

Number 10

AUTUMN 2018

R M M

ROMANIAN MATHEMATICAL MAGAZINE

SOLUTIONS

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

SOLUTIONS

RMM AUTUMN EDITION 2018

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Proposed by

Daniel Sitaru - Romania

Hoang Le Nhat Tung – Hanoi – Vietnam

Andrei Ștefan Mihalcea – Romania

Mehmet Şahin – Ankara – Turkey

Marian Ursărescu – Romania

Do Quoc Chinh – Ho Chi Minh – Vietnam

Shivam Sharma-New Delhi-India

D.M. Bătinețu – Giurgiu - Romania

Neculai Stanciu – Romania

Mihály Bencze – Romania



ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Solutions by

Daniel Sitaru – Romania, Nassim Nicholas Taleb – USA

Michael Stergiou-Greece, Ravi Prakash-New Delhi-India

Soumava Chakraborty-Kolkata-India, Boris Colakovic-Belgrade-Serbia

Sanong Huayrerai-Nakon Pathom-Thailand, Andrei Ştefan Mihalcea – Romania, Hoang Le

Nhat Tung – Hanoi – Vietnam, Marian Ursărescu – Romania, Rovsen Pirguliyev-Sumgait-

Azerbaijan, Heikichi Ezakiya-Jakarta-Indonesia, Rade Krenkov-Sturmica-Macedonia, Lahiru

Samarakoon-Sri Lanka, Soumitra Mandal-Chandar Nagore-India, Sagar Kumar-Kolkata-

India, Tran Hong-Vietnam, Chris Kyriazis-Greece, Remus Florin Stanca-Romania

Shivam Sharma-New Delhi-India, Shafiqur Rahman-Bangladesh

D.M. Bătineţu – Giurgiu – Romania, Neculai Stanciu – Romania, Khalef Ruhemi-Jarash-

Jordan, Mihály Bencze – Romania, Ruanghaw Chaokha-Chiangrai-Thailand, Aaditya Joshi-

Mumbai-India

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

JP.136. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$Q = \frac{1}{\sqrt[3]{2x^5 + y^4 - x^2 + 4}} + \frac{1}{\sqrt[3]{2y^5 + z^4 - y^2 + 4}} + \frac{1}{\sqrt[3]{2z^5 + x^4 - z^2 + 4}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Michael Stergiou-Greece

$$Q = \sum_{cyc} \frac{1}{\sqrt[3]{2x^5 + y^4 - x^2 + 4}} \quad (1)$$

The function $f(x) = \sqrt[3]{x}$ is concave so (1) becomes:

$$Q \leq 3 \sqrt[3]{\frac{1}{3} \sum \frac{1}{2x^5 + y^4 - x^2 + 4}} \rightarrow \frac{Q^3}{9} \leq \sum_{cyc} \frac{1}{2x^5 + y^4 - x^2 + 4} \quad (2)$$

$$\text{As } x^5 + x^5 + x^2 \geq 3x^4 \quad (2) \rightarrow \frac{Q^3}{9} \leq \sum_{cyc} \frac{1}{3x^4 + y^4 - 2x^2 + 4} \quad (3)$$

Let $x^2 = a, y^2 = b, z^2 = c, abc = 1$ (3) $\rightarrow \frac{Q^3}{9} \leq \sum \frac{1}{3a^2 + b^2 - 2a + 4}$ (4). But $a^2 + b^2 \geq 2ab$
 $a^2 + 1 \geq 2a$

(4) $\rightarrow \frac{Q^3}{9} \leq \frac{1}{2} \sum_{cyc} \frac{1}{a + ab + 1} \leq \dots \leq 1$ (after calculus the sum reduces to 1 using $abc = 1$)

$$\text{Therefore } Q \leq \sqrt[3]{\frac{9}{2}}$$

JP.137. Let $x, y \geq 1$. Prove that:

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)}$$

Proposed by Andrei Ştefan Mihalcea – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Rewrite the inequality: } \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} - 2 \geq \frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow$$

$$\Leftrightarrow \frac{(\sqrt{x}-\sqrt{y})^2}{\sqrt{xy}} \geq \frac{4(\sqrt{x}-\sqrt{y})^2(\sqrt{x}+\sqrt{y})^2}{(2x+xy+1)(2y+xy+1)} \quad (1)$$

If $x = y$, there is nothing to show. Suppose $x \neq y$, then (1) can be written as

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \frac{1}{\sqrt{xy}} &\geq \frac{4(\sqrt{x} + \sqrt{y})^2}{(2x + xy + 1)(2y + xy + 1)} \Leftrightarrow (xy + 1)^2 + 2(x + y)(xy + 1) + 4xy \geq \\ &\geq 4\sqrt{xy}(x + y + 2\sqrt{xy}) \Leftrightarrow x^2y^2 + 1 + 2(x + y)(xy + 1) + 6xy \geq 4\sqrt{xy}(x + y) + 8xy \\ &\Leftrightarrow x^2y^2 - 2xy + 1 + 2(x + y)(xy - 2\sqrt{xy} + 1) \geq 0 \Leftrightarrow \\ &\Leftrightarrow (xy - 1)^2 + 2(x + y)(\sqrt{xy} - 1)^2 \geq 0 \text{ which is true } \forall x, y > 0. \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} &\stackrel{(1)}{\geq} 2 + \frac{4(x - y)^2}{(2x + xy + 1)(2y + xy + 1)} \\ (1) \Leftrightarrow \frac{x}{y} + \frac{y}{x} + 2 &\geq 4 + \frac{16(x - y)^4}{(2x + xy + 1)^2(2y + xy + 1)^2} + \frac{16(x - y)^2}{(2x + xy + 1)(2y + xy + 1)} \Leftrightarrow \\ \Leftrightarrow \frac{(x - y)^2}{xy} &\geq \frac{16(x - y)^4}{(2x + y + 1)^2(2y + xy + 1)^2} + \frac{16(x - y)^2}{(2x + xy + 1)(2y + xy + 1)} \Leftrightarrow \\ \Leftrightarrow \frac{1}{xy} &\geq \frac{16(x - y)^2 + 16(2x + xy + 1)(2y + xy + 1)}{(2x + xy + 1)^2(2y + xy + 1)^2} \quad (\because (x - y)^2 \geq 0) \\ \Leftrightarrow (2x + xy + 1)^2(2y + xy + 1)^2 &\geq 16xy(2x + xy + 1)(2y + xy + 1) + \\ &+ 4xy((2x + xy + 1) - (2y + xy + 1))^2 \Leftrightarrow \\ \Leftrightarrow a^2b^2 &\geq 16xyab + 4xy(a^2 + b^2 - 2ab) \quad \left(\begin{array}{l} \text{where } a = 2x + xy + 1 \\ b = 2y + xy + 1 \end{array} \right) \\ \Leftrightarrow a^2b^2 &\geq 4xy(a + b)^2 \Leftrightarrow ab \stackrel{(1)}{\geq} 2(a + b)\sqrt{xy}. \text{ Now,} \\ ab &= (2x + xy + 1)(2y + xy + 1) = (x(1 + y) + (1 + x))(y(1 + x) + (1 + y)) \\ &\stackrel{A-G}{\geq} \left(2\sqrt{x(1 + x)(1 + y)} \right) \left(2\sqrt{y(1 + x)(1 + y)} \right) = 4\sqrt{xy}(1 + x)(1 + y) = \\ &= 2\sqrt{xy}(2x + 2x + 2y + 2xy) = 2\sqrt{xy}(a + b) \Rightarrow (1) \text{ is true (Proved)} \end{aligned}$$

