot c^{\frac{1}{3}} + d \cdot d^{\frac{1}{3}} \right) \stackrel{\text{Chebyshev}}{\geq} \frac{6}{4} (\sum a) (\sum a^{\frac{1}{3}}) \\ &(\because \text{if WLOG, we assume } a \geq b \geq c \geq d, \text{ then } a^{\frac{1}{3}} \geq b^{\frac{1}{3}} \geq c^{\frac{1}{3}} \geq d^{\frac{1}{3}}) \\ &\stackrel{A-G}{\geq} \frac{6}{4} (\sum a) \left\{ 4 \sqrt[4]{(abcd)^{\frac{1}{3}}} \right\} = 6 (\sum a) (abcd)^{\frac{1}{12}} \stackrel{abcd \geq 1}{\geq} 6 \sum a \\ &\Rightarrow 2(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab \\ &\quad + d^2ac + d^2bc) \stackrel{(viii)}{\geq} 6 \sum a \\ &\quad \text{\& lastly, } 3abcd \stackrel{(ix)}{\geq} 3 \\ &(\text{v}) + (\text{vi}) + (\text{vii}) + (\text{viii}) + (\text{ix}) \Rightarrow (2) \text{ is true} \Rightarrow (1) \text{ is true (proved)} \end{aligned}$$

JP.162. Let  $x, y, z$  be non-negative real numbers such that  $x + y + z = 1$ .

Prove that:

$$(a) \frac{3}{\sqrt{10}} \leq \frac{x}{\sqrt{1+yz}} + \frac{y}{\sqrt{1+zx}} + \frac{z}{\sqrt{1+xy}} \leq 1$$

$$(b) \sqrt{\frac{3}{5}} \leq \frac{x}{\sqrt{1+y+z}} + \frac{y}{\sqrt{1+z+x}} + \frac{z}{\sqrt{1+x+y}} \leq 1$$

When do the equalities occur?

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{WLOG, we may assume } x \geq y \geq z. \text{ Then, } \frac{1}{\sqrt{1+yz}} \geq \frac{1}{\sqrt{1+zx}} \geq \frac{1}{\sqrt{1+xy}}$$

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$$\begin{aligned} \therefore \sum \frac{x}{\sqrt{1+yz}} &\stackrel{\text{Chebyshev}}{\geq} \frac{1}{3} (\sum x) \sum \frac{1}{\sqrt{1+yz}} \stackrel{\text{Bergstrom}}{\geq} \frac{\left(\frac{1}{3}\right) \cdot 9}{\sum \sqrt{1+yz}} \\ &\stackrel{\text{CBS}}{\geq} \frac{3}{\sqrt{3}\sqrt{3+\sum xy}} \stackrel{(\sum x)^2 \geq 3\sum xy}{\geq} \frac{3}{\sqrt{3}\sqrt{3+\frac{(\sum x)^2}{3}}} = \frac{3}{\sqrt{3}\sqrt{3+\frac{1}{3}}} = \frac{3}{\sqrt{10}} \\ \therefore \sum \frac{x}{\sqrt{1+yz}} &\geq \frac{3}{\sqrt{10}}, \text{ equality at } x = y = z = \frac{1}{3} \end{aligned}$$

$$\text{Again, } \sum \frac{x}{\sqrt{1+yz}} \leq 1 \Leftrightarrow \sum \frac{x}{\sqrt{1+yz}} \leq \sum x \Leftrightarrow \sum x \left(1 - \frac{1}{\sqrt{1+yz}}\right) \geq 0 \Leftrightarrow \sum x \left(\frac{\sqrt{1+yz}-1}{\sqrt{1+yz}}\right) \stackrel{(a_1)}{\geq} 0$$

$$\because 1+yz \geq 1 (\because yz \geq 0) \therefore \sqrt{1+yz}-1 \geq 0$$

$$\text{Also, } x \geq 0 \Rightarrow \frac{x(\sqrt{1+yz}-1)}{\sqrt{1+yz}} \stackrel{(i)}{\geq} 0. \text{ Similarly, } \frac{y(\sqrt{1+zx}-1)}{\sqrt{1+zx}}, \frac{z(\sqrt{1+xy}-1)}{\sqrt{1+xy}} \stackrel{(ii),(iii)}{\geq} 0$$

(i)+(ii)+(iii)  $\Rightarrow$  (a<sub>1</sub>) is true

$$\therefore \sum \frac{x}{\sqrt{1+yz}} \leq 1, \text{ equality when } x = 0 (y, z \neq 0) \text{ or } y = 0 (z, x \neq 0) \text{ or } z = 0 (x, y \neq 0)$$

$$\text{or } x = y = 0 (z = 1) \text{ or } y = z = 0 (x = 1) \text{ or } z = x = 0 (y = 1).$$

$$\text{Again, } \because x \geq y \geq z, \therefore \frac{1}{\sqrt{1+y+z}} \geq \frac{1}{\sqrt{1+z+x}} \geq \frac{1}{\sqrt{1+x+y}}$$

$$\therefore \sum \frac{x}{\sqrt{1+y+z}} \stackrel{\text{Chebyshev}}{\geq} \frac{1}{3} (\sum x) \sum \frac{1}{\sqrt{1+y+z}} \stackrel{\text{Bergstrom}}{\geq} \frac{\left(\frac{1}{3}\right) \cdot 9}{\sum \sqrt{1+y+z}}$$

$$\stackrel{\text{CBS}}{\geq} \frac{3}{\sqrt{3}\sqrt{3+2\sum x}} = \frac{3}{\sqrt{3}\sqrt{5}} = \sqrt{\frac{3}{5}}$$

$$\therefore \sum \frac{x}{\sqrt{1+y+z}} \geq \sqrt{\frac{3}{5}} \text{ equality at } x = y = z = \frac{1}{3}$$

$$\text{Moreover, } \sum \frac{x}{\sqrt{1+y+z}} \leq 1 \Leftrightarrow \sum \frac{x}{\sqrt{1+y+z}} \leq \sum x \Leftrightarrow$$

$$\Leftrightarrow \sum x \left(1 - \frac{1}{\sqrt{1+y+z}}\right) \geq 0 \Leftrightarrow \sum x \left(\frac{\sqrt{1+y+z}-1}{\sqrt{1+y+z}}\right) \stackrel{(b_1)}{\geq} 0$$

$$\because y+z \geq 0 \therefore 1+y+z \geq 1 \Rightarrow \sqrt{1+y+z}-1 \geq 0$$

$$\& \because x \geq 0 \therefore x \left(\frac{\sqrt{1+y+z}-1}{\sqrt{1+y+z}}\right) \stackrel{(iv)}{\geq} 0$$

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$$\text{Similarly, } y \left( \frac{\sqrt{1+z+x}-1}{\sqrt{1+z+x}} \right)^{(v)} \geq 0, z \left( \frac{\sqrt{1+x+y}-1}{\sqrt{1+x+y}} \right)^{(vi)} \geq 0$$

(iv)+(v)+(vi)  $\Rightarrow$  (b<sub>1</sub>) is true

$$\therefore \sum \frac{x}{\sqrt{1+y+z}} \leq 1, \text{ equality when } x = y = 0, z = 1 \text{ or } y = z = 0, x = 1 \text{ or}$$

$$z = x = 0, y = 1 \text{ (Hence proved)}$$

**JP.163. Let  $ABC$  be an acute triangle with standard notations. Prove that:**

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \leq \frac{a+b+c}{4r}$$

*Proposed by Nguyen Viet Hung – Hanoi – Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$\therefore m_a \leq R(1 + \cos A)$  etc, for acute-angled triangle,

$$\therefore LHS \leq \sum \frac{2R \cos \frac{A}{2}}{a} = \sum \frac{2Rs(s-a)}{abc} = \frac{2Rs}{4Rrs} \sum (s-a) = \frac{s}{2r} = \frac{2s}{4r} = \frac{a+b+c}{4r} \text{ (proved)}$$

**JP.164. If  $a, b \geq 0$  then:**

$$\left( a + b + \sqrt{a^2 + b^2} \right)^2 \geq 6\sqrt{3}ab$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\forall a, b \geq 0, (a+b)^2 \geq 4ab \text{ } (\because (a-b)^2 \geq 0) \Rightarrow |a+b| \geq 2\sqrt{ab} \Rightarrow a+b \stackrel{(1)}{\geq} 2\sqrt{ab}$$

$$(\because a+b \geq 0 \text{ as } a, b \geq 0)$$

$$\text{Also, } \forall a, b \geq 0, a^2 + b^2 \geq 2ab \Rightarrow \sqrt{a^2 + b^2} \stackrel{(2)}{\geq} \sqrt{2ab}$$

$$(1) + (2) \Rightarrow LHS \geq \left( (2 + \sqrt{2})\sqrt{ab} \right)^2 = (6 + 4\sqrt{2})ab \stackrel{?}{\geq} 6\sqrt{3}ab$$

$$\Leftrightarrow (6 + 4\sqrt{2} - 6\sqrt{3})ab \stackrel{?}{\geq} 0 \rightarrow \text{true} \because ab \geq 0 \text{ \& } 6 + 4\sqrt{2} - 6\sqrt{3} > 0 \text{ (Proved)}$$

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JP.165. If  $a, b, c \geq 0$  then:

$$4(a + b + c) \leq (3\sqrt{3} - 2) (\sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{c^2 + a^2})$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\because a, b \geq 0; 2(a^2 + b^2) \geq (a + b)^2 \quad (\because (a - b)^2 \geq 0)$$

$$\Rightarrow \sqrt{a^2 + b^2} \stackrel{(1)}{\geq} \frac{|a + b|}{\sqrt{2}} = \frac{a + b}{\sqrt{2}} \quad (\because a + b \geq 0 \text{ as } a, b \geq 0)$$

$$\text{Similarly, } \forall b, c \geq 0, \sqrt{b^2 + c^2} \stackrel{(2)}{\geq} \frac{b+c}{\sqrt{2}} \quad \& \quad \sqrt{c^2 + a^2} \stackrel{(3)}{\geq} \frac{c+a}{\sqrt{2}}$$

$$(1)+(2)+(3) \Rightarrow RHS \geq \left(\frac{3\sqrt{3}-2}{\sqrt{2}}\right) (2 \sum a) = (3\sqrt{6} - 2\sqrt{2})(\sum a) \stackrel{?}{\geq} 4 \sum a$$

$$\Leftrightarrow (3\sqrt{6} - 2\sqrt{2} - 4)(\sum a) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because \sum a \geq 0 \text{ (as } a, b, c \geq 0) \&$$

$$3\sqrt{6} - 2\sqrt{2} - 4 > 0 \text{ (Hence proved)}$$

SP.151. Let  $a, b, c$  be positive real numbers such that:  $a + b + c = 3$ .

Prove that:

$$\frac{a}{\sqrt{2(b^4 + c^4) + 7bc}} + \frac{b}{\sqrt{2(c^4 + a^4) + 7ca}} + \frac{c}{\sqrt{2(a^4 + b^4) + 7ab}} \geq \frac{1}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a^4 + b^4 \leq 2(a^2 - ab + b^2)^2 \Leftrightarrow a^4 + b^4 + 6a^2b^2 - 4ab(a^2 + b^2) \geq 0$$

$$\Leftrightarrow (a^2 + b^2)^2 + 4a^2b^2 - 4ab(a^2 + b^2) \geq 0 \Leftrightarrow (a^2 + b^2 - 2ab)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore \sqrt{a^4 + b^4} \leq \sqrt{2}|a^2 - ab + b^2| = \sqrt{2}(a^2 - ab + b^2)$$

$$\left(\because a^2 - ab + b^2 = \frac{1}{4}(a + b)^2 + \frac{3}{4}(a - b)^2 > 0\right)$$

$$\Rightarrow \sqrt{2(a^4 + b^4)} \leq 2a^2 - 2ab + 2b^2 \Rightarrow \sqrt{2(a^4 + b^4)} + 7ab \leq 2a^2 + 5ab + 2b^2 =$$

$$= (2a + b)(a + 2b) \stackrel{G-A}{\leq} \frac{(2a + b + a + 2b)^2}{4} = \frac{9}{4}(a + b)^2 \Rightarrow$$

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$$\Rightarrow \frac{1}{\sqrt{2(a^4 + b^4) + 7ab}} \geq \frac{4}{9} \cdot \frac{1}{(a+b)^2} \Rightarrow \frac{c}{\sqrt{2(a^4 + b^4) + 7ab}} \stackrel{(1)}{\geq} \frac{4}{9} \cdot \frac{c}{(a+b)^2}$$

$$\text{Similarly, } \frac{a}{\sqrt{2(b^4 + c^4) + 7bc}} \stackrel{(2)}{\geq} \frac{4}{9} \cdot \frac{a}{(b+c)^2} \quad \& \quad \frac{b}{\sqrt{2(c^4 + a^4) + 7ca}} \stackrel{(3)}{\geq} \frac{4}{9} \cdot \frac{b}{(c+a)^2}$$

$$(1) + (2) + (3) \Rightarrow LHS \stackrel{(4)}{\geq} \frac{4}{9} \sum \frac{a}{(b+c)^2}$$

WLOG, we may assume  $a \geq b \geq c$

$$\text{Now, } \frac{a}{b+c} \geq \frac{b}{c+a} \Leftrightarrow a^2 + ac \geq b^2 + bc \Leftrightarrow (a-b)(a+b+c) \geq 0 \rightarrow \text{true} \because a \geq b \geq c$$

$$\therefore \frac{a}{b+c} \geq \frac{b}{c+a}. \text{ Similarly, } \frac{b}{c+a} \geq \frac{c}{a+b} \Rightarrow \frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b} \quad \& \text{ also, } \because a \geq b \geq c, \frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$

$$\begin{aligned} \therefore \text{ by Chebyshev \& using (4), } LHS &\geq \frac{4}{9} \cdot \frac{1}{3} \left( \sum \frac{a}{b+c} \right) \left( \sum \frac{1}{b+c} \right) \stackrel{\text{Nesbitt}}{\geq} \frac{4}{9} \cdot \frac{1}{3} \cdot \frac{3}{2} \left( \sum \frac{1}{b+c} \right) \stackrel{\text{Bergstrom}}{\geq} \\ &\geq \frac{2}{9} \cdot \frac{9}{2 \sum a} = \frac{1}{\sum a} = \frac{1}{3} \quad (\text{Proved}) \end{aligned}$$

**Solution 2 by Amit Dutta-Jamshedpur-India**

**Using Cauchy Schwarz's Inequality**

$$\left( \sqrt{2(b^4 + c^4)} + 2bc \right)^2 \leq 2 \left( 2(b^4 + c^4 + 2b^2c^2) \right) \leq 4(b^2 + c^2)^2 \Rightarrow$$

$$\sqrt{2(b^4 + c^4)} + 2bc \leq 2(b^2 + c^2)$$

$$\sqrt{2(b^4 + c^4)} + 7bc \leq 2b^2 + 5bc + 2c^2 \leq (b+2c)(c+2b)$$

$$\Rightarrow P \geq \sum_{cyc} \frac{a}{(b+2c)(c+2b)} \stackrel{\text{AM-GM}}{\geq} \sum_{cyc} \frac{a}{\left( \frac{3b+3c}{2} \right)^2}$$

$$\Rightarrow P \geq \sum_{cyc} \frac{4a}{9(b+c)^2} \geq \frac{4}{9} \sum_{cyc} \frac{a}{(b+c)^2}$$

$$P \geq \frac{4}{9} \sum_{cyc} \frac{a^2}{a(b+c)^2}$$

$$P \stackrel{\text{Bergstrom}}{\geq} \frac{4}{9} \left\{ \frac{(a+b+c)^2}{a(b+c)^2 + b(c+a)^2 + c(a+b)^2} \right\}$$

$$\because a(b+c)^2 + b(c+a)^2 + c(a+b)^2 =$$

$$= a(b^2 + c^2 + 2bc) + b(c^2 + a^2 + 2ac) + c(a^2 + b^2 + 2ab)$$

$$= (ab^2 + ba^2 + abc) + (ac^2 + ca^2 + abc) + (bc^2 + cb^2 + abc) + 3abc$$

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$$= ab(a + b + c) + ac(a + b + c) + bc(a + b + c) + 3abc$$

$$= (ab + bc + ac)(a + b + c) + 3abc$$

Also, by AM-GM:  $3abc \leq \frac{(a+b+c)^3}{a}$

$$\therefore a(b + c)^2 + b(c + a)^2 + c(a + b)^2 \leq (ab + bc + ac)(a + b + c) + \frac{(a + b + c)^3}{9}$$

Also,  $(a + b + c)^2 \geq 3(ab + bc + ac)$

$$(ab + bc + ac) \leq \frac{(a + b + c)^2}{3}$$

$$\Rightarrow a(b + c)^2 + b(c + a)^2 + c(a + b)^2 \leq \frac{(a + b + c)^3}{3} + \frac{(a + b + c)^3}{9} \leq \frac{4}{9}(a + b + c)^3$$

$$\therefore P \geq \frac{4}{9} \left[ \frac{(a + b + c)^2}{a(b + c)^2 + b(c + a)^2 + c(a + b)^2} \right]$$

$$P \geq \frac{4}{9} \left[ \frac{(a + b + c)^2}{\frac{4}{9}(a + b + c)^3} \right]$$

$$P \geq \left( \frac{1}{a + b + c} \right) \geq \frac{1}{3}$$

when,  $a = b = c$  }  $\Leftarrow$   $\begin{cases} \text{Equality holds when} \\ b + 2c = c + 2b, a + 2b = b + 2a, c + 2a = a + 2c \\ (a + b + c)^2 = 3(ab + bc + ac) \\ (a + b + c)^3 = 27abc \end{cases}$