JP.138. Let $a, b, c > 0$, with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$(4a - 3)(4b - 3)(4c - 3) \geq 243\sqrt[3]{abc}$$

Proposed by Andrei Ştefan Mihalcea – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbia

$$(4a - 3)(4b - 3)(4c - 3) \geq 243\sqrt[3]{abc} \Leftrightarrow (4a - 3)^3(4b - 3)^3(4c - 3)^3 \geq 243^3abc$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$(4a - 3)^3 = 64a^3 - 144a^2 + 108a - 27 \geq 243a \Leftrightarrow (a - 3)(8a + 3)^2 \geq 0 \Leftrightarrow a \geq 3 \quad (*)$$

$$(4b - 3)^3 = 64b^3 - 144b^2 + 108b - 27 \geq 234b \Leftrightarrow (b - 3)(8b + 3)^2 \geq 0 \Leftrightarrow b \geq 3 \quad (**)$$

$$(4c - 3)^3 = 64c^3 - 144c^2 + 108c - 27 \geq 243c \Leftrightarrow (c - 3)(8c + 3)^2 \geq 0 \Leftrightarrow c \geq 3 \quad (***)$$

$$(*) \times (**) \times (***) \Rightarrow abc \geq 27 \text{ true because of } 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \stackrel{AM-GM}{\geq} \frac{3}{\sqrt[3]{abc}} \Rightarrow abc \geq 27$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, give $a = x^3, b = y^3, c = z^3$. Hence $\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} = 1 \Leftrightarrow$

$$\Leftrightarrow x^3y^3 + y^3z^3 + z^3x^3 = (xyz)^3 \Leftrightarrow xyz \geq 3. \text{ We want to show that}$$

$$64(xyz)^3 + 36(x^3 + y^3 + z^3) \geq 243xyz + 48((xy)^3 + (yz)^3 + (zx)^3).$$

$$\text{Iff } 16(xyz)^3 + 36(x^3 + y^3 + z^3) \geq 243xyz + 2y$$

$$\text{Iff } 16(xyz)^3 \geq 135xyz + 2y$$

$$\text{Iff } xyz(16((xyz)^2 - 135)) \geq 24$$

$$\text{Iff } (16(xyz)^2 - 135) \geq 9$$

$$\text{Iff } 16(xyz)^2 \geq 144: xyz \geq 3$$

$$\text{Hence } (4x^3 - 3)(4y^3 - 3)(4z^3 - 3) \geq 243xyz$$

$$\text{That is } (4a - 3)(4b - 3)(4c - 3) \geq 243\sqrt[3]{abc}$$

Therefore it is to be true.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS = 64abc - 48 \sum ab + 36 \sum a - 27 = 64abc - 48abc + 36 \sum a - 27$$

$$(\because \sum ab = abc)$$

$$\stackrel{A-G}{\geq} 16abc + 108\sqrt[3]{abc} - 27 \stackrel{?}{\geq} 243\sqrt[3]{abc} \Leftrightarrow 16t^3 - 135t - 27 \stackrel{?}{\geq} 0 \quad (t = \sqrt[3]{abc})$$

$$\Leftrightarrow (t - 3)(16t^2 + 48t + 9) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t = \sqrt[3]{abc} \geq 3 \text{ (proved)}$$

Proof of $t \geq 3$:

$$\sum ab = abc \stackrel{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2} \Rightarrow a^3b^3c^3 \geq 27a^2b^2c^2 \Rightarrow \sqrt[3]{abc} = t \geq 3$$

Solution 4 by proposer

From $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ we get $abc = ab + bc + ca \geq 3\sqrt[3]{(abc)^2}$. Now, we have:

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$(abc)^3 \geq 27(abc)^2$ |: $(abc)^2$ getting $abc \geq 27$ and from that $\sqrt[3]{abc} \geq 3$. Note that $a > 1$ because $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ and we have $3a - 3 > 0$, $3b - 3 > 0$ and $3c - 3 > 0$.

Using Hölder's inequality we get:

$$(4a - 3)(4b - 3)(4c - 3) = [a + (3a - 3)][b + (3b - 3)][c + (3c - 3)] \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{(a-1)(b-1)(c-1)} \right)^3$$

$$(4a - 3)(4b - 3)(4c - 3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{abc - (ab + bc + ca) + (a + b + c) - 1} \right)^3$$

$$(4a - 3)(4b - 3)(4c - 3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{abc - abc + 3\sqrt[3]{abc} - 1} \right)^3$$

$$(4a - 3)(4b - 3)(4c - 3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{3 \cdot \sqrt[27]{27} - 1} \right)^3$$

$$(4a - 3)(4b - 3)(4c - 3) \geq (\sqrt[3]{abc} + 3 + 3)^3$$

Using AM-GM we have: $(4a - 3)(4b - 3)(4c - 3) \geq \left(3\sqrt[3]{\sqrt[3]{abc} \cdot 3 \cdot 3} \right)^3$

$$(4a - 3)(4b - 3)(4c - 3) \geq 3^3 \cdot \sqrt[3]{abc} \cdot 9$$

$$(4a - 3)(4b - 3)(4c - 3) \geq 243\sqrt[3]{abc}$$

JP.139. Let x, y, z be positive real numbers such that: $x^2 + y^2 + z^2 = 3$. Find the minimum of the expression:

$$P = \frac{x}{\sqrt[4]{\frac{y^8 + z^8}{2} + 3yz}} + \frac{y}{\sqrt[4]{\frac{z^8 + x^8}{2} + 3zx}} + \frac{z}{\sqrt[4]{\frac{x^8 + y^8}{2} + 3xy}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

- Using Cauchy and Bunhiacopxki inequality. We have:

$$\left(\sqrt{2(y^8 + z^8)} + 2y^2z^2 \right)^2 \leq 2(2(y^8 + z^8) + 4y^4z^4) = 4(y^4 + z^4)^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{2(y^8 + z^8)} + 2y^2z^2 \leq 2(y^4 + z^4)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Leftrightarrow \sqrt{2(y^8 + z^8)} \leq 2(y^4 - y^2z^2 + z^4) \Leftrightarrow \sqrt[4]{\frac{y^8+z^8}{2}} \leq \sqrt{y^4 - y^2z^2 + z^4} \quad (1)$$

- Other:

$$\begin{aligned} \sqrt{y^4 - y^2z^2 + z^4} &= \sqrt{(y^2 + z^2)^2 - (yz\sqrt{3})^2} = \sqrt{(y^2 - yz\sqrt{3} + z^2)(y^2 + yz\sqrt{3} + z^2)} \\ &= \sqrt{(2 + \sqrt{3})(y^2 - yz\sqrt{3} + z^2) \cdot (2 - \sqrt{3})(y^2 + yz\sqrt{3} + z^2)} \leq \\ &\leq \frac{(2 + \sqrt{3})(y^2 - yz\sqrt{3} + z^2) + (2 - \sqrt{3})(y^2 + yz\sqrt{3} + z^2)}{2} \\ &= \frac{2(2y^2 - 3yz + 2z^2)}{2} = 2y^2 - 3yz + 2z^2 \\ &\Rightarrow \sqrt{y^4 - y^2z^2 + z^4} \leq 2y^2 - 3yz + 2z^2 \quad (2) \end{aligned}$$

$$\text{- Let (1), (2): } \Rightarrow \sqrt[4]{\frac{y^8+z^8}{2}} \leq 2y^2 - 3yz + 2z^2 \Leftrightarrow \sqrt[4]{\frac{y^8+z^8}{2}} + 3yz \leq 2(y^2 + z^2)$$

$$\Leftrightarrow \frac{1}{\sqrt[4]{\frac{y^8+z^8}{2}+3yz}} \geq \frac{1}{2(y^2+z^2)} \Leftrightarrow \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}+3yz}} \geq \frac{x}{2(y^2+z^2)} = \frac{x}{2(3-x^2)} \quad (\text{Let } x^2 + y^2 + z^2 = 3) \quad (3)$$

$$\text{- We have: } \frac{x}{3-x^2} - \frac{x^2}{2} = x \left(\frac{1}{3-x^2} - \frac{x}{2} \right) = \frac{x(x^3-3x+2)}{2(3-x^2)} = \frac{x(x-1)^2(x+2)}{2(3-x^2)} \geq 0 \quad (\text{because } x > 0; (x-1)^2 \geq 0)$$

$$\Leftrightarrow \frac{x}{3-x^2} - \frac{x^2}{2} \geq 0 \Leftrightarrow \frac{x}{3-x^2} \geq \frac{x^2}{2}. \text{ Let (3): } \Rightarrow \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}+3yz}} \geq \frac{x^2}{4} \quad (4)$$

$$+ \text{ Similar: } \frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}+3zx}} \geq \frac{y^2}{4}; \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}+3xy}} \geq \frac{z^2}{4} \quad (5)$$

- Let (4), (5) and $x^2 + y^2 + z^2 = 3$:

$$\begin{aligned} \Rightarrow P &= \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}+3yz}} + \frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}+3zx}} + \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}+3xy}} \geq \frac{x^2+y^2+z^2}{4} = \frac{3}{4} \Rightarrow \\ &\Rightarrow Q_{\min} = \frac{3}{4} \end{aligned}$$

+ Equality occurs if:

$$\left\{ \begin{array}{l} x, y, z > 0; x^2 + y^2 + z^2 = 3 \\ \sqrt{2(x^8 + y^8)} = 2x^2y^2; \sqrt{2(y^8 + z^8)} = 2y^2z^2; \sqrt{2(z^8 + x^8)} = 2z^2x^2 \Leftrightarrow x = y = z = 1 \\ x - 1 = y - 1 = z - 1 = 0 \end{array} \right.$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

JP.140. Let $a, b, c > 0$. Prove that:

$$\sum \frac{\sqrt{a+b}}{a} \leq \left(\sum \frac{1}{a} \right) \sqrt{\sum a - \frac{\sum ab}{\sum a}}$$

Proposed by Andrei Ştefan Mihalcea – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{LHS} &= \frac{bc\sqrt{a+b} + ca\sqrt{b+c} + ab\sqrt{c+a}}{abc} = \frac{\sqrt{bc(a+b)}\sqrt{bc} + \sqrt{ca(b+c)}\sqrt{ca} + \sqrt{ab(c+a)}\sqrt{ab}}{abc} \stackrel{\text{CBS}}{\leq} \\ &\leq \frac{\sqrt{\sum ab} \sqrt{3abc + \sum a^2 b}}{abc} \stackrel{?}{\leq} \frac{\sum ab}{abc} \sqrt{\sum a - \frac{\sum ab}{\sum a}} \Leftrightarrow 3abc + \sum a^2 b \stackrel{?}{\leq} \\ &\leq \sum ab \left(\frac{\sum a^2 + \sum ab}{\sum a} \right) \Leftrightarrow \\ &\Leftrightarrow (\sum a^2 + \sum ab)(\sum ab) \stackrel{?}{\geq} (\sum a)(3abc + \sum a^2 b) \Leftrightarrow ab^2 + bc^2 + ca^3 \stackrel{?}{\geq} \text{(1)} \\ &\geq abc(\sum a). \text{ But } ab^3 + bc^3 + ca^3 = abc \left(\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \right) \stackrel{\text{Bergstrom}}{\geq} abc \frac{(\sum a)^2}{\sum a} = \\ &= abc(\sum a) \Rightarrow \text{(1) is true (proved)} \end{aligned}$$

JP.141. Let $a, b, c > 0$. Prove that:

$$\left(\sum \sqrt{\frac{b+c}{a}} \right)^2 \leq \frac{2(\sum ab)^3}{3a^2 b^2 c^2}$$

Proposed by Andrei Ştefan Mihalcea – Romania

Solution 1 by proposer

$$\begin{aligned} \text{Let be } g: (0, \infty) \rightarrow \mathbb{R}; g(x) &= \sqrt{ab + bc + ca - \frac{abc}{x}} \\ g(x) &= \left(ab + bc + ca - \frac{abx}{x} \right)^{\frac{1}{2}}; g'(x) = \frac{abc}{2x^2} \left(ab + bc + ca - \frac{abc}{x} \right)^{-\frac{1}{2}} \\ g''(x) &= \frac{abc}{2} \left[\left(-\frac{2}{x^3} \left(ab + bc + ca - \frac{abc}{x} \right) \right)^{-\frac{1}{2}} - \frac{abc}{2x^4} \left(ab + bc + ca - \frac{abc}{x} \right) \right] \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$g''(x) < 0$; g - concave. Denote $q = ab + bc + ca$; $r = abc$

$$\sum_{cyc} \left(\frac{1}{a} f(a) \right) = \sum_{cyc} \left(\frac{1}{a} \sqrt{ab + bc} \right) = \sum_{cyc} \sqrt{\frac{b+c}{a}} \leq$$

$$\stackrel{JENSEN}{\leq} \frac{q}{r} f\left(\frac{3r}{q}\right) = \frac{q}{r} \sqrt{q - \frac{qr}{3r}} = \frac{q}{r} \sqrt{\frac{2q}{3}} = \frac{ab + bc + ca}{abc} \sqrt{\frac{2(ab + bc + ca)}{3}}$$

$$\left(\sum_{cyc} \sqrt{\frac{b+c}{a}} \right)^2 \leq \frac{2(\sum_{cyc} ab)^3}{3a^2b^2c^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall a, b, c > 0, \left(\sum \sqrt{\frac{b+c}{a}} \right)^2 \leq \frac{2(\sum ab)^3}{3a^2b^2c^2}$$

$$LHS \stackrel{CBS}{\leq} \left(\sqrt{\sum(b+c)} \sqrt{\sum \frac{1}{a}} \right)^2 = \frac{2(\sum a)(\sum ab)}{abc} \stackrel{?}{\leq} \frac{2(\sum ab)^3}{3a^2b^2c^2} \Leftrightarrow (\sum ab)^2 \geq 3abc(\sum a)$$

$$\Leftrightarrow \sum a^2b^2 \geq abc(\sum a) \rightarrow \text{true} \because \sum x^2 \geq \sum xy \quad (\text{where } x = ab, y = bc, z = ca)$$