SP.152. Let  $a, b, c$  be positive real numbers. Find the minimum value of:

$$P = \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{1}{\sqrt{2(b^4 + c^4)}} + \frac{1}{\sqrt{2(c^4 + a^4)}} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Firstly,  $a^4 + b^4 \leq 2(a^2 - ab + b^2)^2 \Leftrightarrow a^4 + b^4 + 6a^2b^2 - 4ab(a^2 + b^2) \geq 0 \Leftrightarrow$

$$\Leftrightarrow (a^2 + b^2)^2 + 4a^2b^2 - 4ab(a^2 + b^2) \geq 0 \Leftrightarrow (a^2 + b^2 - 2ab)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore \sqrt{a^4 + b^4} \leq \sqrt{2}|a^2 - ab + b^2| = \sqrt{2}(a^2 - ab + b^2)$$

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$$\left( \because a^2 - ab + b^2 = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2 > 0 \right)$$

$$\therefore \sqrt{2(a^4 + b^4)} \leq 2a^2 - 2ab + 2b^2 \Rightarrow \frac{1}{\sqrt{2(a^4 + b^4)}} \geq \frac{1}{2} \left( \frac{1}{a^2 + b^2 - ab} \right)$$

$$\therefore \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{a^2}{b} \geq \frac{1}{2} \left( \frac{1}{a^2 + b^2 - ab} \right) + \frac{a^2}{2b} + \frac{a^2}{2b}$$

$$\stackrel{A-G}{\geq} 3 \sqrt[3]{\frac{a^4}{8b^2(a^2 + b^2 - ab)}} = \frac{3}{2} \sqrt[3]{\frac{a^6}{(a^2 + b^2 - ab)ab \cdot ab}}$$

$$= \frac{3a^2}{2} \cdot \frac{1}{\sqrt[3]{(a^2 + b^2 - ab)ab \cdot ab}} \stackrel{A-G}{\geq} \frac{3a^2}{2} \cdot \frac{3}{a^2 + b^2 - ab + 2ab}$$

$$= \frac{9a^2}{2(a^2 + b^2 + ab)} \therefore \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{a^2}{b} \stackrel{(1)}{\geq} \frac{9}{2} \left( \frac{a^2}{a^2 + b^2 + ab} \right)$$

$$\text{Similarly, } \frac{1}{\sqrt{2(b^4 + c^4)}} + \frac{b^2}{c} \stackrel{(2)}{\geq} \frac{9}{2} \left( \frac{b^2}{b^2 + c^2 + bc} \right) \quad \& \quad \frac{1}{\sqrt{2(c^4 + a^4)}} + \frac{c^2}{a} \stackrel{(3)}{\geq} \frac{9}{2} \left( \frac{c^2}{c^2 + a^2 + ca} \right)$$

$$(1)+(2)+(3) \Rightarrow p \geq \frac{9}{2} \left( \frac{a^2}{a^2 + b^2 + ab} + \frac{b^2}{b^2 + c^2 + bc} + \frac{c^2}{c^2 + a^2 + ca} \right) \stackrel{?}{\geq} \frac{9}{2} \Leftrightarrow$$

$$\Leftrightarrow a^2(b^2 + c^2 + bc)(c^2 + a^2 + ca) + b^2(c^2 + a^2 + ca)(a^2 + b^2 + ab) + c^2(a^2 + b^2 + ab)(b^2 + c^2 + bc) - (a^2 + b^2 + ab)(b^2 + c^2 + bc)(c^2 + a^2 + ca) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow a^4b^2 + b^4c^2 + c^4a^2 \stackrel{?}{\geq} a^2b^3c + b^2c^3a + c^2a^3b \rightarrow \text{true}$$

$$\therefore x^2 + y^2 + z^2 \geq xy + yz + zx \text{ where } x = a^2b, y = b^2c, z = c^2a \Rightarrow P \geq \frac{9}{2} \text{ equality at}$$

$$a = b = c = 1 \Rightarrow P_{\min} = \frac{9}{2} \text{ (where } a = b = c = 1 \text{) (Answer)}$$

**Solution 2 by proposer**

*\* By Cauchy Schwarz inequality we have:*

$$\begin{aligned} \left( \sqrt{2(a^4 + b^4)} + 2ab \right)^2 &\leq (1^2 + 1^2) \cdot (2(a^4 + b^4) + 4a^2b^2) = 4(a^4 + 2a^2b^2 + b^4) \\ &= 4(a^2 + b^2)^2 \end{aligned}$$

$$\Leftrightarrow \sqrt{2(a^4 + b^4)} + 2ab \leq 2(a^2 + b^2) \Leftrightarrow \sqrt{2(a^4 + b^4)} \leq 2(a^2 - ab + b^2)$$

$$\Leftrightarrow \frac{1}{\sqrt{2(a^4 + b^4)}} \geq \frac{1}{2(a^2 - ab + b^2)}$$

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$$+ \text{ Similar: } \frac{1}{\sqrt{2(b^4+c^4)}} \geq \frac{1}{2(b^2-bc+c^2)}; \frac{1}{\sqrt{2(c^4+a^4)}} \geq \frac{1}{2(c^2-ca+a^2)}$$

- Hence:

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}} + \frac{1}{\sqrt{2(b^4+c^4)}} + \frac{1}{\sqrt{2(c^4+a^4)}} \geq \frac{1}{2(a^2-ab+b^2)} + \frac{1}{2(b^2-bc+c^2)} + \frac{1}{2(c^2-ca+a^2)} \quad (1)$$

- By inequality  $(m + n + p)^2 \leq 3(m^2 + n^2 + p^2)$  (equality occurs if  $m = n = p$ ). We

$$\text{have: } \left( \frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \right)^2 \leq 3 \left( \frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \right)$$

$$\Leftrightarrow \frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \geq \frac{\left( \frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \right)^2}{3} \quad (2)$$

- Let (1), (2):

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}} + \frac{1}{\sqrt{2(b^4+c^4)}} + \frac{1}{\sqrt{2(c^4+a^4)}} \geq \frac{\left( \frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \right)^2}{2 \cdot 3} \quad (3)$$

- Other, inequality:  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{x+y+z}$  ( $x = y = z > 0$ ). We have:

$$\frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \geq \frac{9}{\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2}} \quad (4)$$

- Let (3), (4):

$$\begin{aligned} \Rightarrow & \frac{1}{\sqrt{2(a^4+b^4)}} + \frac{1}{\sqrt{2(b^4+c^4)}} + \frac{1}{\sqrt{2(c^4+a^4)}} \\ & \geq \frac{\left( \frac{9}{\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2}} \right)^2}{6} \\ \Leftrightarrow & \frac{1}{\sqrt{2(a^4+b^4)}} + \frac{1}{\sqrt{2(b^4+c^4)}} + \frac{1}{\sqrt{2(c^4+a^4)}} \geq \frac{27}{2(\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2})^2} \quad (5) \end{aligned}$$

\* By Cauchy Schwarz inequality we have:

$$\begin{aligned} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} &= \left( \frac{a^2}{b} - a + b \right) + \left( \frac{b^2}{c} - b + c \right) + \left( \frac{c^2}{a} - c + a \right) \\ &= \frac{a^2 - ab + b^2}{b} + \frac{b^2 - bc + c^2}{c} + \frac{c^2 - ca + a^2}{a} \end{aligned}$$

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$$\Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{(\sqrt{a^2-ab+b^2})^2}{b} + \frac{(\sqrt{b^2-bc+c^2})^2}{c} + \frac{(\sqrt{c^2-ca+a^2})^2}{a} \geq \frac{(\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2})^2}{b+c+a} \quad (6)$$

- Other variant:

$$\begin{aligned} & \sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} \\ &= \sqrt{\frac{3(a-b)^2}{4} + \frac{(a+b)^2}{4}} + \sqrt{\frac{3(b-c)^2}{4} + \frac{(b+c)^2}{4}} + \sqrt{\frac{3(c-a)^2}{4} + \frac{(c+a)^2}{4}} \geq \\ &\geq \sqrt{\frac{(a+b)^2}{4}} + \sqrt{\frac{(b+c)^2}{4}} + \sqrt{\frac{(c+a)^2}{4}} = \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2} = a+b+c \\ &\Rightarrow a+b+c \leq \sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} \quad (7) \end{aligned}$$

$$\text{- Let (6), (7): } \Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2})^2}{\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2}}$$

$$\Leftrightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2}$$

- Let (5), (8):

$$\begin{aligned} \Rightarrow P &= \frac{1}{\sqrt{2(a^4+b^4)}} + \frac{1}{\sqrt{2(b^4+c^4)}} + \frac{1}{\sqrt{2(c^4+a^4)}} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \\ &\geq \frac{27}{2(\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2})^2} \\ &\quad + (\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2}) \end{aligned}$$

$$\text{- Let: } \sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} = t > 0$$

- Therefore, by AM-GM inequality we have:

$$\begin{aligned} \Rightarrow P &\geq \frac{27}{2t^2} + t = \frac{27}{2t^2} + \frac{t}{2} + \frac{t}{2} \geq 3 \cdot \sqrt[3]{\frac{27}{2t^2} \cdot \frac{t}{2} \cdot \frac{t}{2}} = 3 \cdot \sqrt[3]{\frac{27}{8t^2}} = 3 \cdot \frac{3}{2} = \frac{9}{2} \\ &\Rightarrow P \geq \frac{9}{2} \Rightarrow P_{\min} = \frac{9}{2}. \text{ Equality occurs if:} \end{aligned}$$

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$$\left\{ \begin{array}{l} \sqrt{2(a^4 + b^4)} = 2ab; \sqrt{2(b^4 + c^4)} = 2bc; \sqrt{2(c^4 + a^4)} = 2ca \\ \frac{1}{\sqrt{a^4 - ab + b^2}} = \frac{1}{\sqrt{b^2 - bc + c^2}} = \frac{1}{\sqrt{c^2 - ca + a^2}} \\ a - b = b - c = c - a = 0 \\ \frac{27}{2t^2} = \frac{t}{2} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} a = b = c > 0 \\ t = 3 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} a = b = c = 0 \\ \sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2} = 3 \end{array} \right. \Leftrightarrow a = b = c = 1$$

**SP.153. Solve the system of equations:**

$$\left\{ \begin{array}{l} 2 \left( \frac{x^3}{y^2} + \frac{y^3}{x^2} \right) = \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \\ 16x^5 - 20x^3 + 5\sqrt{xy} = \sqrt{\frac{y+1}{2}} \end{array} \right.$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$2 \left( \frac{x^3}{y^2} + \frac{y^3}{x^2} \right) \stackrel{(1)}{=} \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \quad \& \quad 16x^5 - 20x^3 + 5\sqrt{xy} \stackrel{(2)}{=} \sqrt{\frac{y+1}{2}}$$

Of course,  $x, y \neq 0$  &  $xy > 0 \Rightarrow x, y < 0$  or  $x, y > 0$ . If  $x, y < 0$ , then LHS of (1)  $< 0$ , but

RHS of (1)  $> 0 \Rightarrow x, y < 0$  is impossible  $\therefore x, y > 0$ . Now,  $x^4 + y^4 \leq 2(x^2 - xy + y^2)^2$

$$\Leftrightarrow x^4 + y^4 + 6x^2y^2 - 6xy(x^2 + y^2) \geq 0 \Leftrightarrow (x^2 + y^2)^2 + 4x^2y^2 - 4xy(x^2 + y^2) \geq 0$$

$\Leftrightarrow$

$$\Leftrightarrow (x^2 + y^2 - 2xy)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore \sqrt{x^4 + y^4} \leq \sqrt{2}|x^2 - xy + y^2| = \sqrt{2}(x^2 - xy + y^2)$$

$$\left( \because x^2 - xy + y^2 = \frac{1}{4}(x+y)^2 + \frac{3}{4}(x-y)^2 > 0 \right)$$

$$\Rightarrow \sqrt[4]{\frac{x^4 + y^4}{2}} \leq \sqrt{x^2 - xy + y^2} \Rightarrow \sqrt[4]{8(x^4 + y^4)} \leq 2\sqrt{x^2 - xy + y^2} \Rightarrow$$

$$\Rightarrow \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \leq 2 \left( \sqrt{x^2 - xy + y^2} + \sqrt{xy} \right) \stackrel{CBS}{\leq} 2\sqrt{2}\sqrt{x^2 - xy + y^2 + xy} =$$

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$$= 2\sqrt{2}\sqrt{x^2 + y^2} \stackrel{?}{\underset{(a)}{\leq}} \frac{2(x^2 + y^2)^2}{xy(x+y)} \Leftrightarrow \frac{(x^2 + y^2)^4}{x^2y^2(x+y)^2} \stackrel{?}{\geq} 2(x^2 + y^2) \Leftrightarrow$$

$$\Leftrightarrow (x^2 + y^2)^3 \stackrel{?}{\underset{(b)}{\geq}} 2x^2y^2(x+y)^2$$

Now,  $(x^2 + y^2)^3 = (x^2 + y^2)(x^2 + y^2)^2 \stackrel{\text{Chebyshev}}{\geq} \frac{1}{2}(x+y)^2(x^2 + y^2) \stackrel{A-G}{\geq}$

$$\geq \frac{1}{2}(x+y)^2 \cdot 4x^2y^2 = 2x^2y^2(x+y)^2 \Rightarrow (b) \text{ is true} \Rightarrow (a) \text{ is true} \Rightarrow$$

$$\Rightarrow \text{RHS of (1)} \stackrel{(i)}{\leq} \frac{2(x^2+y^2)^2}{xy(x+y)}, \text{ equality at } x = y.$$

Again,  $\text{LHS of (1)} = 2 \left( \frac{x^4}{xy^2} + \frac{y^4}{x^2y} \right) \stackrel{\text{Bergstrom}}{\underset{(ii)}{\geq}} 2 \frac{(x^2+y^2)^2}{xy(x+y)}, \text{ equality at } x = y.$

(i), (ii)  $\Rightarrow \text{LHS of (1)} = \text{RHS of (1)} = \frac{2(x^2+y^2)^2}{xy(x+y)}$  &  $\therefore$  respective equalities occur at  $x = y$

$$\therefore x = y$$

Putting  $y = x$  in (2), we get:  $16x^5 - 20x^3 + 5x = \sqrt{\frac{x+1}{2}} \Rightarrow$

$$\Rightarrow 16x^5 - 20x^3 + 5x - 1 = \sqrt{\frac{x+1}{2}} - 1 \Rightarrow (x-1)(4x^2 + 2x - 1)^2 =$$

$$= \frac{\frac{x+1}{2} - 1}{\sqrt{\frac{x+1}{2}} + 1} = \frac{x-1}{2\left(1 + \sqrt{\frac{x+1}{2}}\right)}. \text{ One possibility is } x = 1 \Rightarrow x = y = 1 \text{ is a solution when}$$

$x \neq 1, 2(4x^2 + 2x - 1)^2 \left(1 + \sqrt{\frac{x+1}{2}}\right) = 1.$  Let  $\sqrt{\frac{x+1}{2}} = t.$  Then, we have:

$$(2 + 2t)(4(2t^2 - 1)^2 + 2(2t^2 - 1) - 1)^2 - 1 = 0 \Rightarrow$$

$$\Rightarrow (2t + 1)(8t^3 - 6t + 1)(32t^5 + 16t^4 - 32t^3 - 12t^2 + 6t + 1) = 0$$

The equations yield two acceptable solutions:  $t = \cos^2 \frac{2\pi}{9} \Rightarrow \sqrt{\frac{x+1}{2}} = \cos \frac{2\pi}{9} \Rightarrow x = \cos \frac{4\pi}{9}$

$$\& t \approx .84125 \Rightarrow x \approx .415415$$

$\therefore$  all possible solutions are

$$(x = y = 1), \left(x = y = \cos \frac{4\pi}{9}\right), (x = y \approx .415415)$$

(Answer)

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SP.154. Let  $x, y, z$  be positive real numbers such that:  $x + y + z = 3$ . Find the minimum value of:

$$P = \frac{\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} + 1}{4(\sqrt{x} + \sqrt{y} + \sqrt{z})} + \frac{8}{3}(x^2 + y^2 + z^2)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

$$\begin{aligned} \sum (3\sqrt[3]{x} + x) &= \sum (\sqrt[3]{x} + \sqrt[3]{x} + \sqrt[3]{x} + x) \stackrel{\text{Cauchy}}{\geq} \sum 4\sqrt[3]{x} \Leftrightarrow \sum \sqrt[3]{x} \geq \frac{4\sum\sqrt{x} - 3}{3} \\ P &\geq \frac{1}{3} \cdot \frac{\sum\sqrt{x}}{\sum\sqrt{x}} + \frac{8}{3} \sum x^2 \geq \frac{1}{3} + \frac{8}{3} \cdot \frac{(x+y+z)^2}{3} = \frac{1}{3} + 8 = \frac{25}{3} \\ &\Rightarrow P_{\min} = \frac{25}{3} \Leftrightarrow a = b = c = 1. \end{aligned}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For  $x, y, z > 0$  and  $x + y + z = 3$ , we give  $x = a^6, y = b^6, z = c^6$ , where  $a, b, c > 0$ , we

$$\text{get } \frac{\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} + 1}{4(\sqrt{x} + \sqrt{y} + \sqrt{z})} + \frac{8}{3}(x^2 + y^2 + z^2)$$

$$\frac{a^2 + b^2 + c^2 + 1}{4(a^3 + b^3 + c^3)} + \frac{8}{3}(a^{12} + b^{12} + c^{12}) \geq \frac{25}{3}$$

$$\text{Iff } \frac{3(a^2 + b^2 + c^2 + 1)}{4(a^3 + b^3 + c^3)} + 8(a^{12} + b^{12} + c^{12}) \geq 25$$

$$\text{Iff } \frac{3(a^2 + b^2 + c^2) + 3}{(a^3 + b^3 + c^3)} + 32(a^{12} + b^{12} + c^{12}) \geq 100$$

$$\begin{aligned} \text{Iff } 3(a^2 + b^2 + c^2) + (a^6 + b^6 + c^6) + 32(a^{12} + b^{12} + c^{12})(a^3 + b^3 + c^3) &\geq \\ &\geq 100(a^3 + b^3 + c^3) \text{ and it is to be true because} \end{aligned}$$

$$1. a^6 + b^6 + c^6 \geq a^3 + b^3 + c^3$$

$$2. (a^{12} + b^{12} + c^{12})(a^3 + b^3 + c^3) \geq 3(a^3 + b^3 + c^3)$$

$$3. 3(a^2 + b^2 + c^2) + (a^{12} + b^{12} + c^{12})(a^3 + b^3 + c^3) \geq$$

$$\geq 2\sqrt{3(a^2 + b^2 + c^2)(a^{12} + b^{12} + c^{12})(a^3 + b^3 + c^3)}$$

$$\geq 2\sqrt{3(a^7 + b^7 + c^7)(a^7 + b^7 + c^7)(a^3 + b^3 + c^3)}$$

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$$\begin{aligned} &\geq 6\sqrt{(a^7 + b^7 + c^7)(a^3 + b^3 + c^3)} \\ &\geq 6(a^3 + b^3 + c^3) \end{aligned}$$

Therefore, it is to be true (it's minimum is  $8\frac{1}{3}$ )

**SP.155. Let  $x, y, z$  be positive real numbers such that:  $xyz = 1$ . Find the minimum value of:**

$$P = \frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2} + \frac{xy + yz + zx}{3} \quad (1)$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

*Solution by proposer*

\* *Lemma: Let  $a, b, c$  be positive real numbers we have inequality:*

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \quad (2)$$

$$(2): a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)$$

$$\Leftrightarrow a^4 + b^4 + c^4 + abc(a + b + c) - ab(a^2 + b^2) - bc(b^2 + c^2) - ca(c^2 + a^2) \geq 0$$

$$\Leftrightarrow a^2(a^2 - ab - ac + bc) + b^2(b^2 - bc - ba + ca) + c^2(c^2 - ca - cb + ab) \geq 0$$

$$\Leftrightarrow a^2(a - b)(a - c) + b^2(b - a)(b - c) + c^2(c - a)(c - b) \geq 0 \quad (3)$$

- Supposed  $a \geq b \geq c > 0$ .

$$+ \text{ We have: } \begin{cases} c \leq a \\ c \leq b \end{cases} \Leftrightarrow \begin{cases} c - a \leq 0 \\ c - b \leq 0 \end{cases} \Rightarrow (c - a)(c - b) \geq 0 \Leftrightarrow c^2(c - a)(c - b) \geq 0 \quad (4)$$

$$+ \text{ Let: } a^2(a - b)(a - c) + b^2(b - a)(b - c) = (a - b)[a^2(a - c) - b^2(b - c)]$$

$$\Leftrightarrow a^2(a - b)(a - c) + b^2(b - a)(b - c) = (a - b)[(a^3 - b^3) - c(a^2 - b^2)]$$

$$= (a - b)[(a - b)(a^2 + ab + b^2) - c(a - b)(a + b)]$$

$$= (a - b)(a - b)(a^2 + ab + b^2 - ac - bc)$$

$$= (a - b)^2(a^2 + ab + b^2 - ac - bc) \quad (5)$$

- Because  $a \geq b \geq c > 0$  then  $a - c \geq 0; b - c \geq 0$

$$+ \text{ Hence: } a^2 + ab + b^2 - ac - bc = a(a - c) + b(b - c) + ab \geq ab >$$

$$0; (a - b)^2 \geq 0; \forall a, b \in \mathbb{R}$$

$$\Rightarrow (a - b)^2(a^2 + ab + b^2 - ac - bc) \geq 0. \text{ Let (5): } \Rightarrow a^2(a - b)(a - c) +$$

$$b^2(b - a)(b - c) \geq 0 \quad (6)$$

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- Let (4), (6):  $\Rightarrow a^2(a-b)(a-c) + b^2(b-a)(b-c) + c^2(c-a)(c-b) \geq 0$

$\Rightarrow$  Inequality (3) true  $\Rightarrow$  (2) true and lemma get the result.

\* Let  $(a, b, c) = (x, y, z)$ :

$$\Rightarrow x^4 + y^4 + z^4 + xyz(x+y+z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)$$

(7)

- By AM-GM inequality we have:

$$\begin{aligned} xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) &\geq xy \cdot 2xy + yz \cdot 2yz + zx \cdot 2zx \\ &= 2(x^2y^2 + y^2z^2 + z^2x^2) \end{aligned}$$

$$\Leftrightarrow xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \geq 2(x^2y^2 + y^2z^2 + z^2x^2) \quad (8)$$

- Let (7), (8):  $\Rightarrow x^4 + y^4 + z^4 + xyz(x+y+z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$

$$\Leftrightarrow x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2)$$

$$\geq 4(x^2y^2 + y^2z^2 + z^2x^2) - xyz(x+y+z)$$

$$\Leftrightarrow (x^2 + y^2 + z^2)^2 \geq 4(x^2y^2 + y^2z^2 + z^2x^2) - xyz(x+y+z)$$

$$\Leftrightarrow \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \geq 1 \quad (9)$$

\* By Cauchy Schwarz inequality we have:

$$\begin{aligned} &\frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2} \\ &= \frac{\left(\frac{x^2}{2y^2 - yz + 2z^2}\right)^2}{x} + \frac{\left(\frac{y^2}{2z^2 - zx + 2x^2}\right)^2}{y} + \frac{\left(\frac{z^2}{2x^2 - xy + 2y^2}\right)^2}{z} \geq \\ &\geq \frac{\left(\frac{x^2}{2y^2 - yz + 2z^2} + \frac{y^2}{2z^2 - zx + 2x^2} + \frac{z^2}{2x^2 - xy + 2y^2}\right)^2}{x+y+z} \quad (10) \end{aligned}$$

$$\text{- Other: } \frac{x^2}{2y^2 - yz + 2z^2} + \frac{y^2}{2z^2 - zx + 2x^2} + \frac{z^2}{2x^2 - xy + 2y^2}$$

$$\begin{aligned} &= \frac{x^4}{2x^2y^2 - x^2yz + 2x^2z^2} + \frac{y^4}{2y^2z^2 - y^2zx + 2y^2x^2} + \frac{z^4}{2z^2x^2 - z^2xy + 2z^2y^2} \geq \\ &\geq \frac{(x^2 + y^2 + z^2)^2}{(2x^2y^2 - x^2yz + 2x^2z^2) + (2y^2z^2 - y^2zx + 2y^2x^2) + (2z^2x^2 - z^2xy + 2z^2y^2)} \end{aligned}$$

$$\Leftrightarrow \frac{x^2}{2y^2 - yz + 2z^2} + \frac{y^2}{2z^2 - zx + 2x^2} + \frac{z^2}{2x^2 - xy + 2y^2} \geq \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \quad (11)$$

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- Let (9), (11):  $\Rightarrow \frac{x^2}{2y^2 - yz + 2z^2} + \frac{y^2}{2z^2 - zx + 2x^2} + \frac{z^2}{2x^2 - xy + 2y^2} \geq 1$  (12)

- Let (10), (12):

$$\begin{aligned} \Rightarrow \frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2} &\geq \frac{1}{x + y + z} \\ \Rightarrow P = \frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2} \\ &+ \frac{xy + yz + zx}{3} \geq \\ &\geq \frac{1}{x + y + z} + \frac{xy + yz + zx}{3} \quad (13) \end{aligned}$$

- By inequality:  $(mn + np + pm)^2 \geq 3mnp(m + n + p)$  and AM-GM inequality and:  $xyz = 1$ . We have:

$$\begin{aligned} \frac{1}{x + y + z} + \frac{xy + yz + zx}{3} &= \left( \frac{1}{x + y + z} + \frac{xy + yz + zx}{9} + \frac{xy + yz + zx}{9} \right) + \frac{xy + yz + zx}{9} \geq \\ &\geq 3 \cdot \sqrt[3]{\frac{1}{x + y + z} \cdot \frac{xy + yz + zx}{9} \cdot \frac{xy + yz + zx}{9}} + \frac{3 \cdot \sqrt[3]{xy \cdot yz \cdot zx}}{9} \\ \Rightarrow \frac{1}{x + y + z} + \frac{xy + yz + zx}{3} &\geq 3 \cdot \sqrt[3]{\frac{(xy + yz + zx)^2}{81(x + y + z)}} + \frac{3 \sqrt[3]{(xyz)^2}}{9} \geq 3 \sqrt[3]{\frac{3xyz(x + y + z)}{81(x + y + z)}} + \frac{3 \cdot 1}{9} \\ \Rightarrow \frac{1}{x + y + z} + \frac{xy + yz + zx}{3} &\geq 3 \sqrt[3]{\frac{3 \cdot 1}{81}} + \frac{3}{9} = 1 + \frac{1}{3} = \frac{4}{3} \Leftrightarrow \frac{1}{x + y + z} + \frac{xy + yz + zx}{3} \geq \frac{4}{3} \quad (14) \end{aligned}$$

- Let (13), (14):  $\Rightarrow P \geq \frac{4}{3} \Rightarrow P_{\min} = \frac{4}{3}$ . Equality occurs if:

$$\Leftrightarrow \begin{cases} x = y = z > 0 \\ xyz = 1 \\ \frac{1}{2y^2 - yz + 2z^2} = \frac{1}{2z^2 - zx + 2x^2} = \frac{1}{2x^2 - xy + 2y^2} \Leftrightarrow x = y = z = 1. \\ \frac{1}{x + y + z} = \frac{xy + yz + zx}{9} \end{cases}$$

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SP.156. Find all the polynomials  $P \in \mathbb{R}[x]$  having the property

$$P(x) = P\left(x + \sqrt{x^2 + 1}\right), \forall x \in \mathbb{R}$$

*Proposed by Marian Ursărescu – Romania*

*Solution 1 by Ravi Prakash-New Delhi-India*

Let  $P(x)$  be a polynomial of degree  $m$  where  $m \in \mathbb{N}$ .

If  $m = 1$ , let  $P(x) = ax + b, a \neq 0$ , then  $ax + b = a(x + \sqrt{x^2 + 1}) + b \quad \forall x \in \mathbb{R}$

$$\Rightarrow a\sqrt{x^2 + 1} = 0, \forall x \in \mathbb{R} \Rightarrow a = 0$$

**A contradiction.**

**Assume  $m \geq 2$ .**

**Choose a sequence  $m_1 > m_2 > \dots > m_m$  of positive integers such that**

$$m_{k+1} > m_k + \sqrt{m_k^2 + 1} \text{ for } 1 \leq k \leq m - 1.$$

**For  $1 \leq r \leq m$**

$$P(m_r) = P\left(m_r + \sqrt{m_r^2 + 1}\right) \text{ (given)}$$

**By the Rolle's theorem  $\exists \alpha_r \in (m_r, m_r + \sqrt{m_r^2 + 1})$  such that**

$P'(\alpha_r) = 0 \quad (1 \leq r \leq m) \Rightarrow P'(x)$  has at least  $m$  zeros. But  $P'(x)$  is a polynomial of degree  $(m - 1)$ . **A contradiction.**

$\therefore$  there is no polynomial of degree  $\geq 1$ , satisfying the given condition.

Thus,  $P(x)$  satisfies the given condition if and only if  $P(x)$  is a constant

*Solution 2 by Tran Hong-Dong Thap-Vietnam*

Suppose  $\deg P(x) \geq 1: P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0;$

$$(a_n, a_{n-1}, \dots, a_0 \in \mathbb{R} \text{ and } a_n \neq 0)$$

Let  $x = -t (\forall t \in \mathbb{R}) \Rightarrow P(-t) = P(-t + \sqrt{t^2 + 1}) \quad \forall t \in \mathbb{R}$

$$\Leftrightarrow P(-t) = P\left(\frac{1}{\sqrt{t^2 + 1} + t}\right) \forall t \in \mathbb{R} \quad (*)$$

$n = 2k (k \in \mathbb{N}^*)$  then

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$$\lim_{t \rightarrow +\infty} P(-t) = \begin{cases} +\infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases} \text{ but } \lim_{t \rightarrow +\infty} P\left(\frac{1}{\sqrt{t^2 + 1} + t}\right) = P(0) = a_0$$

$n = 2k + 1$  ( $k \in \mathbb{N}$ ) then

$$\lim_{t \rightarrow +\infty} P(-t) = \begin{cases} -\infty & \text{if } a_n > 0 \\ +\infty & \text{if } a_n < 0 \end{cases} \text{ but } \lim_{t \rightarrow +\infty} P\left(\frac{1}{\sqrt{t^2 + 1} + t}\right) = P(0) = a_0$$

$$\Rightarrow \deg P(x) = 0 \Rightarrow P(x) = C, \forall x \in \mathbb{R} \quad (C: \text{const})$$

**SP.157. Let  $f: [0, +\infty) \rightarrow [0, +\infty)$  a derivable function and  $a > 1$ . If**

**$f'(x)(f(x) + x^2 + 2x + a) = 1, \forall x \geq 0$  then:  $\lim_{x \rightarrow \infty} f(x)$  exists and it is finite.**

*Proposed by Marian Ursărescu – Romania*

*Solution by Tran Hong-Vietnam*

$$f'(x)(f(x) + x^2 + 2x + a) = 1$$

$$\Rightarrow f'(x) > 0 \forall x \geq 0 \quad (\text{because } f(x) \geq 0 \forall x \geq 0, a > 1)$$

$$\Rightarrow f(x) \nearrow \text{ on } [0, +\infty)$$

$$\Rightarrow \lim_{x \rightarrow +\infty} f(x) = l \in [0, +\infty) \text{ or } \lim_{x \rightarrow +\infty} f(x) = +\infty.$$

$$\text{If } \lim_{x \rightarrow +\infty} f(x) = +\infty \Rightarrow \lim_{x \rightarrow +\infty} f'(x) = \lim_{x \rightarrow +\infty} \frac{1}{f(x) + x^2 + 2x + a} = 0$$

$$\Rightarrow \exists \alpha > 0: 0 < f'(x) \leq \frac{1}{x^2 + 1} \quad (\forall x \geq \alpha) \Rightarrow 0 < \int_0^x f'(t) dt \leq \int_0^x \frac{1}{t^2 + 1} dt$$

$$\Rightarrow 0 < f(x) - f(\alpha) \leq \tan^{-1}(x) - \tan^{-1}(\alpha) \Rightarrow f(x) \leq \tan^{-1}(x) + f(\alpha) - \tan^{-1}(\alpha)$$

$$\Rightarrow \lim_{x \rightarrow +\infty} f(x) \leq \frac{\pi}{2} + f(\alpha) - \tan^{-1}(\alpha)$$

which is contrary with  $\lim_{x \rightarrow +\infty} f(x) = +\infty$ . So, we have  $\lim_{x \rightarrow +\infty} f(x) = l \in [0, +\infty)$ .

*Proved.*

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SP.158. Prove that for any real numbers  $a_1, a_2, \dots, a_n$ , the inequality holds:

$$\sum_{i=1}^n a_i (a_i + a_{i+1}) (a_i^2 + a_{i+1}^2) \dots (a_i^{2^k} + a_{i+1}^{2^k}) \geq 0$$

Where  $k$  is a positive integer and  $a_{n+1} = a_1$ . When does the equality occur?