(Proved)

JP.142. Let $a, b, c \geq 1$. Prove that:

$$\sum \sqrt{\frac{a-1}{bc}} \leq \left(\sum \frac{1}{ab} \right) \sqrt{abc - \frac{\sum a}{3}}$$

Proposed by Andrei Ştefan Mihalcea – Romania

Solution by proposer

$$\text{Let be } f: (0, \infty) \rightarrow \mathbb{R}; f(x) = \sqrt{abc - \frac{abc}{x}}$$

$$f(x) = \left(abc - \frac{abc}{x} \right)^{\frac{1}{2}}; f'(x) = \frac{1}{2} \cdot \frac{abc}{x^2} \left(abc - \frac{abc}{x} \right)^{-\frac{1}{2}}$$

$$f''(x) = \frac{abc}{2} \left[-\frac{2}{x^3} \left(abc - \frac{abc}{x} \right)^{-\frac{1}{2}} - \frac{abc}{2x^4} \left(abc - \frac{abc}{x} \right)^{-\frac{3}{2}} \right]$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$f''(x) < 0$; f concave. Denote $p = a + b + c$; $r = abc$

$$\sum_{cyc} \left(\frac{f(ab)}{ab} \right) = \sum_{cyc} \sqrt{\frac{a-1}{bc}} \stackrel{JENSEN}{\leq} \frac{p}{r} f\left(\frac{3r}{p}\right) = \frac{p}{r} \sqrt{r - \frac{pr}{3r}} = \frac{p}{r} \sqrt{r - \frac{p}{3}}$$

$$abc \sum_{cyc} \sqrt{\frac{a-1}{bc}} \leq \left(\sum_{cyc} a \right) \cdot \sqrt{abc - \frac{1}{3} \sum_{cyc} a}$$

$$\sum_{cyc} \sqrt{\frac{a-1}{bc}} \leq \left(\sum_{cyc} \frac{1}{ab} \right) \sqrt{abc - \frac{a+b+c}{3}}$$

JP.143. In any ABC triangle the following relationship holds:

$$\frac{w_a^2}{h_b \cdot h_c} + \frac{w_b^2}{h_c \cdot h_a} + \frac{w_c^2}{h_a \cdot h_b} \leq \left(\frac{R}{r} \right)^2 - 1$$

all notations are usual sense.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$\because w_a^2 \leq s(s-a), \text{ etc} \therefore LHS \leq \sum \frac{s(s-a)bc}{4s^2} = \frac{s}{4s^2} (s \sum ab - 12Rrs)$$

$$= \frac{s^2}{4r^2 s^2} (s^2 - 8Rr + r^2)$$

$$\stackrel{?}{\leq} \frac{R^2 - r^2}{r^2} \Leftrightarrow s^2 \stackrel{?}{\leq} 4R^2 + 8Rr - 5r^2. \text{ Now, } s^2 \stackrel{Gerretsen}{\leq}$$

$$4R^2 + 4Rr + 3r^2 \stackrel{?}{\leq} 4R^2 + 8Rr - 5r^2 \Leftrightarrow 4Rr \stackrel{?}{\geq} 8r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (Euler)}$$

JP.144. In any ABC triangle the following relationship holds:

$$\frac{a}{w_a} + \frac{b}{w_b} + \frac{c}{w_c} \geq \frac{4s}{3R}$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{a^2}{aw_a} \stackrel{Bergstrom}{\geq} \frac{4s^2}{\sum aw_a}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

WLOG, we may assume $a \geq b \geq c \therefore w_a \leq w_b \leq w_c$

$$(1) \Rightarrow LHS \stackrel{\text{Chebyshev}}{\underset{(2)}{\geq}} \frac{4s^2}{\frac{1}{3}(2s) \sum w_a} = \frac{2s \cdot 3}{\sum w_a}$$

$$\text{Now, } \sum w_a \stackrel{w_a \leq \sqrt{s(s-a)}, \text{ etc}}{\underset{(3)}{\leq}} \sqrt{3s} \sqrt{3s-2s} = \sqrt{3}s$$

$$(2), (3) \Rightarrow LHS \geq \frac{2s \cdot 3}{\sqrt{3}s} = \frac{2 \cdot 3}{\sqrt{3}} = \frac{2 \cdot 3 \sqrt{3}}{3} = \frac{2 \cdot 3 \sqrt{3} R}{3R} \stackrel{\text{Mitrinovic}}{\geq} \frac{2 \cdot 2s}{3R} = \frac{4s}{3R} \text{ (Done)}$$

JP.145. In any ABC triangle the following relationship holds:

$$\frac{m_a}{r+r_a} + \frac{m_b}{r+r_b} + \frac{m_c}{r+r_c} \leq \frac{s}{2r}$$

all notations are usual sense.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} LHS &= \sum \frac{m_a}{\frac{\Delta}{s} + \frac{\Delta}{s-a}} = \sum \frac{m_a s(s-a)}{rs(b+c)} \stackrel{m_a < \frac{b+c}{2}, \text{ etc}}{<} \sum \left(\frac{b+c}{2} \right) \cdot \frac{(s-a)}{r(b+c)} = \\ &= \frac{\sum(s-a)}{2r} = \frac{3s-2s}{2r} = \frac{s}{2r} \therefore LHS < \frac{s}{2r} \Rightarrow LHS \leq \frac{s}{2r} \text{ (proved)} \end{aligned}$$

JP.146. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$P = \frac{1}{\sqrt[3]{2(x^5-x^3+4)}} + \frac{1}{\sqrt[3]{2(y^5-y^3+4)}} + \frac{1}{\sqrt[3]{2(z^5-z^3+4)}}.$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

$$* \text{ We have: } x^5 - x^3 - 2x + 2 = x^4(x-1) + x^3(x-1) - 2(x-1) = (x-1)(x^4 + x^3 - 2)$$

$$= (x-1) \left(x^3(x-1) + 2x^2(x-1) + 2x(x-1) + 2(x-1) \right)$$

$$= (x-1)^2(x^3 + 2x^2 + 2x + 2) \geq 0$$

$$\Rightarrow x^5 - x^3 - 2x + 2 \geq 0 \Leftrightarrow x^5 - x^3 + 4 \geq 2(x+1) \Leftrightarrow \frac{1}{\sqrt[3]{2(x^5-x^3+4)}} \leq \frac{1}{\sqrt[3]{2 \cdot 2(x+1)}}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