*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

*Solution by Tran Hong-Vietnam*

$$\text{With } n = 1: a_1(a_1 + a_2)(a_1^2 + a_2^2) \dots (a_1^{2^k} + a_2^{2^k})$$

$$\stackrel{(a_2=a_1)}{=} 2a_1^2 \cdot 2a_1^2 \dots 2a_1^{2^k} \geq 0 \text{ (true)}$$

Suppose it true with  $1, 2, \dots, n$ ; we prove with  $n + 1$ ;

$$\text{Let: } a_{n+1} = \max\{a_i | i = 1, 2, \dots, n + 1\} \Rightarrow a_{n+1} \geq a_1$$

$$U_{n+1} = \sum_{i=1}^{n+1} a_i (a_i + a_{i+1}) (a_i^2 + a_{i+1}^2) \dots (a_i^{2^k} + a_{i+1}^{2^k})$$

$$= \sum_{i=1}^n a_i (a_i + a_{i+1}) (a_i^2 + a_{i+1}^2) \dots (a_i^{2^k} + a_{i+1}^{2^k}) + a_{n+1} (a_{n+1} + a_{n+2}) (a_{n+1}^2 + a_{n+2}^2) \dots$$

$$\dots (a_{n+2}^{2^k} + a_{n+2}^{2^k})$$

$$U_{n+1} = \sum_{i=1}^n a_i (a_i + a_{i+1}) (a_i^2 + a_{i+1}^2) \dots (a_i^{2^k} + a_{i+1}^{2^k}) +$$

$$+ a_{n+1} (a_{n+1} + a_1) (a_{n+1}^2 + a_1^2) \dots (a_{n+1}^{2^k} + a_1^{2^k});$$

$$\text{If } a_{n+1} \geq a_1 \geq 0 \text{ then } U_{n+1} \geq 0$$

$$\text{If } a_1 \leq a_{n+1} \leq 0 \text{ then } U_{n+1} \geq 0$$

If  $a_{n+1} \geq 0 > a_1$  then we have:

$$U_{n+1} = \sum_{i=1}^n a_i (a_i + a_{i+1}) (a_i^2 + a_{i+1}^2) \dots (a_i^{2^k} + a_{i+1}^{2^k}) +$$

$$+ \frac{a_{n+1}}{a_{n+1} - a_1} \cdot (a_{n+1}^2 - a_1^2) (a_{n+1}^2 + a_1^2) \dots (a_{n+1}^{2^k} + a_1^{2^k})$$

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$$= \sum_{i=1}^n a_i (a_i + a_{i+1})(a_i^2 + a_{i+1}^2) \dots (a_i^{2^k} + a_{i+1}^{2^k}) + a_{n+1} \cdot \frac{a_{n+1}^{2^{k+1}} - a_1^{2^{k+1}}}{a_{n+1} - a_1} \geq 0$$

$\Rightarrow$  Proved. Equality  $\Leftrightarrow a_i = 0$  ( $i = 1, 2, \dots, n$ )

**SP.159. Prove that in any triangle  $ABC$  with standard notations, the inequality holds:**

$$(a + b + c)(a^2 + b^2 + c^2) \geq 2(b + c)h_a^2 + 2(c + a)h_b^2 + 2(a + b)h_c^2$$

*Proposed by Nguyen Viet Hung – Hanoi – Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum a \sum a^2 \stackrel{(1)}{\geq} 2 \sum (b + c)h_a^2$$

$$(1) \Leftrightarrow 2s(s^2 - 4Rr - r^2) \geq \sum (2s - a) \frac{b^2 c^2}{4R^2}$$

$$\Leftrightarrow 4R^2 \cdot 2s(s^2 - 4Rr - r^2) \geq 2s \left\{ \left( \sum ab \right)^2 - 16Rrs^2 \right\} - 4Rrs \left( \sum ab \right)$$

$$\Leftrightarrow 4R^2(s^2 - 4Rr - r^2) \geq (s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 2Rr(s^2 + 4Rr + r^2)$$

$$\Leftrightarrow s^4 - s^2(4R^2 + 10Rr - 2r^2) + 16R^3r + 12R^2r^2 + 6Rr^3 + r^4 \stackrel{(2)}{\leq} 0$$

Now,  $s^2 \geq m - n \Rightarrow s^2 - m + n \stackrel{(2)}{\geq} 0$  &  $s^2 \leq m + n \Rightarrow s^2 - m - n \stackrel{(b)}{\leq} 0$ , where

$$m = 2R^2 + 10Rr - r^2 \text{ \& \ } n = 2(R - 2r)\sqrt{R^2 - 2Rr}$$

$$(a).(b) \Rightarrow s^4 - 2ms^2 + m^2 - n^2 \leq 0 \Rightarrow$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + 64R^3r + 48R^2r^2 + 12Rr^3 + r^4 \stackrel{(3)}{\leq} 0$$

(2), (3)  $\Rightarrow$  in order to prove (2), it suffices to show:

$$-s^2(4R^2 + 10Rr - 2r^2) + 16R^3r + 12R^2r^2 + 6Rr^3 \leq -s^2(4R^2 + 20Rr - 2r^2) +$$

$$+ 64R^3r + 48R^2r^2 + 12Rr^3 \Leftrightarrow 5Rs^2 \stackrel{(4)}{\leq} 24R^3 + 18R^2r + 3Rr^2$$

$$\text{Now, LHS of (4)} \stackrel{\text{Gerretsen}}{\leq} 5R(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} 24R^3 + 18R^2r + 3Rr^2$$

$$\Leftrightarrow 2R^2 - Rr - 6r^2 \stackrel{?}{\leq} 0 \Leftrightarrow (R - 2r)(2R + 3r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \text{ (proved)}$$

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SP.160. Let  $a, b, c$  be positive real numbers such that  $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ .

Prove that:

$$3(a^3b + b^3c + c^3a) \geq (a + b + c)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum a \geq \sum \frac{1}{a} \Leftrightarrow \frac{abc(\sum a)}{\sum ab} \stackrel{(1)}{\geq} 1$$

$$\text{Now, } 3(a^3b + b^3c + c^3a) = 3abc \left( \frac{a^3}{ac} + \frac{b^3}{ab} + \frac{c^3}{bc} \right) \geq$$

$$\stackrel{\text{Holder}}{\geq} 3abc \frac{(\sum a)^3}{3(\sum ab)} = \frac{abc(\sum a)}{\sum ab} \cdot (\sum a)^2 \stackrel{\text{by (1)}}{\geq} (\sum a)^2$$

(Proved)

SP.161. Prove that the following inequalities holds for all real numbers

$$a, b, c \in [0, 1]$$

$$(a) \quad (1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \leq (1 - abc + a^2b^2c^2)$$

$$(b) \quad (1 - a + a^2)^2(1 - b + b^2)^2(1 - c + c^2)^2 \leq \\ \leq (1 - ab + a^2b^2)(1 - bc + b^2c^2)(1 - ca + c^2a^2)$$

When does the equality occur?

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Ravi Prakash-New Delhi-India

$$(1 - a + a^2)(1 - b + b^2) - (1 - ab + a^2b^2) = 1 - a + a^2 - b + ab - a^2b + b^2 - \\ - ab^2 + a^2b^2 - [1 - ab + a^2b^2]$$

$$= (a + b)^2 - (a + b) - ab(a + b) = (a + b)[a + b - ab - 1] =$$

$$= -(a + b)(1 - a)(1 - b) \leq 0$$

Equality when  $a = b = 0$  or  $a = 1$  or  $b = 1$

$$\therefore (1 - a + a^2)(1 - b + b^2) \leq 1 - ab + a^2b^2$$

Equality when  $a = b = 0$  or  $a = 1$  or  $b = 1$ .

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$$\begin{aligned} &\Rightarrow (1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \leq (1 - ab + a^2b^2)(1 - c + c^2) \leq \\ &\leq 1 - abc + a^2b^2c^2. \text{ Equality when } a = b = c = 0 \text{ or when at least two of } a, b, c \text{ are} \\ &\text{equal to 1. Next, } (1 - a + a^2)^2(1 - b + b^2)^2(1 - c + c^2)^2 = \\ &= [(1 - a + a^2)(1 - b + b^2)][(1 - b + b^2)(1 - c + c^2)][(1 - c + c^2)(1 - a + a^2)] \\ &= [(1 - ab + a^2b^2) - (a + b)(1 - a)(1 - b)] \\ &\quad [(1 - bc + b^2c^2) - (b + c)(1 - b)(1 - c)] \\ &\quad [(1 - ca + c^2 + a^2) - (c + a)(1 - c)(1 - a)] \\ &\leq (1 - ab + a^2b^2)(1 - bc + b^2c^2)(1 - ca + c^2a^2) \\ &\quad \text{with equality iff } (a + b)(1 - a)(1 - b) = 0 \\ &\quad (b + c)(1 - b)(1 - c) = 0; (c + a)(1 - a)(1 - c) = 0 \\ &\Leftrightarrow a = b = 0, c = 0 \text{ or } a = b = 0, c = 1 \text{ or } a = 1, b = c = 0 \text{ or } a = 0, b = 1, c = 0 \\ &\quad \text{or } a = 1, b = 1 \text{ or } a = 1, c = 1 \text{ or } b = 1, c = 1 \end{aligned}$$

SP.162. If  $m \in [0, \infty)$ ,  $x, y, z, t \in (0, \infty)$ , then in any triangle  $ABC$ , with usual notations holds:

$$\sum_{\text{cyclic}} \frac{(xa^2 + ym_b^2)^{m+1}}{(zw_b^2 + th_c^2)^m} \geq \frac{(4x + 3y)^{m+1}}{3^m(z + t)^m} \sqrt{3}S$$

Proposed by D.M. Băţineţu – Giurgiu, Neculai Stanciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Firstly, } \sum w_a^2 &\stackrel{(1)}{\leq} \sum s(s - a) = s^2. \text{ Now, LHS} \stackrel{\text{Radon}}{\geq} \frac{(x \sum a^2 + y \sum m_a^2)^{m+1}}{(z \sum w_a^2 + t \sum h_a^2)^m} = \frac{(x \sum a^2 + \frac{3}{4}y \sum a^2)^{m+1}}{(z \sum w_a^2 + t \sum h_a^2)^m} \geq \\ &\geq \frac{(\sum a^2)^{m+1} (4x+3y)^{m+1}}{4^{m+1} (z+t)^m (\sum w_a^2)^m} \quad (\because h_a \leq w_a \text{ etc, } \Rightarrow h \sum h_a^2 \leq \sum w_a^2) \\ &\geq \frac{(\sum a^2)^{m+1} (4x+3y)^{m+1}}{(4s^2)^m \cdot 4(z+t)^m} \quad (\text{using (1)}) \\ &\geq \frac{(\sum a^2)^{m+1} (4x+3y)^{m+1}}{(3 \sum a^2)^m \cdot 4(z+t)^m} \quad (\because 4s^2 = (\sum a)^2 \leq 3 \sum a^2) \\ &= \frac{(\sum a^2)(4x+3y)^{m+1}}{3^m \cdot 4(z+t)^m} \stackrel{\text{Ionescu}}{\geq} \stackrel{\text{Weitzenbock}}{\geq} \frac{4\sqrt{3}S(4x+3y)^{m+1}}{3^m \cdot 4(z+t)^m} = \frac{(4x+3y)^{m+1}}{3^m(z+t)^m} \sqrt{3}S \quad (\text{Proved}) \end{aligned}$$

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SP.163. If  $m \in [0, \infty)$ ,  $x, y, z, t \in (0, \infty)$ , then in any triangle  $ABC$ , with usual notations holds:

$$\sum_{\text{cyclic}} \frac{(xa^2 + yb^2)^{m+1}}{(zw_b^2 + tw_c^2)^m} \geq \frac{4^{m+1}(x+y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m} S$$

Proposed by D.M. Băținețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

Firstly,  $\sum w_a^2 \stackrel{(1)}{\leq} \sum s(s-a) = s^2$ . Now, LHS  $\stackrel{\text{Radon}}{\geq} \frac{(x \sum a^2 + y \sum a^2)^{m+1}}{(z \sum w_a^2 + t \sum w_a^2)^m} = \frac{(x+y)^{m+1} (\sum a^2)^{m+1}}{(z+t)^t (\sum w_a^2)^m}$

by (1)  $\frac{(x+y)^{m+1} (\sum a^2)^{m+1}}{(z+t)^m (s^2)^m} \geq \frac{(x+y)^{m+1} (\sum a^2)^{m+1}}{(z+t)^m \left(\frac{3}{4} \sum a^2\right)^m} \left(\because s^2 \leq \frac{3 \sum a^2}{4}\right)$

$= \frac{4^m (x+y)^{m+1} (\sum a^2)^{m+1}}{(z+t)^m 3^m \cdot (\sum a^2)^m} = \frac{4^m (x+y)^{m+1} \sum a^2}{(z+t)^m \cdot 3^m}$

$\stackrel{\text{Ionescu-Wetizenbock}}{\geq} \frac{4^m (x+y)^{m+1} \cdot 4\sqrt{3}S}{(z+t)^m \cdot 3^m} = \frac{4^{m+1} (x+y)^{m+1} 3^{\frac{1}{2}} S}{3^m (z+t)^m} = \frac{4^{m+1} (x+y)^{m+1}}{3^{m-\frac{1}{2}} (z+t)^m} S \text{ (Proved)}$

SP.164. If  $a, b, c > 0$  then:

$$(a+b)\sqrt{a^2 + b^2 - ab} + (b+c)\sqrt{b^2 + c^2 - bc} + (c+a)\sqrt{c^2 + a^2 - ca} \geq 2(ab + bc + ca)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a^2 + b^2 - ab = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2 \geq \frac{1}{4}(a+b)^2 \Rightarrow \sqrt{a^2 + b^2 - ab} \stackrel{(1)}{\geq} \frac{a+b}{2}$$

$$\text{Similarly, } \sqrt{b^2 + c^2 - bc} \stackrel{(2)}{\geq} \frac{b+c}{2} \text{ \& } \sqrt{c^2 + a^2 - ca} \stackrel{(3)}{\geq} \frac{c+a}{2}$$

$$(1), (2), (3) \Rightarrow LHS \geq \frac{\sum(a+b)^2}{2} \stackrel{?}{\geq} 2 \sum ab \Leftrightarrow 2 \sum a^2 + 2 \sum ab \stackrel{?}{\geq} 4 \sum ab \Leftrightarrow \sum a^2 \stackrel{?}{\geq} \sum ab$$

→ true (Proved)

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SP.165. If  $a, b, c \geq 0$  then:

$$\begin{aligned} (a+b)\sqrt{a^2+b^2} + (b+c)\sqrt{b^2+c^2} + (c+a)\sqrt{c^2+a^2} &\geq \\ &\geq (2\sqrt{3}-1)(ab+bc+ca) \end{aligned}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall a, b \geq 0, 2(a^2 + b^2) \geq (a + b)^2 (\because (a - b)^2 \geq 0)$$

$$\Rightarrow \sqrt{a^2 + b^2} \stackrel{(1)}{\geq} \frac{|a + b|}{\sqrt{2}} = \frac{a + b}{\sqrt{2}} (\because a + b \geq 0 \text{ as } a, b \geq 0)$$

$$\text{Similarly, } \forall b, c \geq \sqrt{b^2 + c^2} \stackrel{(2)}{\geq} \frac{b+c}{\sqrt{2}} \text{ \& } \forall c, a \geq 0, \sqrt{c^2 + a^2} \stackrel{(3)}{\geq} \frac{c+a}{\sqrt{2}}$$

$$(1), (2), (3) \Rightarrow LHS \geq \frac{1}{\sqrt{2}} \sum (a + b)^2 \stackrel{?}{\geq} (2\sqrt{3} - 1) \sum ab$$

$$\Leftrightarrow 2 \sum a^2 + 2 \sum ab \stackrel{?}{\geq} (2\sqrt{6} - \sqrt{2}) \sum ab \Leftrightarrow 2 \sum a^2 \stackrel{?}{\geq} (2\sqrt{6} - \sqrt{2} - 2) \sum ab$$

$$\text{Now, } \because 2 > 2\sqrt{6} - \sqrt{2} - 2$$

$$\therefore 2 - (2\sqrt{6} - \sqrt{2} - 2) > 0$$

$$\therefore \left( \sum a^2 \right) \{ 2 - (2\sqrt{6} - \sqrt{2} - 2) \} \geq 0 (\because \sum a^2 \geq 0)$$

$$\Rightarrow 2 \sum a^2 \geq (2\sqrt{6} - \sqrt{2} - 2) \sum a^2 \geq (2\sqrt{6} - \sqrt{2} - 2) \sum ab \Rightarrow (a) \text{ is true (Proved)}$$

Solution 2 by Michael Sterghiou-Greece

If  $a, b, c \geq 0$  then:

$$\sum_{cyc} (a + b) \sqrt{a^2 + b^2} \geq (2\sqrt{3} - 1) \sum_{cyc} ab \quad (1)$$

$$\sum_{cyc} (a + b) \sqrt{a^2 + b^2} \stackrel{AM-GM}{\geq} \sum_{cyc} 2\sqrt{ab} \cdot \sqrt{2ab} = \sum_{cyc} 2\sqrt{2}ab = 2\sqrt{2} \cdot \sum_{cyc} ab$$

$$\geq (2\sqrt{3} - 1) \sum_{cyc} ab$$

$$\left. \begin{array}{l} \sqrt{2} > 1 \\ \sqrt{3} > 1 \end{array} \right\} \rightarrow \sqrt{2} + \sqrt{3} > 2 \rightarrow 1 - \frac{2}{\sqrt{2} + \sqrt{3}} > 0 \rightarrow 1 - 2 \cdot \frac{(\sqrt{3})^2 (\sqrt{2})^2}{\sqrt{3} + \sqrt{2}} > 0$$

$$\rightarrow 1 \cdot (\sqrt{3} - \sqrt{2}) > 0$$

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→  $2\sqrt{2} - (2\sqrt{3} - 1) > 0$ . Done.

**UP.151. Given real numbers  $a_1, a_2, \dots, a_n \in [0, 1]$ . Find the maximum and minimum possible value of**

$$a_1 + a_2 + \dots + a_n + (1 - a_1)(1 - a_2) \dots (1 - a_n).$$

*Proposed by Nguyen Viet Hung – Hanoi – Vietnam*

**Solution 1 by proposer**

We fix the variables  $a_2, a_3, \dots, a_n$  and rewrite the given expression as follows

$$f(a_1) = [1 - (1 - a_2) \dots (1 - a_n)]a_1 + (1 - a_2) \dots (1 - a_n) + (a_2 + \dots + a_n)$$

Now, we consider the following two cases:

Case 1. If  $a_2, \dots, a_n$  are all zero, then  $f(a_1) = 1$ .

Case 2. If there is at least one of these numbers  $a_2, \dots, a_n$  is non zero. Then  $f(a_1)$  is a first degree function (which is called linear function) with the variable  $a_1$  and the leading coefficient is positive, hence  $f(0) \leq f(a_1) \leq f(1)$ . Similarly, we also have  $f(0) \leq f(a_i) \leq f(1)$  for each  $i \in \{2, 3, \dots, n\}$ . Let  $F(a_1, a_2, \dots, a_n)$  denotes the given expression, then we obtain  $1 = F(0, 0, \dots, 0) \leq F(a_1, a_2, \dots, a_n) \leq F(1, 1, \dots, 1) = n$ .

Combining the above two cases we deduce that  $\min F(a_1, a_2, \dots, a_n) = 1$  when  $(a_1, a_2, \dots, a_n) = (x, 0, \dots, 0)$  and its permutations;  $\max F(a_1, a_2, \dots, a_n) = n$  when  $(a_1, a_2, \dots, a_n) = (1, 1, \dots, 1)$

**Solution 2 by Michael Sterghiou-Greece**

$$A = a_1 + a_2 + \dots + a_n + (1 - a_1)(1 - a_2) \cdot \dots \cdot (1 - a_n) \quad (1)$$

From Weierstrass product inequality we have  $\prod_{k=1}^n (1 - a_k) \geq 1 - \sum_{k=1}^n a_k$  (\*) which gives immediately  $\min A = 1$  when  $a_k = 0 \forall k \in \{1, 2, \dots, n\}$ . Now,

$$\prod_{k=1}^n (1 - a_k) \leq \left[ \frac{\sum_{k=1}^n (1 - a_k)}{n} \right]^n \text{ from AM-GM given } 1 - a_k \geq 0 \forall k \text{ or}$$

$$\prod_{k=1}^n (1 - a_k) \leq \left( 1 - \frac{S_n}{n} \right)^n \text{ where } S_n = \sum_{k=1}^n a_k \quad (1) \text{ becomes } A \leq S_n + \left( 1 - \frac{S_n}{n} \right)^n$$

$$\text{Let } S_n = x, x \in [0, n]: f(x) = x + \left( 1 - \frac{x}{n} \right)^n. f'(x) = 1 - \left( 1 - \frac{x}{n} \right)^{n-1} > 0 \text{ as } \frac{x}{n} \leq 1,$$

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$$0 \leq \left(1 - \frac{x}{n}\right)^{n-1} \leq 1 \text{ so } f(x) \uparrow. \text{ This means } f(x) \leq f(n) \text{ or}$$

$$S_n + \left(1 - \frac{S_n}{n}\right)^n \leq n + \left(1 - \frac{n}{n}\right)^n = n. \text{ Therefore } A \leq n \text{ and } A_{\max} = n \text{ when } a_k = 1$$

$$\forall k \in \{1, 2, \dots, n\}$$

\* proved easily by induction over n.

UP.152. Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that:

$$\sum_{1 \leq i < j \leq n} a_i a_j \geq \frac{n(n-1)}{2}$$

Prove that:

$$\sum_{i=1}^n \frac{1}{a_i^2 + n - 1} \leq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

$$\begin{aligned} \sum \frac{1}{a_i^2 + n - 1} &= \frac{1}{n-1} \sum_{i=1}^n \left(1 - \frac{a_i^2}{a_i^2 + n - 1}\right) = \frac{1}{n-1} \left[n - \sum_{i=1}^n \frac{a_i^2}{a_i^2 + n - 1}\right] \quad (*) \\ \sum_{i=1}^n \frac{a_i^2}{a_i^2 + n - 1} &\stackrel{(Schwarz)}{\geq} \frac{(\sum_{i=1}^n a_i)^2}{\sum_{i=1}^n a_i^2 + n(n-1)} = \frac{\sum_{i=1}^n a_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j}{\sum_{i=1}^n a_i^2 + n(n-1)} \\ &\geq \frac{\sum_{i=1}^n a_i^2 + 2n \left(\frac{n-1}{2}\right)}{\sum_{i=1}^n a_i^2 + n(n-1)} = 1 \Rightarrow (*) \leq \frac{1}{n-1} (n-1) = 1 \text{ (Proved)} \end{aligned}$$

Solution 2 by Nassim Nicholas Taleb-USA

Solution with probabilistic commentary:

$$\text{There are } \frac{n(n-1)}{2} \text{ distinct terms of } \sum_{j=1}^n \sum_{i=1}^{j-1} a_i a_j, \sum_{j=1}^n \sum_{i=1}^{j-1} 1 = \frac{n(n-1)}{2}$$

Note that we have the lower or upper half of a symmetric square matrix minus the diagonal  $a_i^2$ .

The average term  $a_i a_j \geq 1$ , allora the mean  $a_i^2 \geq 1$ . In probabilistic terms:  $\mathbb{E}(a_i a_{j \neq i}) \leq$

$$\mathbb{E}(a^2)$$

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**Proof: (Cauchy – Schwarz):**  $\left(\frac{2}{n(n-1)} \sum_{j=1}^n \sum_{i=1}^{j-1} a_i a_j\right)^2 \leq \left(\frac{1}{n} \sum_{j=1}^n a_i^2\right) \left(\frac{1}{n} \sum_{j=1}^n a_j^2\right) =$   
 $\left(\frac{1}{n} \sum_{j=1}^n a_i^2\right)^2$

hence, since all  $a_i$  are positive:  $\frac{1}{n} \sum_{j=1}^n a_i^2 \geq 1$

by Harmonic/Arithmetic Mean inequality,  $\frac{1}{n} \sum_{i=1}^n \frac{1}{a_i^2+n-1} \geq \frac{n}{\sum_{i=1}^n (a_i^2+n-1)} = \frac{n}{\sum_{i=1}^n a_i^2+n^2-n}$

allora  $\frac{n^2}{\sum_{i=1}^n a_i^2+n^2-n} \leq 1$  when  $\frac{1}{n} \sum_{j=1}^n a_i^2 \geq 1$ .

**UP.153. Find:**

$$\Omega = \int_0^1 \int_0^\infty \left(\frac{x^{100}}{100^x}\right) (y \ln y)^{100} dx dy$$

*Proposed by Ekpo Samuel – Nigeria*

**Solution 1 by Yen Tung Chung-Taichung-Taiwan**

$$\Omega = \int_0^1 \int_0^\infty \left(\frac{x^{100}}{100^x}\right) (y \ln y)^{100} dx dy = \left(\int_0^1 (y \ln y)^{100} dy\right) \left(\int_0^\infty \frac{x^{100}}{100^x} dx\right)$$

$$= \left(\frac{100!}{101^{101}}\right) \left(\frac{100!}{(\ln 100)^{101}}\right) = \frac{(100!)^2}{(101 \ln 100)^{101}}$$

where

$$\text{let } t = -\ln y \Rightarrow y = e^{-t}, dy = -e^{-t} dt$$

$$\int_0^1 (y \ln y)^{100} dy = \int_\infty^0 (-te^{-t})^{100} (-e^{-t} dt) = \int_0^\infty t^{100} e^{-101t} dt = \frac{\Gamma(101)}{101^{101}} = \frac{100!}{101^{101}}$$

$$\int_0^\infty \frac{x^{100}}{100^x} dx = \int_0^\infty x^{100} e^{-x \ln 100} dx = \frac{\Gamma(101)}{(\ln 100)^{101}} = \frac{100!}{(\ln 100)^{101}}$$

**Solution 2 by Zaharia Burghilea-Romania**

$$\Omega = \int_0^1 \int_0^\infty \left(\frac{x^{100}}{100^x}\right) (y \ln y)^{100} dx dy = \int_0^1 (y \ln y)^{100} dy \int_0^\infty \frac{x^{100}}{100^x} dx$$

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$$I_1 = \int_0^1 (y \ln y)^{100} dy$$

Consider:

$$I(n) = \int_0^1 x^n dx = \frac{1}{n+1}$$

$$I'(n) = \int_0^1 x^n \ln x dx = -\frac{1}{(n+1)^2}$$

$$I''(n) = \int_0^1 x^2 \ln^2 x dx = -1 \cdot (-2) \frac{1}{(n+1)^3}$$

$$I^{(k)}(n) = \int_0^1 x^n \ln^k x dx = (-1)^k k! \frac{1}{(n+1)^{k+1}}$$

$$I_1 = I^{(100)}(100) = \int_0^1 x^{100} \ln^{100} x dx = 100! \frac{1}{101^{101}}$$

$$I_2 = \int_0^{\infty} \frac{x^{100}}{100^x} dx$$

$$\text{substitute } x = \log_{100} t \rightarrow dx = \frac{1}{\ln 100} \frac{1}{t} dt$$

$$I_2 = \frac{1}{\ln 100} \int_1^{\infty} \frac{1}{t} \left( \frac{\ln t}{\ln(100)} \right)^{100} \frac{1}{t} dt = \frac{1}{\ln^{101}(100)} \int_1^{\infty} \frac{\ln^{100} t}{t^2} dt$$

$$\ln t = y \rightarrow \frac{1}{t} dt = dy$$

$$I_2 = \frac{1}{\ln^{101}(100)} \int_0^{\infty} y^{100} e^{-y} dy = \frac{100!}{\ln^{101}(100)}$$

$$\Omega = I_1 \cdot I_2 = \frac{(101!)}{101^{101}} \cdot \frac{100!}{\ln^{101}(100)} = \frac{(101!)^2}{(101 \ln 100)^{100}}$$

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**Solution 3 by Nelson Javier Villaherrera-El Salvador**

$$\begin{aligned}
 \Omega &= \int_0^1 \int_0^\infty \frac{x^{100}}{100^x} [y \ln(y)]^{100} dx dy = \left[ \int_0^\infty \frac{x^{100}}{e^{x \ln(100)}} dx \right] \int_0^1 \{-y[-\ln(y)]\}^{100} dy = \\
 &= \left[ \int_0^\infty x^{100} e^{-x \ln(100)} dx \right] \int_0^1 \{y[-\ln(y)]\}^{100} dy = \\
 &= [L\{x^{100}\}]_{1-\ln(100)} \int_0^\infty u^{100} e^{-100u} (-e^{-u}) du = \frac{100!}{\ln^{101}(100)} \int_0^\infty u^{100} e^{-100u} du = \\
 &= \frac{100!}{[2 \ln(10)]^{101}} L\{u^{100}\}|_{x-101} = \frac{100!}{[2 \ln(10)]^{101}} \cdot \frac{100!}{101^{101}} = \frac{(100!)^2}{[202 \ln(10)]^{101}}
 \end{aligned}$$

**UP.154. If  $(a_n)_{n \geq 1} \subset (0, \infty)$ ;  $n \in \mathbb{N}^*$ ;  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 \cdot a_n} = a > 0$  then find:**

$$\Omega = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^n \left[ \left( \sqrt[2k]{k!} + \sqrt[2k+2]{(k+1)!} \right)^2 \right];$$

**[\*] - is great integer function.**

**Proposed by D.M. Bătinețu – Giurgiu; Daniel Sitaru – Romania**

**Solution by proposers**

$$\text{Let be: } v_n = \sqrt[2n]{n!} + \sqrt[2n+2]{(n+1)!}; n \in \mathbb{N}^*$$

$$x_n = \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^n \left[ \left( \sqrt[2k]{k!} + \sqrt[2k+2]{(k+1)!} \right)^2 \right] = \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^n [v_k^2]; n \in \mathbb{N}^*$$

$$\Omega = \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt[n]{a_n}} \cdot \frac{1}{n^2} \sum_{k=1}^n [v_k^2]$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{\sqrt[n]{a_n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^{2n}}{a_n}} \stackrel{CDA}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^{2n+2}}{a_{n+1}} \cdot \frac{a_n}{n^{2n}} = \lim_{n \rightarrow \infty} \frac{n^2 a_n}{a_{n+1}} \cdot \left( \frac{n+1}{n} \right)^{2n+2} = \frac{e^2}{a}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = e.$$

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$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n [v_k^2] &\stackrel{CS}{=} \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{n+1} [v_k^2] - \sum_{k=1}^n [v_k^2]}{(n+1)^2 - n^2} = \lim_{n \rightarrow \infty} \frac{[v_{n+1}^2]}{2n+1} \\ & [v_n^2] \leq v_n^2 < [v_n^2] + 1; n \in \mathbb{N}^* \\ & \frac{[v_n^2]}{2n+1} \leq \frac{v_n^2}{2n+1} < \frac{[v_n^2]}{2n+1} + \frac{1}{2n+1}; n \in \mathbb{N}^* \\ \lim_{n \rightarrow \infty} \frac{[v_n^2]}{2n+1} &= \lim_{n \rightarrow \infty} \frac{v_n^2}{2n+1} = \lim_{n \rightarrow \infty} \frac{\left( {}^{2n}\sqrt{n!} + {}^{2n+2}\sqrt{(n+1)!} \right)^2}{2n+1} = \\ &= \lim_{n \rightarrow \infty} \left( \frac{{}^n\sqrt{n!}}{2n+1} + 2 \sqrt{\frac{{}^n\sqrt{n!}}{n}} \cdot \sqrt{\frac{{}^{n+1}\sqrt{(n+1)!}}{n+1}} \cdot \frac{\sqrt{n(n+1)}}{2n+1} + \frac{{}^{n+1}\sqrt{(n+1)!}}{n+1} \cdot \frac{n+1}{2n+1} \right) = \\ &= \frac{1}{2} \left( \frac{1}{e} + \frac{2}{e} + \frac{1}{e} \right) = \frac{2}{e} \text{ then:} \\ \lim_{n \rightarrow \infty} x_n &= \frac{e^2}{a} \cdot \frac{2}{e} = \frac{2e}{a} \end{aligned}$$

UP.155. If  $m \in \mathbb{N}; x, y, z > 0$  then in  $\Delta ABC$  the following relationship holds:

$$3m + \left( \frac{a^2 x^2}{yz} \right)^{m+1} + \left( \frac{b^2 y^2}{zx} \right)^{m+1} + \left( \frac{c^2 z^2}{xy} \right)^{m+1} \geq 4(m+1)\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\begin{aligned} (1+x)^\alpha &> 1 + \alpha x, \text{ Bernoulli} \Rightarrow x^\alpha > 1 + \alpha(x-1) \Rightarrow \\ &\Rightarrow \left( \frac{a^2 x^2}{yz} \right)^{m+1} > 1 + (m+1) \left( \frac{a^2 x^2}{yz} - 1 \right) \\ &\left( \frac{b^2 y^2}{xz} \right)^{m+1} > 1 + (m+1) \left( \frac{b^2 y^2}{xz} - 1 \right) \\ &\left( \frac{c^2 z^2}{xy} \right)^{m+1} > 1 + (m+1) \left( \frac{c^2 z^2}{xy} - 1 \right) \\ &\Rightarrow 3m + \sum \left( \frac{a^2 x^2}{yz} \right)^{m+1} > 3 + (m+1) \left( \sum \frac{a^2 x^2}{yz} - 3 \right) + 3m = \\ &= 3 + (m+1) \sum \frac{a^2 x^2}{yz} - 3(m+1) + 3m = (m+1) \sum \frac{a^2 x^2}{yz} \geq 4(m+1)\sqrt{3}S \end{aligned}$$

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$$\Rightarrow \sum \frac{a^2 x^2}{yz} \geq 4\sqrt{3}S$$

$$\sum \frac{a^2 x^2}{yz} \stackrel{(Ma \geq Mg)}{\geq} \sqrt[3]{\prod \frac{a^2 x^2}{yz}} = \sqrt[3]{a^2 b^2 c^2} = 4\sqrt{3}S$$

$$3^3 \sqrt{a^2 b^2 c^2} \geq 4\sqrt{3}S, 27(abc)^2 \geq (4\sqrt{3}S)^3 \Rightarrow$$

$\Rightarrow (abc)^2 \geq \left(\frac{4\sqrt{3}S}{3}\right)^3, (abc)^2 \geq \left(\frac{4S}{\sqrt{3}}\right)^3$ , true as in the inequality G. Polya and G. Szegä.

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$3m + \left(\frac{a^2 x^2}{yz}\right)^{m+1} + \left(\frac{b^2 y^2}{zx}\right)^{m+1} + \left(\frac{c^2 z^2}{xy}\right)^{m+1} \stackrel{(1)}{\geq} 4(m+1)\sqrt{3}S$$

\* Case (1)  $m \in \mathbb{N}^*$

$$\frac{m + \left(\frac{a^2 x^2}{yz}\right)^{m+1}}{m+1} = \frac{\underbrace{1+1+\dots+1}_{m \text{ times}} + \left(\frac{a^2 x^2}{yz}\right)^{m+1}}{m+1} \stackrel{A-G}{\geq} \sqrt[1]{\left(\frac{a^2 x^2}{yz}\right)^{m+1}} = \frac{a^2 x^2}{yz}$$

$$\Rightarrow m + \left(\frac{a^2 x^2}{yz}\right)^{m+1} \stackrel{(a)}{\geq} (m+1) \left(\frac{a^2 x^2}{yz}\right)$$

Similarly,  $\forall m \in \mathbb{N}^*, \left(\frac{b^2 y^2}{zx}\right)^{m+1} + m \stackrel{(b)}{\geq} (m+1) \left(\frac{b^2 y^2}{zx}\right)$  &  $\forall m \in \mathbb{N}^*,$

$$\left(\frac{c^2 z^2}{xy}\right)^{m+1} + m \stackrel{(c)}{\geq} (m+1) \left(\frac{c^2 z^2}{xy}\right)$$

$$(a)+(b)+(c) \Rightarrow 3m + \left(\frac{a^2 x^2}{yz}\right)^{m+1} + \left(\frac{b^2 y^2}{zx}\right)^{m+1} + \left(\frac{c^2 z^2}{xy}\right)^{m+1}$$

$$\geq (m+1) \left(\frac{a^2 x^2}{yz} + \frac{b^2 y^2}{zx} + \frac{c^2 z^2}{xy}\right) \stackrel{?}{\geq} 4(m+1)\sqrt{3}S$$

$$\Leftrightarrow \frac{a^2 x^2}{yz} + \frac{b^2 y^2}{zx} + \frac{c^2 z^2}{xy} \stackrel{?}{\geq} 4\sqrt{3}S \quad (2)$$

$$\forall m, n, p \geq 0, a^2 m + b^2 n + c^2 p \stackrel{(i)}{\geq} 4S \sqrt{mn + np + pm}$$

Taking  $m = \frac{x^2}{yz}, n = \frac{y^2}{zx}$  &  $p = \frac{z^2}{xy}$  & using (i), we get

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$$\text{LHS of (2)} \geq 4S \sqrt{\frac{xy}{z^2} + \frac{yz}{x^2} + \frac{zx}{y^2}} \stackrel{(A-G)}{\geq} 4S\sqrt{3} \Rightarrow (2) \text{ is true} \Rightarrow (1) \text{ is true.}$$

\* Case (2)  $m = 0$ , Then (1)  $\Leftrightarrow$  (2), which has been just proved. This completes the proof.