- Therefore, by AM-GM inequality for three positive real numbers:

$$\Rightarrow \frac{1}{\sqrt[3]{2(x^5 - x^3 + 4)}} \leq \frac{1}{\sqrt[3]{4(x+1)}} \leq \frac{1}{3} \left(\frac{1}{\sqrt{2(x+1)}} + \frac{1}{\sqrt{2(x+1)}} + \frac{1}{2} \right) = \frac{1}{3} \left(\sqrt{\frac{2}{x+1}} + \frac{1}{2} \right)$$

$$+ \text{ Similar: } \frac{1}{\sqrt[3]{2(y^5 - y^3 + 4)}} \leq \frac{1}{3} \left(\sqrt{\frac{2}{y+1}} + \frac{1}{2} \right); \frac{1}{\sqrt[3]{2(z^5 - z^3 + 4)}} \leq \frac{1}{3} \left(\sqrt{\frac{2}{z+1}} + \frac{1}{2} \right)$$

- Therefore:

$$\Rightarrow P = \frac{1}{\sqrt[3]{2(x^5 - x^3 + 4)}} + \frac{1}{\sqrt[3]{2(y^5 - y^3 + 4)}} + \frac{1}{\sqrt[3]{2(z^5 - z^3 + 4)}} \leq \frac{1}{3} \left(\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2}{z+1}} \right) + \frac{1}{2} \quad (1)$$

- Other, because $xyz = 1$ then exist positive real numbers a, b, c such that:

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$$

+ Therefore, by inequality Cauchy Schwarz:

$$\begin{aligned} \left(\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2}{z+1}} \right)^2 &= \left(\sqrt{\frac{2}{\frac{a}{b}+1}} + \sqrt{\frac{2}{\frac{b}{c}+1}} + \sqrt{\frac{2}{\frac{c}{a}+1}} \right)^2 = 2 \left(\sqrt{\frac{b}{a+b}} + \sqrt{\frac{c}{b+c}} + \sqrt{\frac{a}{c+a}} \right)^2 \\ &= 2 \left(\sqrt{\frac{b}{(a+b)(b+c)}} \cdot \sqrt{b+c} + \sqrt{\frac{c}{(b+c)(c+a)}} \cdot \sqrt{c+a} + \sqrt{\frac{a}{(c+a)(a+b)}} \cdot \sqrt{a+b} \right)^2 \leq \\ &\leq 2((b+c) + (c+a) + (a+b)) \cdot \left(\frac{b}{(a+b)(b+c)} + \frac{c}{(b+c)(c+a)} + \frac{a}{(c+a)(a+b)} \right) \\ &\leq 4(a+b+c) \cdot \frac{b(c+a)+c(a+b)+a(b+c)}{(a+b)(b+c)(c+a)} = \frac{8(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)} \quad (2) \end{aligned}$$

$$- \text{ We have: } 9(a+b)(b+c)(c+a) - 8(a+b+c)(ab+bc+ca) = a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0$$

$$\Rightarrow 9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca) \Leftrightarrow \frac{8(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq 9$$

$$\text{By (2): } \Rightarrow \left(\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2}{z+1}} \right)^2 \leq 9 \Rightarrow \sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2}{z+1}} \leq 3 \quad (3)$$

$$- \text{ Let (1), (3): } \Rightarrow P \leq \frac{1}{3} \cdot 3 + \frac{1}{2} = \frac{3}{2} \Rightarrow P_{\max} = \frac{3}{2}. \text{ Equality occurs if: } \begin{cases} xyz = 1 \\ x = y = z > 0 \end{cases} \Leftrightarrow$$

$$x = y = z = 1$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a > 0$, we have this fact:

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$a^5 - a^3 + 4 \geq a^2 + 3 \text{ because } a^5 - a^3 - a^2 + 1 = a^3(a^2 - 1) - (a^2 - 1) = \\ = (a^2 - 1)(a^3 - 1) = (a^2 - 1)(a - 1)(a^2 + a + 1) = (a - 1)^3(a + 1)(a^2 + a + 1) \geq 0$$

$$\text{Hence } \frac{1}{\sqrt[3]{2(x^5-x^3+4)}} + \frac{1}{\sqrt[3]{2(y^5-y^3+4)}} + \frac{1}{\sqrt[3]{2(z^5-z^3+4)}} \leq \frac{1}{\sqrt[3]{2(x^2+3)}} + \frac{1}{\sqrt[3]{2(y^3+3)}} + \frac{1}{\sqrt[3]{2(z^3+3)}} \leq$$

$$\leq \sqrt{\frac{3^2}{2} \left(\frac{1}{x^2+3} + \frac{1}{y^2+3} + \frac{1}{z^2+3} \right)} \leq \frac{3}{2}$$

$$\text{Iff } \frac{9}{2} \left(\frac{1}{x^2+3} + \frac{1}{y^2+3} + \frac{1}{z^2+3} \right) \leq \frac{27}{8}$$

$$\text{Iff } \frac{1}{x^2+3} + \frac{1}{y^2+3} + \frac{1}{z^2+3} \leq \frac{3}{4}$$

$$\text{Iff } \frac{b^2}{a^2+3b^2} + \frac{c^2}{b^2+3c^2} + \frac{a^2}{c^2+3a^2} \leq \frac{3}{4}; xyz = 1; x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a} \text{ and it is be true.}$$

$$\text{Because } 5(a^4b^2 + b^4c^2 + c^4a^2) + 3(a^4c^2 + c^4b^2 + b^4a^2) \geq 24(abc)^2 \Rightarrow$$

$$\Rightarrow 9(a^4b^2 + b^4c^2 + c^4a^2) + 27(a^4c^2 + c^4b^2 + b^4a^2) + 84(abc)^2 \geq$$

$$\geq 4(a^4b^2 + b^4c^2 + c^4a^2) + 24(a^4c^2 + c^4b^2 + b^4a^2) + 108(abc)^2 \Rightarrow$$

$$\Rightarrow \frac{a^2}{c^2+3a^2} + \frac{b^2}{a^2+3b^2} + \frac{c^2}{b^2+3c^2} \leq \frac{3}{4}$$

Therefore it is to be true.

JP.148. Let a, b, c be positive real numbers such that: $ab + bc + ca = 12$. Prove that:

$$\frac{a^3 + b^3}{2b^2 - bc + 2c^2} + \frac{b^3 + c^3}{2c^2 - ca + 2a^2} + \frac{c^3 + a^3}{2a^2 - ab + 2b^2} \geq 4$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu – Romania

Because $ab + bc + ca = 12 \Rightarrow \exists x, y, z > 0$ such that:

$$a = \frac{2\sqrt{3}x}{\sqrt{xy + yz + zx}}, b = \frac{2\sqrt{3}y}{\sqrt{xy + yz + zx}}, c = \frac{2\sqrt{3}z}{\sqrt{xy + yz + zx}}$$

Inequality becomes:

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum \left(\frac{\frac{24\sqrt{3}(x^3 + y^3)}{(\sqrt{xy + yz + zx})^3}}{\frac{24y^2 - 12yz + 24z^2}{xy + yz + zx}} \right) \geq 4 \Leftrightarrow 2\sqrt{3} \sum \frac{x^3 + y^3}{\sqrt{xy + yz + zx}(2y^2 - yz + z^2)} \geq 4 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{x^3 + y^3}{2y^3 - yz + 2z^2} \geq \frac{2}{\sqrt{3}} \sqrt{xy + yz + zx} \quad (1)$$

$$\text{But } (x + y + z)^2 \geq 3(xy + yz + zx) \Rightarrow \sqrt{xy + yz + zx} \leq \frac{x+y+z}{\sqrt{3}} \quad (2)$$

$$\text{From (1) + (2) we must show: } \sum \frac{x^3 + y^3}{2y^2 - yz + 2z^2} \geq \frac{2}{3}(x + y + z) \quad (3)$$

$$\text{But } 2y^2 - yz + 2z^2 \leq 3(y^2 - yz + z^2) \quad (4) \text{ (because } \Leftrightarrow y^2 - 2yz + z^2 \geq 0)$$

$$\sum \frac{x^3 + y^3}{y^2 - yz + z^2} \geq 2(x + y + z) \quad (5)$$

But this inequality its proposed and solved by Vasile Cîrtoaje in 2009, solved by S.O.S method. (Or its solved used $4(x^3 + y^3) \geq (x + y)^3$ and Hölder's inequality)

$$\text{Completion: We must (5): } \sum \frac{x^3 + y^3}{y^2 - yz + z^2} \geq 2(x + y + z)$$

$$\text{We show: (6) } \sum \frac{x^3}{y^2 - yz + z^2} \geq x + y + z \text{ and } \sum \frac{y^3}{y^2 - yz + z^2} \geq x + y + z \quad (7)$$

$$\text{From (6) + (7) } \Rightarrow (5)$$

$$\text{For (6): } \sum \frac{x^3}{y^2 - yz + z^2} = \sum \frac{x^4}{x(y^2 - yz + z^2)} \geq \frac{(x^2 + y^2 + z^2)^2}{\sum x(y^2 - yz + z^2)}$$

(from Cauchy's or Bergström's inequality) \Rightarrow

$$\text{We must show: } \frac{(x^2 + y^2 + z^2)^2}{\sum x(y^2 - yz + z^2)} \geq x + y + z \Leftrightarrow$$

$$\Leftrightarrow (x^2 + y^2 + z^2)^2 \geq (x + y + z) \cdot \sum x(y^2 - yz + z^2) \Leftrightarrow$$

$$\Leftrightarrow (x^2 + y^2 + z^2)^2 - (x + y + z) \sum x(y^2 - yz + z^2) \geq 0 \quad (8)$$

Now we use Cîrtoaje's theorem: If $f_n(x, y, z)$ is a symmetric and homogeneous polynom of degree 4 then $f_4(x, y, z) \geq 0 \forall x, y, z \in \mathbb{R} \Leftrightarrow f_4(x_1, 1, 1) \geq 0 \forall x \in \mathbb{R}$ in

$$\text{our case: } f_4(x, y, z) = (x^2 + y^2 + z^2)^2 - (x + y + z) \sum x(y^2 - yz + z^2)$$

$$f_4(x_1, 1, 1) = x^4 - 2x^3 + x^2 = x^2(x - 1)^2 \geq 0 \text{ true } \Rightarrow (6) \text{ its true.}$$

$$\text{For (7): } \sum \frac{y^3}{y^2 - yz + z^2} \geq x + y + z \Leftrightarrow \frac{y^3}{y^2 - yz + z^2} + \frac{z^3}{z^2 - zx + x^2} + \frac{x^3}{x^2 - xy + y^2} \geq x + y + z \Leftrightarrow$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \sum x^3 (y^2 - yz + z^2)(z^2 - xz + x^2) &\geq (x + y + z) \prod (x^2 - xy + y^2) \Leftrightarrow \\ &\Leftrightarrow \sum x^3 (y^2 - yz + z^2)(z^2 - xz + x^2) - \\ &-(x + y + z)(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - xz + x^2) \geq 0 \quad (9) \end{aligned}$$

Now again use Cîrtoaje's theorem: If $f_5(x, y, z)$ it's a symmetric polynomial function of degree 5 then: $f_5(x, y, z) \geq 0 \forall x, y, z \geq 0 \Leftrightarrow f_5(0, 4, 4) \geq 0$. In our case:

$$\begin{aligned} f_5(x, y, z) &= \sum x^3 (y^2 - yz + z^2)(z^2 - xz + x^2) - \\ &-(x + y + z)(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - xz + x^2) \\ f_5(0, 4, 4) &= 2y^7 - 2y^7 \geq 0 \text{ true} \Rightarrow (9) \text{ its true} \Rightarrow (7) \text{ its true.} \end{aligned}$$

Vasile Cartoaje proof:

Let a, b, c be non-negative real numbers. Prove that:

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq a + b + c$$

Solution. Applying Cauchy - Schwarz inequality, we have:

$$\sum_{cyc} \frac{a^3}{b^2 - bc + c^2} = \sum_{cyc} \frac{a^4}{a(b^2 - bc + c^2)} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{cyc} a(b^2 - bc + c^2)}$$

It remains to prove that: $(\sum_{cyc} a^2)^2 \geq (\sum_{cyc} a(b^2 - bc + c^2))(\sum_{cyc} a)$ or

$$\sum_{cyc} a^4 + 2 \sum_{cyc} a^2 b^2 \geq (a + b + c) \sum_{cyc} a^2 (b + c) - 3abc \sum_{cyc} a \text{ or}$$

$$\sum_{cyc} a^4 + abc \sum_{cyc} a \geq \sum_{cyc} a^3 (b + c)$$

This is exactly the fourth degree-Schur's inequality, so we are done.

Equality holds for $a = b = c$ or $a = b, c = 0$ up to permutation.

Solution 2 by Michael Sterghiou-Greece

$$\sum_{cyc} \frac{a^3 + b^3}{2b^2 - bc + 2c^2} \geq 4 \quad (1)$$

$$(1) \rightarrow \sum_{cyc} \frac{a^3}{2b^2 - bc + 2c^2} + \sum_{cyc} \frac{b^3}{2b^2 - bc + 2c^2} \geq 4. \text{ But}$$

$$\sum_{cyc} \frac{a^3}{2b^2 - bc + 2c^2} \stackrel{BCS}{\geq} \frac{(a^2 + b^2 + c^2)^2}{\sum_{cyc} 2ab^2 - abc + 2ac^2} \quad (2)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Let $(\sum_{cyc} a, \sum_{cyc} ab, abc) = (p, q, r)$ with $q = 12$

$$(2) \equiv \frac{(p^2-2q)^2}{24p-9r} \geq \frac{(p^2-2q)^2}{24p-9 \cdot \frac{48p-p^3}{9}} = \frac{p^2-24}{p} \quad (3) \text{ because from the 3rd degree Schur}$$

$$q = 12 \leq \frac{4p^3 + 9r}{4p} \rightarrow r \geq \frac{48p - p^3}{9}$$

$$\text{Now } \sum_{cyc} \frac{b^3}{2b^2-bc+2c^2} \stackrel{\text{Holder}}{\geq} \frac{p^3}{3 \cdot \sum_{cyc} (2b^2-bc+2c^2)} = \frac{p^3}{12(p^2-27)} \quad (4)$$

$$\text{It suffices to prove that } \frac{p^2-24}{p} + \frac{p^3}{12(p^2-27)} - 4 \geq 0 \quad (5)$$

$$(5) \rightarrow 13p^4 - 48p^3 - 612p^2 + 1296p + 7776 \geq 0 \text{ or}$$

$(p-6)^2(13p^2 + 108p + 216) \geq 0$ which holds as $p^2 \geq 3q \rightarrow p \geq 6$ and the trinomial is clearly ≥ 0 for $p \geq 6$. Done!