**Solution 3 by Marian Ursărescu-Romania**

$$m + \left(\frac{a^2x^2}{yz}\right)^{m+1} = \underbrace{1 + 1 + \dots + 1}_{m \text{ times}} + \left(\frac{a^2x^2}{yz}\right)^{m+1} \geq (m+1) \sqrt[m+1]{\left(\frac{a^2x^2}{yz}\right)^{m+1}} \Rightarrow$$

$$\left. \begin{aligned} m + \left(\frac{a^2x^2}{yz}\right)^{m+1} &\geq (m+1) \frac{a^2x^2}{yz} \\ m + \left(\frac{b^2y^2}{zx}\right)^{m+1} &\geq (m+1) \frac{b^2y^2}{zx} \\ m + \left(\frac{c^2z^2}{xy}\right)^{m+1} &\geq (m+1) \frac{c^2z^2}{xy} \end{aligned} \right\} \Rightarrow$$

$$3m + \left(\frac{a^2x^2}{yz}\right)^{m+1} + \left(\frac{b^2y^2}{zx}\right)^{m+1} + \left(\frac{c^2z^2}{xy}\right)^{m+1} \geq (m+1) \left(\frac{a^2x^2}{yz} + \frac{b^2y^2}{zx} + \frac{c^2z^2}{xy}\right) \quad (1)$$

$$\text{From (1) we must show: } \frac{a^2x^2}{yz} + \frac{b^2y^2}{zx} + \frac{c^2z^2}{xy} \geq 4\sqrt{3}S \quad (2)$$

$$\text{But } \frac{a^2x^2}{yz} + \frac{b^2y^2}{zx} + \frac{c^2z^2}{xy} \geq 3\sqrt[3]{a^2b^2c^2} \quad (3)$$

$$\text{From (2)+(3) we must show: } \sqrt[3]{a^2b^2c^2} \geq \frac{4}{\sqrt{3}}S \Leftrightarrow$$

$$a^2b^2c^2 \geq \frac{64}{3\sqrt{3}}S^3 \Leftrightarrow 16R^2S^2 \geq \frac{64}{3\sqrt{3}}S^3 \Leftrightarrow R^2 \geq \frac{4}{3\sqrt{3}}pr \quad (4)$$

$$\text{But } \left. \begin{aligned} R &\geq 2r \\ R &\geq \frac{2}{3\sqrt{3}}p \end{aligned} \right\} \Rightarrow R^2 \geq \frac{4}{3\sqrt{3}}rp \Rightarrow (4) \text{ its true.}$$

**Solution 4 by Tran Hong-Vietnam**

We have

$$\frac{a^2x^2}{yz} + \frac{b^2y^2}{zx} + \frac{c^2z^2}{xy} \stackrel{\text{(Cauchy)}}{\geq} 3\sqrt[3]{a^2b^2c^2} \stackrel{(1)}{\geq} 4\sqrt{3}S$$

$$(1) \Leftrightarrow \frac{4\sqrt{3}S}{3} \leq \sqrt[3]{a^2b^2c^2} \Leftrightarrow \frac{64\sqrt{3}}{9}S^3 \leq (abc)^2$$

$$\Leftrightarrow \frac{64\sqrt{3}}{9}S^3 \leq 16(RS)^2 \Leftrightarrow \frac{2\sqrt{3}}{9}(a+b+c)r \leq R^2$$

$$\text{(True because: } (a+b+c)\frac{\sqrt{3}}{9} \leq R, 2r \leq R)$$

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$$\text{Let } f(t) = m + t^{m+1} - (m+1)t, t > 0, m \in \mathbb{N}$$

$$\Rightarrow f'(t) = (m+1)t^m - (m+1) = 0 \Leftrightarrow t = 1$$

$$\Rightarrow f(t) \geq f_{\min}(1) = 0 \Rightarrow m + t^{m+1} \geq (m+1)t \quad (*)$$

$$\begin{aligned} *LHS &= 3m + \left(\frac{a^2x^2}{yz}\right)^{m+1} + \left(\frac{b^2y^2}{zx}\right)^{m+1} + \left(\frac{c^2z^2}{xy}\right)^{m+1} = m + \left(\frac{a^2x^2}{yz}\right)^{m+1} + m + \left(\frac{b^2y^2}{zx}\right)^{m+1} + \\ &+ m + \left(\frac{c^2x^2}{xy}\right)^{m+1} \stackrel{(*)}{\geq} (m+1) \left(\frac{a^2x^2}{yz}\right) + (m+1) \left(\frac{b^2y^2}{zx}\right) + (m+1) \left(\frac{c^2z^2}{xy}\right) = \\ &= (m+1) \left[ \frac{a^2x^2}{yz} + \frac{b^2y^2}{zx} + \frac{c^2z^2}{xy} \right] \stackrel{(1)}{\geq} (m+1) \cdot 4\sqrt{3}S = 4\sqrt{3}(m+1)S \end{aligned}$$

Proved.

$$\text{Equality} \Leftrightarrow \frac{a^2x^2}{yz} = \frac{b^2y^2}{zx} = \frac{c^2z^2}{xy} = 1.$$

**UP.156.** If  $a, b, c > 0; b < c; b(a+c) \in \mathbb{N}; b(a+c) \geq 1; n \in \mathbb{N}; n \geq 2; x_k \in [b, c];$

$t_k > 0; k \in \overline{1, n}$  then:

$$\left( \sum_{k=1}^n t_k x_k \right) \left( \sum_{k=1}^n \frac{t_k}{x_k} \right)^{b(a+c)} \leq \left( \frac{a+b+c}{1+b(a+c)} \right)^{1+ab+bc} \cdot \left( \sum_{k=1}^n t_k \right)^{1+ab+bc}$$

Proposed by D.M. Băţineţu-Giurgiu, Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

From A.M-G.M. inequality  $\Rightarrow$

$$\sum_{k=1}^n t_k x_k \left( \sum_{k=1}^n \frac{t_k}{x_k} \right)^{ab+bc} \leq \left( \frac{\sum_{k=1}^n t_k x_k + (ab+bc) \sum_{k=1}^n \frac{t_k}{x_k}}{1+ab+bc} \right)^{1+ab+bc} \quad (1)$$

From (1) we must show this:

$$\left( \frac{\sum_{k=1}^n t_k x_k + (ab+bc) \sum_{k=1}^n \frac{t_k}{x_k}}{1+ab+bc} \right)^{1+ab+bc} \leq \left( \frac{a+b+c}{1+ab+bc} \right)^{1+ab+bc} \left( \sum_{k=1}^n t_k \right)^{1+ab+bc}$$

$$\Leftrightarrow \sum_{k=1}^n t_k x_k + (ab+bc) \sum_{k=1}^n \frac{t_k}{x_k} \leq (a+b+c) \sum_{k=1}^n t_k \quad (2)$$

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$$\text{But } x_k \in [b, c] \Rightarrow b \leq x_k \leq c \Rightarrow (x_k - b)(c - x_k) \geq 0 \Rightarrow x_k^2 - (b + c)x_k + bc \leq 0 \Leftrightarrow$$

$$\Leftrightarrow x_k^2 + bc \leq (b + c)x_k \Rightarrow x_k + \frac{bc}{x_k} \leq b + c \Rightarrow$$

$$t_k x_k + bc \frac{t_k}{x_k} \leq (b + c)t_k \Rightarrow \sum_{k=1}^n t_k x_k + bc \sum_{k=1}^n \frac{t_k}{x_k} \leq (b + c) \sum_{k=1}^n t_k \Rightarrow$$

$$\sum_{k=1}^n t_k x_k + (ab + bc) \sum_{k=1}^n \frac{t_k}{x_k} \leq ab \sum_{k=1}^n \frac{t_k}{x_k} + (b + c) \sum_{k=1}^n t_k \quad (3)$$

From (2)+(3) we must show:

$$ab \sum_{k=1}^n \frac{t_k}{x_k} + (b + c) \sum_{k=1}^n t_k \leq (a + b + c) \sum_{k=1}^n t_k \Leftrightarrow ab \sum_{k=1}^n \frac{t_k}{x_k} \leq a \sum_{k=1}^n t_k \Leftrightarrow$$

$$b \sum_{k=1}^n \frac{t_k}{x_k} \leq \sum_{k=1}^n t_k \quad (4)$$

$$\text{But } x_k \geq b \Rightarrow \frac{1}{x_k} \leq \frac{1}{b} \Rightarrow \frac{t_k}{x_k} \leq \frac{t_k}{b} \Rightarrow$$

$$\sum_{k=1}^n \frac{t_k}{x_k} \leq \frac{1}{b} \sum_{k=1}^n t_k \Rightarrow b \sum_{k=1}^n \frac{t_k}{x_k} \leq \sum_{k=1}^n t_k \Rightarrow$$

(4) is true.

UP.157. If  $m \geq 0$ ;  $x, y > 0$  then in  $\Delta ABC$  the following relationship holds:

$$\frac{h_a}{h_b^m h_c^m (x h_b + y h_c)^m} + \frac{h_b}{h_c^m h_a^m (x h_c + y h_a)^m} + \frac{h_c}{h_a^m h_b^m (x h_a + y h_b)^m} \geq$$

$$\geq \frac{9}{(x + y)^m \cdot s^{2m} \cdot r^{m-1}}$$

Proposed by D.M. Băținețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum h_a = \frac{s^2 + 4Rr + r^2}{2R} \geq 9r \Leftrightarrow s^2 \geq 14Rr - r^2 \Leftrightarrow$$

$$\Leftrightarrow (s^2 - 16Rr + 5r^2) + (2Rr - 4r^2) \geq 0 \rightarrow \text{true}$$

$$\therefore s^2 \stackrel{\text{Gerretsen}}{\geq} 16Rr - 5r^2 \text{ \& } 2Rr \geq 4r^2 \text{ (as } R \stackrel{\text{Euler}}{\geq} 2r)$$

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$$\therefore \sum h_a \stackrel{(1)}{\geq} 9r$$

$$\begin{aligned} \text{Now, LHS} &= \frac{h_a^{m+1}}{(h_a h_b h_c)^m (x h_b + y h_c)^m} + \frac{h_b^{m+1}}{(h_a h_b h_c)^m (x h_c + y h_a)^m} + \frac{h_c^{m+1}}{(h_a h_b h_c)^m (x h_a + y h_b)^m} \geq \\ &\stackrel{\text{Radon}}{\geq} \frac{(\sum h_a)^{m+1}}{[h_a h_b h_c \{(x h_b + x h_c + x h_a) + (y h_c + y h_a + y h_b)\}]^m} \\ &= \frac{(\sum h_a)^{m+1}}{\left(\frac{8r^3 s^3}{4Rrs}\right)^m (x+y)^m (\sum h_a)^m} = \frac{\sum h_a}{\left(\frac{2r^s s^2}{R}\right)^m (x+y)^m} \geq \frac{\sum h_a}{(rs^2)^m (x+y)^m} \left(\because \frac{2r}{R} \stackrel{\text{Euler}}{\leq} 1\right) \\ &\stackrel{\text{by (1)}}{\geq} \frac{9r}{(x+y)^m s^{2m} r^m} = \frac{9}{(x+y)^m s^{2m} r^{m-1}} \quad (\text{Proved}) \end{aligned}$$

**Solution 2 by Marian Ursarescu-Romania**

We must show:

$$\frac{h_a^{m+1}}{h_a^m h_b^m h_c^m (x h_b + y h_c)^m} + \frac{h_b^{m+1}}{(h_a h_b h_c)^m (x h_c + y h_a)^m} + \frac{h_c^{m+1}}{(h_a h_b h_c)^m (x h_a + y h_b)^m} \geq \frac{9}{(x+y)^m s^{2m} r^{m-1}} \quad (1)$$

From Hölder's inequality we have:

$$\begin{aligned} \frac{1}{(h_a h_b h_c)^m} \cdot \sum \frac{h_a^{m+1}}{(x h_b + y h_c)^m} &\geq \frac{1}{(h_a h_b h_c)^m} \cdot \frac{(h_a + h_b + h_c)^{m+1}}{(x+y)^m (h_a + h_b + h_c)^m} \Leftrightarrow \\ \sum \frac{h_a}{h_b^m h_c^m (x h_c + y h_a)^m} &\geq \frac{(h_a + h_b + h_c)}{(h_a h_b h_c)^m (x+y)^m} \quad (2) \end{aligned}$$

$$\text{From (2) we must show: } \frac{h_a + h_b + h_c}{(h_a h_b h_c)^m (x+y)^m} \geq \frac{9}{(x+y)^m s^{2m} r^{m-1}} \Leftrightarrow$$

$$\frac{h_a + h_b + h_c}{(h_a h_b h_c)^m} \geq \frac{9}{s^{2m} r^{m-1}} \quad (3)$$

Because  $h_a + h_b + h_c \geq 9r$  (4). From (3) + (4)  $\Rightarrow$

$$\begin{aligned} \frac{9r}{(h_a h_b h_c)^m} &\geq \frac{9}{s^{2m} r^{m-1}} \Leftrightarrow \frac{1}{(h_a h_b h_c)^m} \geq \frac{1}{s^{2m} r^m} \\ \Leftrightarrow \frac{1}{h_a h_b h_c} &\geq \frac{1}{s^2 r} \Leftrightarrow h_a h_b h_c \leq s^2 r \quad (5) \end{aligned}$$

But  $h_a h_b h_c = \frac{2s^2 r^2}{R}$  (6). From (5) + (6) we must show:

$$\frac{2s^2 r^2}{R} \leq s^2 r \Leftrightarrow 2r \leq R \quad \text{true (Euler)}$$

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**Solution 3 by Soumitra Mandal-Chandar Nagore-India**

$$\begin{aligned} \sum_{cyc} \frac{h_a}{h_b^m h_c^m (xh_b + yh_c)^m} &= \frac{1}{(h_a h_b h_c)^m} \sum_{cyc} \frac{h_a^{m+1}}{(xh_b + yh_c)^m} \\ &\stackrel{\text{RADON'S INEQUALITY}}{\geq} \left(\frac{abc}{8\Delta^3}\right)^m \frac{(h_a + h_b + h_c)^{m+1}}{(\sum_{cyc} (xh_b + yh_c))^m} = \left(\frac{8Rrp}{8r^3 p^3}\right)^m \cdot \frac{\sum_{cyc} h_a}{(x+y)^m} \\ &\geq \left(\frac{8r^2 p}{8r^3 p^3}\right)^m \frac{9r}{(x+y)^m} \left[ \text{since } R \geq 2r \text{ and } \sum_{cyc} h_a \geq 9r \right] = \frac{9}{(x+y)^m p^{2m} r^{m-1}} \end{aligned}$$

(proved)

**UP.158. Find:**

$$\Omega = \int_0^1 x^2 \log(x) \cot\left(\frac{\pi x}{2}\right) dx$$

**Proposed by Arafat Rahman Akib-Dhaka-Bangladesh**

**Solution by proposer**

We know,  $\frac{1}{2} \int_a^b g(x) \cot\left(\frac{ax}{2}\right) dx = \sum_{n=1}^{\infty} \int_a^b g(x) \sin(ax) dx$

put,  $g(x) = x^2 \log(x)$ ,  $a = \pi$  and  $(a, b) = (0, 1)$ . Now,

$$\frac{1}{2} \int_0^1 x^2 \log(x) \cot\left(\frac{\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^1 x^2 \log(x) \sin(\pi n x) dx \quad (1)$$

We have that,  $\int x^2 \log(x) \sin(ax) dx = \frac{3 \cos ax}{a^3} + \frac{x \sin ax}{a^2} + \left(\frac{2 \cos ax + 2ax \sin ax - a^2 x^2 \cos ax}{a^3}\right)$

$$\begin{aligned} \log(x) - \frac{2 Ci(ax)}{a^3} \therefore \int_0^1 x^2 \log(x) \sin(\pi n x) dx &= \\ &= \frac{(-1)^n 3}{n^3 \pi^3} - \frac{3}{n^3 \pi^3} - \frac{2 Ci(n\pi)}{n^3 \pi^3} + \frac{2(\gamma + \log n\pi)}{n^3 \pi^3} \end{aligned}$$

From (1) we get,

$$\frac{1}{2} \int_0^1 x^2 \log(x) \cot\left(\frac{\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \left( \frac{(-1)^n 3}{n^3 \pi^3} - \frac{3}{n^3 \pi^3} - \frac{2 Ci(n\pi)}{n^3 \pi^3} + \frac{2(\gamma + \log(n\pi))}{n^3 \pi^3} \right) =$$

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$$\begin{aligned}
 &= -\frac{3}{\pi^3} [\xi_a(3) + \xi(3)] - \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{Ci(n\pi)}{n^3} + \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{(\gamma + \log n\pi)}{n^3} = \\
 &= -\frac{3}{\pi^3} [\xi_a(3) + \xi(3)] - \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{Ci(n\pi)}{n^3} + \frac{2}{\pi^3} (-\xi'(3) + \gamma\xi(3) + \xi(3) \log(\pi)) \\
 \therefore \int_0^1 x^2 \log(x) \cot\left(\frac{\pi x}{2}\right) dx &= \frac{-6}{\pi^3} [\xi_a(3) + \xi(3)] - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{Ci(n\pi)}{n^3} + \\
 &\quad + \frac{4}{\pi^3} \xi(3) [\gamma + \log \pi] - \frac{4}{\pi^3} - \xi'(3)
 \end{aligned}$$