JP.149. Find all functions: $f: (0, +\infty) \rightarrow \mathbb{R}$ which verify the relationship:

$$\ln(xy) \leq xf(x) + yf(y) \leq xyf(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$\ln(xy) \leq xf(x) + yf(y) \leq xyf(xy), \forall x, y > 0$$

$$\text{Put } x = y = 1, \text{ we get: } 0 \leq f(1) + f(1) \leq f(1) \Rightarrow f(1) = 0.$$

$$\text{Put } y = \frac{1}{x} \text{ to obtain}$$

$$0 \leq xf(x) + \frac{1}{x}f\left(\frac{1}{x}\right) \leq 0 \Rightarrow f\left(\frac{1}{x}\right) = -x^2f(x). \text{ Taking } y = 1, \text{ we get:}$$

$$\ln(x) \leq xf(x), \forall x > 0 \quad (1)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) \leq \frac{1}{x}f\left(\frac{1}{x}\right) = -xf(x), \forall x > 0 \Rightarrow -\ln(x) \leq -xf(x), \forall x > 0$$

$$\Rightarrow xf(x) \leq \ln x, \forall x > 0 \quad (2)$$

$$\text{From (1), (2): } xf(x) = \ln(x), \forall x > 0 \Rightarrow f(x) = \frac{1}{x} \ln(x), \forall x > 0$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

JP.150. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that: $|z_1| = |z_2| = |z_3|$. If $(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 = 0$, then z_1, z_2, z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Let } z = z_1 + z_2 + z_3$$

$$\begin{aligned} (z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 = 0 &\Rightarrow (z - z_3)(z - z_1)(z - z_2) + z_1z_2z_3 = 0 \Rightarrow \\ \Rightarrow z^3 - (z_1 + z_2 + z_3)z^2 + (z_2z_3 + z_3z_1 + z_1z_2)z - z_1z_2z_3 + z_1z_2z_3 = 0 &\Rightarrow \\ \Rightarrow z^3 - z(z^2) + z(z_2z_3 + z_3z_1 + z_1z_2) = 0 &\Rightarrow z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)z_1z_2z_3 = 0. \end{aligned}$$

As $|z_1| = |z_2| = |z_3| = k > 0$, $z_1z_2z_3 \neq 0$. Thus

$$z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = 0 \quad (1)$$

$$\text{Also, } k^2 = z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 \quad (2)$$

From (1), (2): $k^2z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 0 \Rightarrow k^2z\bar{z} = 0$. As $k^2 \neq 0$, $|z|^2 = 0 \Rightarrow z = 0 \Rightarrow$
 $\Rightarrow z_1 + z_2 + z_3 = 0$. Now, $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2 =$
 $= |z_2 - z_3|^2 + |z_2 + z_3|^2 = 2|z_2|^2 + 2|z_3|^2 = 4k^2 \Rightarrow |z_2 - z_3|^2 + k^2 = 4k^2 \Rightarrow$
 $\Rightarrow |z_2 - z_3| = \sqrt{3}k$. Similarly, $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}k$. Thus,
 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \Rightarrow$ triangle with vertices z_1, z_2, z_3 is an equilateral triangle.

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaijan

Lemma: Let $|z_1| = |z_2| = |z_3|$. Points z_1, z_2, z_3 are vertices of an equilateral triangle

**if and only if $z_1 + z_2 + z_3 = 0$. Since $(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 =$
 $= (z_1 + z_2 + z_3)(z_1z_2 + z_1z_3 + z_2z_3)$,**

considering condition, we have: $z_1 + z_2 + z_3 = 0$ or

$z_1z_2 + z_1z_3 + z_2z_3 = 0$. Using Lemma \Rightarrow Q.E.D.

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

SP.136. Let x, y, z be positive real numbers such that: $x^4 + y^4 + z^4 = xy + yz + zx$.

Find the maximum of the expression:

$$P = \sqrt[3]{\frac{x^6 + y^6}{2}} + \sqrt[3]{\frac{y^6 + z^6}{2}} + \sqrt[3]{\frac{z^6 + x^6}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

* Let $x, y, z > 0$, we will prove that inequality:

$$x^4 + y^4 + z^4 + xyz(x + y + z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \quad (1)$$

$$(1) \Leftrightarrow x^4 + y^4 + z^4 + xyz(x + y + z) - xy(x^2 + y^2) - yz(y^2 + z^2) - zx(z^2 + x^2) \geq 0$$

$$\Leftrightarrow x^2(x^2 - xy - xz + yz) + y^2(y^2 - yz - yx + zx) + z^2(z^2 - zx - zy + xy) \geq 0$$

$$\Leftrightarrow x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \geq 0 \quad (2)$$

- Supposed $x \geq y \geq z > 0$

$$+ \text{ We have: } \begin{cases} z \leq x \\ z \leq y \end{cases} \Leftrightarrow \begin{cases} z - x \leq 0 \\ z - y \leq 0 \end{cases} \Rightarrow (z - x)(z - y) \geq 0 \Rightarrow z^2(z - x)(z - y) \geq 0 \quad (3)$$

$$+ \text{ Other: } x^2(x - y)(x - z) + y^2(y - z)(y - x)$$

$$= (x - y)[x^2(x - z) - y^2(y - z)] = (x - y)[(x^3 - y^3) - z(x^2 - y^2)]$$

$$= (x - y)[(x - y)(x^2 + xy + y^2) - z(x - y)(x + y)] = (x - y)^2(x^2 + xy + y^2 - zx - zy) \geq 0 \quad (4)$$

$$\text{(because } x \geq y \geq z > 0, x^2 + xy + y^2 - zx - zy = x(x - z) + y(x - z) + y^2 \geq y^2 > 0$$

$$\text{and } (x - y)^2 \geq 0)$$

$$- \text{ Let } (3), (4): \Rightarrow x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \geq 0$$

$$\Rightarrow \text{Inequality (2) true} \Rightarrow (1) \text{ true.}$$

$$* \text{ We have: } x^6 + y^4 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = (x^2 + y^2)(x^2 - xy\sqrt{3} + y^2)(x^2 + xy\sqrt{3} + y^2)$$

$$- \text{ Therefore, by AM-GM inequality: } \sqrt[3]{\frac{x^6 + y^6}{2}} = \sqrt[3]{\frac{(x^2 + y^2)(x^2 - xy\sqrt{3} + y^2)(x^2 + xy\sqrt{3} + y^2)}{2}}$$