**UP.159.Find:**

$$\Omega = \int_{-1}^{+1} (x^2 + 2x + 1)^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx$$

*Proposed by Ekpo Samuel-Nigeria*

*Solution by proposer*

$$\begin{aligned}
 &\int_{-1}^{+1} [(x+1)(x+1)]^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx \\
 &\int_{-1}^{+1} (1+x)^{\frac{3}{2}} (1+x)^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx \Rightarrow \int_{-1}^{+1} ((1+x)^2)^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx \\
 &(ex x = \cos 2t \quad dx = -2 \sin 2\theta \, d\theta \\
 &\quad \cos^{-1}(x) = 2\theta \\
 &\quad \cos^{-1}(1) = 2\theta; 0 = 2\theta \therefore \theta = 0 \\
 &\quad \cos^{-1}(-1) = 2\theta; \pi = 2\theta; 0 = \frac{\pi}{2} \\
 &\int_{\frac{\pi}{2}}^0 (1 + \cos 2\theta)^3 (1 - (\cos 2\theta))^{\frac{1}{2}} (-2 \sin 2\theta) d\theta
 \end{aligned}$$

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$$-2 \int_{\frac{\pi}{2}}^0 (1 + 2 \cos^2 t - 1)^3 (1 - (2 \cos^2 t - 1))^{\frac{1}{2}} \sin 2t dt$$

$$2 \int_0^{\frac{\pi}{2}} (1 - 1 + 2 \cos^2 t)^3 (1 + 1 - 2 \cos^2 t)^{\frac{1}{2}} \sin 2t dt$$

$$2 \int_0^{\frac{\pi}{2}} 2^3 (\cos t)^6 \left( 2(1 - \cos^2 t)^{\frac{1}{2}} \right) \sin 2t dt$$

$$2 \cdot 2^3 \int_0^{\frac{\pi}{2}} \cos^6 t 2^{\frac{1}{2}} (\sin^2 t)^{\frac{1}{2}} \sin 2t dt$$

$$\sin 2t = 2 \sin t \cos t$$

$$2^{\frac{1}{2}} 2^4 \int_0^{\frac{\pi}{2}} \cos^6 t \sin t 2 \sin t \cos t dt$$

$$2^{4+\frac{1}{2}+1} \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 t dt$$

$$2^{\frac{11}{2}} \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 t dt \Rightarrow 2^{\frac{9}{2}} \left[ 2 \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 \theta d\theta \right]$$

$$\text{or } 2^4 \sqrt{2} \left[ 2 \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 t dt \right]$$

$$16 \sqrt{2} \left[ 2 \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 t dt \right]$$

$$\beta(m, n) = 2 \int_0^{\frac{\pi}{2}} \cos^{2m-1} t \sin^{2n-1} \theta dt$$

$$\therefore 2m - 1 = 1 \text{ and } 2n - 1 = 2$$

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$$2m - 2 \rightarrow +1 \quad 2n = 1 + 1$$

$$2m = 0 \quad 2n = 3$$

$$m = 4 \quad 2n = 3$$

$$n = \frac{3}{2}$$

$$\therefore 10\sqrt{2} \left[ 2 \int_0^{\frac{\pi}{2}} \cos^{2(4)-1} t \sin^{2(\frac{3}{2})-1} t dt \right]$$

$$= 10\sqrt{2} \left[ \beta \left( 4, \frac{3}{2} \right) \right] \Rightarrow 10\sqrt{2} \left( \frac{\sqrt{4} \sqrt{\frac{3}{2}}}{\sqrt{4 + \frac{3}{2}}} \right) = \sqrt{2}(10) \left[ \frac{\sqrt{4} \sqrt{\frac{3}{2}}}{\sqrt{\frac{11}{2}}} \right]$$

$$\sqrt{2} \left[ \frac{16 \times 6 \times \sqrt{\frac{\pi}{2}}}{\frac{945\sqrt{\pi}}{32}} \right] = \sqrt{2} \left[ \frac{3072 \times \sqrt{\frac{\pi}{2}}}{945\sqrt{\pi}} \right] = \sqrt{2} \left[ \frac{1536}{945} \right] = \frac{512\sqrt{2}}{315} \approx 2 \cdot 2987$$

**UP.160. Let  $x, y, z \in (0; +\infty)$  and  $a \geq 1$ . Prove:**

$$\frac{1}{([x] + \{y\} + 1)^a} + \frac{1}{([y] + \{z\} + 1)^a} + \frac{1}{([z] + \{x\} + 1)^a} \geq \frac{3^{a+1}}{(x + y + z + 3)^a}$$

$[x]$  is the integer part of the real number  $x$ ;  $\{x\}$  is the fraction of real numbers  $x$ .

*Proposed by Nguyen Van Nho – Nghe An – Vietnam*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\because x > 0 \therefore [x] \geq 0 \text{ and of course, } \{y\} \geq 0 \Rightarrow [x] + \{y\} + 1 \geq 1 > 0$$

$$\text{Similarly, } [y] + \{z\} + 1 > 0 \text{ and } [z] + \{x\} + 1 > 0$$

$$\text{Now, } \frac{1}{([x] + \{y\} + 1)^a} + \frac{1}{([y] + \{z\} + 1)^a} + \frac{1}{([z] + \{x\} + 1)^a} = \frac{1^{a+1}}{([x] + \{y\} + 1)^a} + \frac{1^{a+1}}{([y] + \{z\} + 1)^a} + \frac{1^{a+1}}{([z] + \{x\} + 1)^a}$$

$$\stackrel{\text{Radon}}{\geq} \frac{3^{a+1}}{([x] + \{x\}) + ([y] + \{y\}) + ([z] + \{z\}) + 3)^a} = \frac{3^{a+1}}{(x + y + z + 3)^a}$$

$$(\because x = [x] + \{x\}, \text{ etc})$$

**Solution 2 by Tran Hong-Vietnam**

$$x, y, z \geq 0 \Rightarrow [x], [y], [z] \geq 0 \text{ and } 0 \leq \{x\}, \{y\}, \{z\} < 1$$

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$\Rightarrow [x] + \{y\} + 1, [y] + \{z\} + 1, [z] + \{x\} + 1 > 0$ . Using Cauchy's inequality we have:

$$\begin{aligned} LHS &\geq \frac{3}{\sqrt[3]{([x] + \{y\} + 1)^\alpha ([y] + \{z\} + 1)^\alpha ([z] + \{x\} + 1)^\alpha}} = \\ &= \frac{3}{\sqrt[3]{([x] + \{y\} + 1)([y] + \{x\} + 1)([z] + \{x\} + 1)^\alpha}} \geq \frac{3}{\sqrt[3]{\left(\frac{3}{x+y+3}\right)^{3\alpha}}} = \frac{(x+y+3)^\alpha}{3^{\alpha-1}} \end{aligned}$$

We need to prove that:  $\frac{(x+y+z+3)^\alpha}{3^{\alpha-1}} \geq \frac{3^{\alpha+1}}{(x+y+z+3)^\alpha} \Leftrightarrow (x+y+3)^{2\alpha} \geq 3^{2\alpha-2} \Leftrightarrow$

$\Leftrightarrow (x+y+z+3)^\alpha \geq 3^{\alpha-1}$ . Which is true because  $x+y+z+3 \geq 3 \Leftrightarrow$

$\Leftrightarrow (x+y+3)^\alpha \geq 3^\alpha \geq 3^{\alpha-1}$  for  $x, y, z \geq 0$  and  $\alpha \geq 1$ . Proved.

**UP.161. Find:**

$$\Omega = \int_0^\infty \int_0^\infty \frac{dx dy}{(x^4 + 2x^2 + 1)(y^2 + 25)^4}$$

Proposed by Ekpo Samuel – Nigeria

Solution 1 by Ravi Prakash-New Delhi-India

$\Omega = (I_1)(I_2)$  where  $I_1 = \int_0^\infty \frac{dx}{x^4+2x^2+1} = \int_0^\infty \frac{dx}{(x^2+1)^2}$ . Put  $x = \tan \theta$

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{\pi}{4}$$

and

$$I_2 = \int_0^\infty \frac{dy}{(y^2 + 25)^4}$$

Put  $y = 5 \tan \varphi$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{5 \sec^2 \varphi}{5^8 \sec^8 \varphi} d\varphi = \frac{1}{5^7} \int_0^{\frac{\pi}{2}} \cos^6 \varphi d\varphi = \frac{1}{5^7} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{5^6} \cdot \frac{\pi}{32}$$

$$\therefore \Omega = \frac{\pi^2}{5^6 2^7} = \frac{\pi^2}{2000000}$$

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*Solution 2 by Togrul Ehmedov-Baku-Azerbaijan*

$$\begin{aligned}
 \int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(x^4 + 2x^2 + 1)(y^2 + 25)^4} &= \int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(x^2 + 1)^2 (y^2 + 25)^4} = \\
 &= \int_0^{\infty} \frac{dy}{(y^2 + 25)^4} \int_0^{\frac{\pi}{2}} \cos^2 z dx \Bigg|_{z=\tan^{-1} x} \\
 &= \frac{\pi}{4} \int_0^{\infty} \frac{dy}{(y^2 + 25)^4} \Bigg|_{y=5 \tan(u)} = \frac{\pi}{4 \cdot 5^7} \int_0^{\frac{\pi}{2}} \cos^6 u du \\
 &= \frac{\pi}{8 \cdot 5^7} \int_0^{\frac{\pi}{2}} (\cos^6 u + \sin^6 u) du = \frac{\pi}{8 \cdot 5^7} \int_0^{\frac{\pi}{2}} \left(1 - \frac{3}{4} \sin^2 2u\right) du \\
 &= \frac{\pi}{8 \cdot 5^7} \int_0^{\frac{\pi}{2}} \left(\frac{5}{8} + \frac{3}{8} \cos(4u)\right) du = \frac{\pi}{8 \cdot 5^7} \left[\frac{5}{8} u + \frac{3}{32} \sin(4u)\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{2 \cdot 10^6}
 \end{aligned}$$

*Solution 3 by Nelson Javier Villaherrera Lopez-El Salvador*

$$\begin{aligned}
 \Omega &= \int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(x^4 + 2x^2 + 1)(y^2 + 25)^4} = \frac{1}{5^8} \int_0^{\infty} \frac{dx}{(x^2 + 1)^2} \int_0^{\infty} \frac{dy}{\left[\left(\frac{y}{5}\right)^2 + 1\right]^4} = \\
 &= \frac{1}{5^7} \int_0^{\frac{\pi}{2}} \frac{\sec^2(\theta) d\theta}{[\tan^2(\theta) + 1]^2} \int_0^{\infty} \frac{dz}{(z^2 + 1)^4} = \frac{1}{5^7} \left[ \int_0^{\frac{\pi}{2}} \cos^2(\theta) d\theta \right] \int_0^{\frac{\pi}{2}} \frac{\sec^2(\theta)}{[\tan^2(\theta) + 1]^4} d\theta \\
 &= \frac{1}{2 \cdot 5^7} \left\{ \int_0^{\frac{\pi}{2}} [1 + \cos(2\theta)] d\theta \right\} \int_0^{\frac{\pi}{2}} \cos^6(\theta) d\theta = \frac{1}{2 \cdot 5^7} \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[ \frac{1 + \cos(2\theta)}{2} \right]^3 d\theta \\
 &= \frac{\pi}{2^5 \cdot 5^7} \int_0^{\frac{\pi}{2}} [1 + 3 \cos(2\theta) + 3 \cos^2(2\theta) + \cos^3(2\theta)] d\theta =
 \end{aligned}$$

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$$\begin{aligned}
 &= \frac{\pi}{2^5 \cdot 5^7} \int_0^{\frac{\pi}{2}} \left\{ 1 + 3 \cos(2\theta) + 3 \frac{1 + \cos(4\theta)}{2} + \cos(2\theta) [1 - \sin^2(2\theta)] \right\} d\theta \\
 &= \frac{\pi}{2^5 \cdot 5^7} \int_0^{\frac{\pi}{2}} \left[ \frac{5}{2} + 4 \cos(2\theta) + \frac{3}{2} \cos(4\theta) - \cos(2\theta) \sin^2(2\theta) \right] d\theta = \\
 &= \frac{\pi}{2^5 \cdot 5^7} \left[ \frac{5}{2} \theta + 2 \sin(2\theta) + \frac{3}{8} \sin(4\theta) - \frac{1}{6} \sin^3(2\theta) \right]_0^{\frac{\pi}{2}} \\
 &= \frac{5}{2^7 \cdot 5^7} \pi^2 = \frac{\pi^2}{2^7 \cdot 5^6} = \frac{\pi^2}{2 \cdot 10^6} = \frac{\pi^2}{2000000}
 \end{aligned}$$

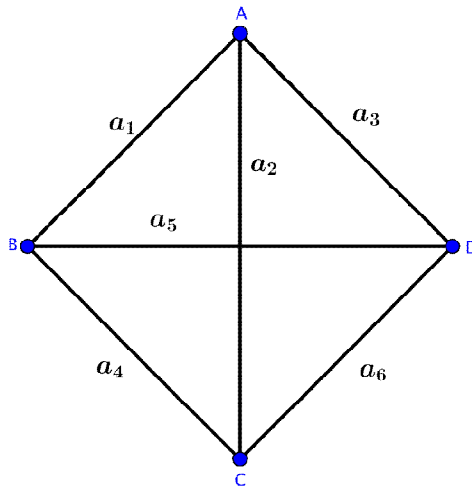
UP.162. If  $ABCD$  tetrahedron;  $AB = a_1$ ;  $AC = a_2$ ;  $AD = a_3$ ;  $BC = a_4$ ;  
 $BD = a_5$ ;  $CD = a_6$  then:

$$\sum_{1 \leq i < j \leq 6} (a_i + a_j)^2 \geq 4\sqrt{3}S[ABCD]$$

$S[ABCD]$  – total area of tetrahedron  $ABCD$

*Proposed by Daniel Sitaru – Romania*

*Solution by Marian Ursărescu – Romania*



$$(a_i + a_j)^2 \geq 3a_i a_j \Rightarrow \text{we must show:}$$

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$$3 \sum_{1 \leq i < j \leq 6} a_i a_j \geq 12\sqrt{3}S[ABCD] \Leftrightarrow$$

$$\sum_{1 \leq i < j \leq 6} a_i a_j \geq 4\sqrt{3}S[ABCD] \quad (1)$$

Now, using Gordon's inequality: in any  $\Delta ABC$  we have:

$$ab + bc + ac \geq 4\sqrt{3} \Rightarrow$$

$$\left. \begin{array}{l} a_1 a_2 + a_1 a_4 + a_2 a_4 \geq 4\sqrt{3}S_{ABC} \\ a_2 a_6 + a_2 a_3 + a_3 a_6 \geq 4\sqrt{3}S_{ACD} \\ a_1 a_5 + a_1 a_3 + a_3 a_5 \geq 4\sqrt{3}S_{ABD} \\ a_4 a_5 + a_4 a_6 + a_5 a_6 \geq 4\sqrt{3}S_{BCD} \end{array} \right\} \Rightarrow \sum_{1 \leq i < j \leq 6} a_i a_j \geq 4\sqrt{3}S[ABCD]$$

UP.163. Let  $a, b, c$  be positive real numbers such that:

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) = 8.$$

Find the minimum value of:

$$P = \frac{a}{b(b+2c+1)(a+3c)^2} + \frac{b}{c(c+2a+1)(b+3a)^2} + \frac{c}{a(a+2b+1)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

\* By Cauchy Schwarz inequality we have:

$$\begin{aligned} P &= \frac{a}{b(b+2c+1)(a+3c)^2} + \frac{b}{c(c+2a+1)(b+3a)^2} + \frac{c}{a(a+2b+1)(c+3b)^2} \\ &= \frac{\left(\frac{a}{a+3c}\right)^2}{ab(b+2c+1)} + \frac{\left(\frac{b}{b+3a}\right)^2}{bc(c+2a+1)} + \frac{\left(\frac{c}{c+3b}\right)^2}{ca(a+2b+1)} \geq \\ &\geq \frac{\left(\frac{a}{a+3} + \frac{b}{b+3a} + \frac{c}{c+3b}\right)^2}{ab(b+2c+1) + bc(c+2a+1) + ca(a+2b+1)} \\ &\Leftrightarrow P \geq \frac{\left(\frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b}\right)^2}{(ab^2+bc^2+ca^2)+6abc+(ab+bc+ca)} \quad (1) \end{aligned}$$

- By inequality:  $x^2 + y^2 \geq \frac{(x+y)^2}{2}$  (equality occurs if  $x = y$ ) we have:

$$8 = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq \frac{(a+b)^2}{2} \cdot \frac{(b+c)^2}{2} \cdot \frac{(c+a)^2}{2} \Leftrightarrow$$

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$$\Leftrightarrow 64 \geq (a+b)^2(b+c)^2(c+a)^2$$

$$\Leftrightarrow 8 \geq (a+b)(b+c)(c+a) \quad (2)$$

- Other, by AM-GM inequality for 2 positive real numbers:

$$(a+b)(b+c)(c+a) \geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ca} = 8\sqrt{(abc)^2} = 8abc \Leftrightarrow$$

$$abc \leq \frac{(a+b)(b+c)(c+a)}{8} \quad (3)$$

- Hence (3):

$$\begin{aligned} &\Rightarrow (a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc \leq \\ &\leq (a+b)(b+c)(c+a) + \frac{(a+b)(b+c)(c+a)}{8} = \frac{9(a+b)(b+c)(c+a)}{8} \end{aligned}$$

$$\Leftrightarrow (a+b+c)(ab+bc+ca) \leq \frac{9(a+b)(b+c)(c+a)}{8} \quad (4)$$

$$\text{- Let (2), (4):} \Rightarrow 8 \geq \frac{8(a+b+c)(ab+bc+ca)}{9} \Leftrightarrow 9 \geq (a+b+c)(ab+bc+ca)$$

$$\Leftrightarrow 9 \geq (ab^2 + bc^2 + ca^2) + (a^2b + b^2c + c^2a) + 3abc \quad (5)$$

- By AM-GM inequality we have:

$$\begin{aligned} a^2b + b^2c + c^2a &\geq 3\sqrt[3]{(a^2b) \cdot (b^2c) \cdot (c^2a)} = 3\sqrt[3]{(abc)^3} = 3abc \Leftrightarrow a^2b + b^2c + \\ &c^2a \geq 3abc \quad (6) \end{aligned}$$

$$\text{- Let (5), (6):} \Rightarrow 9 \geq (ab^2 + bc^2 + ca^2) + 3abc + 3abc \Leftrightarrow (ab^2 + bc^2 + ca^2) + 6abc \leq 9 \quad (7)$$

$$\text{- Other: } a^2 + b^2 + c^2 = \frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} \geq \frac{2ab}{2} + \frac{2bc}{2} + \frac{2ca}{2} = ab + bc + ca$$

$$\begin{aligned} &\Leftrightarrow a^2 + b^2 + c^2 \geq ab + bc + ca \Leftrightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ &\geq 3(ab + bc + ca) \end{aligned}$$

$$\Leftrightarrow (a+b+c)^2 \geq 3(ab+bc+ca) \Leftrightarrow a+b+c \geq \sqrt{3(ab+bc+ca)} \quad (8)$$

$$\text{- Let (4), (8):} \Rightarrow (a+b)(b+c)(c+a) \geq \frac{8\sqrt{3(ab+bc+ca)} \cdot (ab+bc+ca)}{9} \quad (9)$$

- Let (2), (9):

$$\Rightarrow 8 \geq \frac{8\sqrt{3(ab+bc+ca)} \cdot (ab+bc+ca)}{9} \Leftrightarrow 9 \geq \sqrt{3(ab+bc+ca)^3} \Leftrightarrow 81$$

$$\geq 3(ab+bc+ca)^3$$

$$\Leftrightarrow 27 \geq (ab+bc+ca)^3 \Leftrightarrow 3 \geq (ab+bc+ca) \Leftrightarrow (ab+bc+ca) \leq 3 \quad (10)$$

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- Let (7), (10):  $\Rightarrow (ab^2 + bc^2 + ca^2) + 6abc + (ab + bc + ca) \leq 9 + 3$

$\Leftrightarrow (ab^2 + bc^2 + ca^2) + 6abc + (ab + bc + ca) \leq 12 \quad (11)$

- Let (1), (11):  $\Rightarrow P \geq \frac{\left(\frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b}\right)^2}{12} \quad (12)$

- By Cauchy Schwarz inequality we have:

$$\begin{aligned} \frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b} &= \frac{a^2}{a^2+3ca} + \frac{b^2}{b^2+3ab} + \frac{c^2}{c^2+3bc} \\ &\geq \frac{(a+b+c)^2}{(a^2+3ca) + (b^2+3ab) + (c^2+3bc)} \end{aligned}$$

$$\Leftrightarrow \frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b} \geq \frac{(a+b+c)^2}{(a^2+b^2+c^2+2ab+2bc+2ca) + (ab+bc+ca)}$$

$$\Leftrightarrow \frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b} \geq \frac{(a+b+c)^2}{(a+b+c)^2 + (ab+bc+ca)} \quad (13)$$

- Let (8):  $(a+b+c)^2 \geq 3(ab+bc+ca) \Leftrightarrow ab+bc+ca \leq \frac{(a+b+c)^2}{3} \quad (14)$

- Let (13), (14):  $\Rightarrow \frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b} \geq \frac{(a+b+c)^2}{(a+b+c)^2 + \frac{(a+b+c)^2}{3}} = \frac{3(a+b+c)^2}{4(a+b+c)^2} = \frac{3}{4}$

$$\Leftrightarrow \frac{a}{a+3c} + \frac{b}{b+3a} + \frac{c}{c+3b} \geq \frac{3}{4} \quad (15)$$

- Let (12), (15):  $\Rightarrow P \geq \frac{\left(\frac{3}{4}\right)^2}{12} = \frac{9}{16 \cdot 12} = \frac{3}{64} \Rightarrow P \geq \frac{3}{64} \Rightarrow P_{\min} = \frac{3}{64}$

+ Equality occurs if: 
$$\begin{cases} a, b, c > 0; (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) = 8 \\ a = b; b = c; c = a \\ \frac{a}{(a+3c) \cdot ab(b+2c+1)} = \frac{b}{(b+3a) \cdot bc(c+2a+1)} = \frac{c}{(c+3b) \cdot ca(a+2b+1)} \\ \frac{a}{a(a+3c)} = \frac{b}{b(b+3a)} = \frac{c}{c(c+3b)} \end{cases}$$

$$\Leftrightarrow \begin{cases} a = b = c > 0 \\ (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) = 8 \end{cases} \Leftrightarrow \begin{cases} a = b = c > 0 \\ 2a^2 \cdot 2a^2 \cdot 2a^2 = 8 \end{cases} \Leftrightarrow \begin{cases} a = b = c > 0 \\ a^6 = 1 \end{cases} \Leftrightarrow$$

$$a = b = c = 1.$$

Therefore, minimum of  $P: \frac{3}{64}$  then  $a = b = c = 1$ .

UP.164. Solve the equation:  $\sqrt{2(x^4 + 1)} + 2\sqrt{3x - 2x^4} = 7 - 3x$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

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**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned}
 x^4 + y^4 &\leq 2(x^2 - xy + y^2)^2 \Leftrightarrow x^4 + y^4 + 6x^2y^2 - 4xy(x^2 + y^2) \geq 0 \Leftrightarrow \\
 &\Leftrightarrow (x^2 + y^2)^2 + 4x^2y^2 - 4xy(x^2 + y^2) \geq 0 \Leftrightarrow (x^2 + y^2 - 2xy)^2 \geq 0 \rightarrow \text{true} \\
 \therefore x^4 + 1 &\leq 2(x^2 - x + 1)^2; (y = 1) \Rightarrow \sqrt{x^4 + 1} \leq \sqrt{2}|x^2 - x + 1| \Rightarrow \sqrt{x^4 + 1} \\
 &\leq \sqrt{2}(x^2 - x + 1) \\
 &\left( \because x^2 - x + 1 = \frac{1}{4}(x + 1)^2 + \frac{3}{4}(x - 1)^2 > 0 \right) \\
 &\Rightarrow \sqrt{2(x^4 + 1)} \stackrel{(1)}{\leq} 2x^2 - 2x + 2 \text{ (equality when } x = 1)
 \end{aligned}$$

Now, for LHS to be defined, we must have:  $x(3 - 2x^3) \geq 0$ . If  $x \leq 0$ , we then must have

$$3 - 2x^3 \leq 0 \Rightarrow x \geq \sqrt[3]{\frac{3}{2}} \rightarrow \text{not possible} \therefore x \geq 0 \text{ \& \& \& we must have}$$

$$3 - 2x^3 \geq 0 \Rightarrow x \leq \sqrt[3]{\frac{3}{2}}$$

$$\begin{aligned}
 \text{Now, } 2x^2 + x + 2\sqrt{3x - 2x^4} &\stackrel{(2)}{\leq} 5 \Leftrightarrow (2x^2 + x)^2 + 4(3x - 2x^4) + 4(2x^2 + \\
 &\mathbf{x3x - 2x4 \leq 25}
 \end{aligned}$$

$$\Leftrightarrow 4(2x^2 + x)\sqrt{3x - 2x^4} \stackrel{(3)}{\leq} 4x^4 - 4x^3 - x^2 - 12x + 25$$

Let  $f(x) = 4x^4 - x^2 - (4x^3 + 12x - 25)$ , which is a quadratic in  $x^2$ .

$$\begin{aligned}
 \text{Discriminant} &= 1 + 4(4)(4x^3 + 12x - 25) = 64x^3 + 192x - 399 < 64\left(\frac{3}{2}\right) + 192\left(\frac{3}{2}\right) - \\
 &\mathbf{399}
 \end{aligned}$$

$$\left( \because x^3 \leq \frac{3}{2} \text{ \& } x \leq \sqrt[3]{\frac{3}{2}} < \frac{3}{2} \right)$$

$$\begin{aligned}
 = -15 < 0 \therefore f(x) > 0 \therefore (3) \Leftrightarrow 16(3x - 2x^4)(2x^2 + x)^2 &\leq (4x^4 - 4x^3 - x^2 - 12x + \\
 &\mathbf{252}
 \end{aligned}$$

$$\Leftrightarrow 144x^8 + 96x^7 + 40x^6 - 280x^5 + 105x^4 - 224x^3 + 94x^2 - 600x + 625 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (x - 1)^2(144x^6 + 384x^5 + 664x^4 + 664x^3 + 769x^2 + 650x + 625) \geq 0$$

$\rightarrow \text{true} \because x \geq 0$  (equality when  $x = 1$ )

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$$\therefore (3) \text{ \& \textit{hence} (2) is true} \Rightarrow 2\sqrt{3x - 2x^4} \stackrel{(4)}{\leq} 5 - 2x^2 - x$$

$$(1)+(4) \Rightarrow \sqrt{2(x^4 + 1)} + 2\sqrt{3x - 2x^4} \leq 7 - 3x, \text{ equality when } x = 1.$$

$$\& \therefore \sqrt{2(x^4 + 1)} + 2\sqrt{3x - 2x^4} = 7 - 3x, \therefore x = 1, \text{ which is the only solution (answer)}$$

**Solution 2 by Amit Dutta-Jamshedpur-India**

*Using Cauchy – Schwarz's Inequality:*

$$\left(\sqrt{2(x^4 + y^4)} + 2xy\right)^2 \leq 2\left(2(x^4 + y^4 + 2x^2y^2)\right) \leq 4(x^2 + y^2)^2 \Rightarrow$$

$$\Rightarrow \left(\sqrt{2(x^4 + y^4)} + 2xy\right) \leq 2(x^2 + y^2)$$

$$\sqrt{2(x^4 + y^4)} \leq 2(x^2 - xy + y^2)$$

$$\text{Putting } y = 1: \sqrt{2(x^4 + 1)} \leq 2(x^2 - x + 1) \quad (1)$$

$$\left( \begin{array}{l} \text{Equality holds when} \\ \frac{1}{\sqrt{2(x^4 + y^4)}} = \frac{1}{2xy} \text{ or } \frac{1}{\sqrt{2(x^4 + 1)}} = \frac{1}{2x} \\ \Rightarrow x = 1 \quad (a) \end{array} \right)$$

$$\therefore GM \leq AM$$

$$2\sqrt{3x - 2x^4} \leq 2\sqrt{x(3 - 2x^3)} \leq (x + 3 - 2x^3) \quad (2)$$

$$\left( \begin{array}{l} \text{Equality holds when} \\ x = 3 - 2x^3 \Rightarrow x = 1 \\ (b) \end{array} \right)$$

*Adding (1) & (2):*

$$\sqrt{2(x^4 + 1)} + 2\sqrt{3x - 2x^4} \leq 2(x^2 - x + 1) + x + 3 - 2x^3 \Rightarrow$$

$$\Rightarrow 7 - 3x \leq 2x^2 - x + 5 - 2x^3 \Rightarrow 2x^3 - 2x^2 - 2x + 2 \leq 0 \Rightarrow x^3 - x^2 - x + 1 \leq 0 \Rightarrow$$

$$\Rightarrow x^2(x - 1) - 1(x - 1) \leq 0 \Rightarrow (x^2 - 1)(x - 1) \leq 0 \Rightarrow (x + 1)(x - 1)^2 \leq 0$$

$$\therefore (x + 1) > 0 \quad \{ \because x > 0 \text{ from domain} \}$$

$$\Rightarrow (x - 1)^2 \leq 0 \Rightarrow (x - 1)^2 = 0$$

$$\Rightarrow x = 1 \quad (c)$$

**From (a), (b) & (c) we can conclude that the only real solution is  $x = 1$ .**

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UP.165. Solve the system of equation: 
$$\begin{cases} \frac{1}{\sqrt{a^3}} + \frac{1}{\sqrt{b^3}} + \frac{1}{\sqrt{c^3}} = 3 \\ \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{(a^3+b^3+c^3)^2}{3} \end{cases} \quad (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

\* Hence (1), by AM-GM inequality for three positive real numbers we have:

$$3 = \frac{1}{\sqrt{a^3}} + \frac{1}{\sqrt{b^3}} + \frac{1}{\sqrt{c^3}} \geq 3 \cdot \sqrt[3]{\frac{1}{\sqrt{a^3} \cdot \sqrt{b^3} \cdot \sqrt{c^3}}} = \frac{3}{\sqrt[6]{(abc)^3}} = \frac{3}{\sqrt{abc}} \Leftrightarrow 3 \geq \frac{3}{\sqrt{abc}} \Leftrightarrow \sqrt{abc} \geq 1 \Leftrightarrow abc \geq 1$$

$$\text{Hence (2):} \Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^3c+b^3a+c^3b}{abc} \leq \frac{a^3c+b^3a+c^3b}{1} = a^3c + b^3a + c^3b \quad (3)$$

- By AM-GM inequality, we have:

$$a^3c + b^3a + c^3b = a^3ac + b^2ba + c^2cb \leq \frac{a^3 + (ac)^2}{2} + \frac{b^4 + (ba)^2}{2} + \frac{c^4 + (cb)^2}{2}$$

$$\Leftrightarrow a^3c + b^3a + c^3b \leq \frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2} \quad (4)$$

$$\text{- Hence (3), (4):} \Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \leq \frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2} \quad (5)$$

- Other, by AM-GM inequality:

$$\frac{a^6 + a^6 + 1}{2} + \frac{b^6 + b^6 + 1}{2} + \frac{c^6 + c^6 + 1}{2} \geq \frac{3\sqrt[3]{a^6 \cdot a^6 \cdot 1}}{2} + \frac{3\sqrt[3]{b^6 \cdot b^6 \cdot 1}}{2} + \frac{3\sqrt[3]{c^6 \cdot c^6 \cdot 1}}{2}$$

$$= \frac{3(a^4 + b^4 + c^4)}{2}$$

$$\Leftrightarrow a^6 + b^6 + c^6 + \frac{3}{2} \geq \frac{3(a^4+b^4+c^4)}{2} \Leftrightarrow 2(a^6 + b^6 + c^6) + 3 \geq 3(a^4 + b^4 + c^4) \quad (6)$$

$$(a^3b^3 + a^3b^3 + 1) + (b^3c^3 + b^3c^3 + 1) + (c^3a^3 + c^3a^3 + 1) \geq$$

$$\geq 3\sqrt[3]{(a^3b^3)(a^3b^3) \cdot 1} + 3\sqrt[3]{(b^3c^3)(b^3c^3) \cdot 1} + 3\sqrt[3]{(c^3a^3)(c^3a^3) \cdot 1}$$

$$= 3(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow 2(a^3b^3 + b^3c^3 + c^3a^3) + 3 \geq 3(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow 4(a^3b^3 + b^3c^3 + c^3a^3) + 6 \geq 6(a^2b^2 + b^2c^2 + c^2a^2) \quad (7)$$

$$\text{- Let (6), (7):} \Rightarrow 2(a^6 + b^6 + c^6) + 3 + 4(a^3b^3 + b^3c^3 + c^3a^3) + 6 \geq$$

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$$\begin{aligned} &\geq 3(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) \\ \Leftrightarrow &2(a^6 + b^6 + c^6 + 2a^3b^3 + 2b^3c^3 + 2c^3a^3) + 9 \geq \\ &\geq 3(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) \\ \Leftrightarrow &2(a^3 + b^3 + c^3)^2 \geq 3(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) - 9 \quad (8) \end{aligned}$$

- By AM-GM inequality and (2). We have:

$$\begin{aligned} 3(a^2b^2 + b^2c^2 + c^2a^2) &\geq 3 \cdot 3 \cdot \sqrt[3]{(a^2b^2)(b^2c^2)(c^2a^2)} = 9\sqrt[3]{(abc)^4} \geq 9\sqrt[3]{1^4} = 9 \\ \Leftrightarrow &6(a^2b^2 + b^2c^2 + c^2a^2) - 9 \geq 3(a^2b^2 + b^2c^2 + c^2a^2) \quad (9) \end{aligned}$$

- Let (8),(9):  $\Rightarrow 2(a^3 + b^3 + c^3)^2 \geq 3(a^4 + b^4 + c^4) + 3(a^2b^2 + b^2c^2 + c^2a^2)$

$$\Leftrightarrow 2(a^3 + b^3 + c^3)^2 \geq 3(a^4 + b^4 + c^4 + a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow \frac{a^4 + b^4 + c^4 + a^2b^2 + b^2c^2 + c^2a^2}{2} \leq \frac{(a^3 + b^3 + c^3)^2}{3} \quad (10)$$

$$\text{- (5), (10):} \Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \leq \frac{(a^3 + b^3 + c^3)^2}{3} \quad (11)$$

- (1), (11):  $\Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{(a^3 + b^3 + c^3)^2}{3}$  occurs if:  $\begin{cases} a = b = c > 0 \\ \frac{1}{\sqrt{a^3}} + \frac{1}{\sqrt{b^3}} + \frac{1}{\sqrt{c^3}} = 3 \end{cases} \Leftrightarrow a = b = c = 1.$

Solution of equation is:  $(a, b, c) = (1, 1, 1).$