$$= \sqrt[3]{\frac{(x^2 + y^2)}{2} \cdot (2 + \sqrt{3})(x^2 - xy\sqrt{3} + y^2)(2 - \sqrt{3})(x^2 + xy\sqrt{3} + y^2)}$$

$$\leq \frac{\frac{x^2 + y^2}{2} + (2 + \sqrt{3})(x^2 - xy\sqrt{3} + y^2) + (2 - \sqrt{3})(x^2 + xy\sqrt{3} + y^2)}{3} = \frac{3x^2 - 3xy + 3y^2}{2}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Rightarrow \sqrt[3]{\frac{x^6+y^6}{2}} \leq \frac{3x^2-4xy+3y^2}{2}. \text{ Similar: } \sqrt[3]{\frac{y^6+z^6}{2}} \leq \frac{3y^2-4yz+3z^2}{2}; \sqrt[3]{\frac{z^6+x^6}{2}} \leq \frac{3z^2-4zx+3x^2}{2}$$

$$\Rightarrow P = \sqrt[3]{\frac{x^6+y^6}{2}} + \sqrt[3]{\frac{y^6+z^6}{2}} + \sqrt[3]{\frac{z^6+x^6}{2}} \leq \frac{3x^2-4xy+3y^2}{2} + \frac{3y^2-4yz+3z^2}{2} + \frac{3z^2-4zx+3x^2}{2}$$

$$\Leftrightarrow P \leq 3(x^2+y^2+z^2) - 2(xy+yz+zx) \quad (5)$$

$$\text{*We will prove: } 3(x^2+y^2+z^2) - 2(xy+yz+zx) \leq 3 \quad (6)$$

$$\Leftrightarrow 3(x^2+y^2+z^2) - 2(xy+yz+zx) \leq \frac{3(x^4+y^4+z^4)}{xy+yz+zx} \quad (x^4+y^4+z^4 = xy+yz+zx \text{ then } \frac{x^4+y^4+z^4}{xy+yz+zx} = 1)$$

$$\Leftrightarrow (3(x^2+y^2+z^2) - 2(xy+yz+zx))(xy+yz+zx) \leq 3(x^4+y^4+z^4)$$

$$\Leftrightarrow 3(x^2+y^2+z^2)(xy+yz+zx) \leq 3(x^4+y^4+z^4) + 2(xy+yz+zx)^2$$

$$\Leftrightarrow 3xy(x^2+y^2) + 3yz(y^2+z^2) + 3zx(z^2+x^2) + 3xyz(x+y+z) \leq$$

$$\leq 3(x^4+y^4+z^4) + 2(x^2y^2+y^2z^2+z^2x^2) + 4xyz(x+y+z)$$

$$\Leftrightarrow 3(x^4+y^4+z^4) + xyz(x+y+z) + 2(x^2y^2+y^2z^2+z^2x^2) \geq 3xy(x^2+y^2) + 3yz(y^2+z^2) + 3zx(z^2+x^2) \quad (7)$$

- By AM-GM inequality for 2 real numbers:

$$x^2y^2 + y^2z^2 + z^2x^2 = \frac{x^2(y^2+z^2)}{2} + \frac{y^2(z^2+x^2)}{2} + \frac{z^2(x^2+y^2)}{2} \geq \frac{x^2 \cdot 2yz}{2} + \frac{y^2 \cdot 2zx}{2} + \frac{z^2 \cdot 2xy}{2}$$

$$\Rightarrow x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x+y+z) \quad (8)$$

$$\Rightarrow 3(x^4+y^4+z^4) + xyz(x+y+z) + 2(x^2y^2+y^2z^2+z^2x^2) \geq 3(x^4+y^4+z^4 + xyz(x+y+z)) \quad (9)$$

- Let (1), (9):

$$\Rightarrow 3(x^4+y^4+z^4) + xyz(x+y+z) + 2(x^2y^2+y^2z^2+z^2x^2) \geq 3xy(x^2+y^2) + 3yz(y^2+z^2) + 3zx(z^2+x^2)$$

$$\Rightarrow (7) \text{ True} \Rightarrow \text{Inequality (6) true.}$$

$$\text{- Let (5), (6): } \Rightarrow P \leq 3 \Rightarrow P_{\max} = 3$$

$$\text{+ Equality occurs if: } \Leftrightarrow \begin{cases} x, y, z > 0 \\ x^4 + y^4 + z^4 = xy + yz + zx \\ x = y = z \\ x^2 = y^2 + z^2 \end{cases} \Leftrightarrow x = y = z = 1$$

SP.137. Let $a, b, c > 0$ such that $a + b + c = 3$. Prove that:

$$\frac{a}{\sqrt[3]{4(b^6+c^6)+7bc}} + \frac{b}{\sqrt[3]{4(c^6+a^6)+7ca}} + \frac{c}{\sqrt[3]{4(a^6+b^6)+7ab}} + \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{7}{12} \quad (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by proposer

* *We have:*

$$\begin{aligned} b^6 + c^6 &= (b^2 + c^2)(b^4 - b^2c^2 + c^4) = (b^2 + c^2) \left[(b^2 + c^2)^2 - (bc\sqrt{3})^2 \right] \\ &= (b^2 + c^2)(b^2 - bc\sqrt{3} + c^2)(b^2 + bc\sqrt{3} + c^2) \end{aligned}$$

- *By inequality AM-GM for three positive real numbers:*

$$\begin{aligned} \sqrt[3]{4(b^6 + c^6)} &= \sqrt[3]{(b^2 + c^2) \cdot 2(2 + \sqrt{3})(b^2 - bc\sqrt{3} + c^2) \cdot 2(2 - \sqrt{3})(b^2 + bc\sqrt{3} + c^2)} \leq \\ &\leq \frac{(b^2 + c^2) + 2(2 + \sqrt{3})(b^2 - bc\sqrt{3} + c^2) + 2(2 - \sqrt{3})(b^2 + bc\sqrt{3} + c^2)}{3} = \frac{9b^2 - 12bc + 9c^2}{3} \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \sqrt[3]{4(b^6 + c^6)} &\leq 3b^2 - 4bc + 3c^2 \Leftrightarrow \sqrt[3]{4(b^6 + c^6)} + 7bc \leq 3b^2 + 3bc + 3c^2 \\ \Leftrightarrow \frac{1}{\sqrt[3]{4(b^6 + c^6) + 7bc}} &\geq \frac{1}{3(b^2 + bc + c^2)} \Leftrightarrow \frac{a}{\sqrt[3]{4(b^6 + c^6) + 7bc}} \geq \frac{a}{3(b^2 + bc + c^2)} \quad (2) \end{aligned}$$

$$+ \text{ *Similar:* } \frac{b}{\sqrt[3]{4(c^6 + a^6) + 7ca}} \geq \frac{b}{3(c^2 + ca + a^2)} \quad (3)$$

$$\frac{c}{\sqrt[3]{4(a^6 + b^6) + 7ab}} \geq \frac{c}{3(a^2 + ab + b^2)} \quad (4)$$

$$\begin{aligned} - \text{ *Then (2), (3), (4):* } \Rightarrow &\frac{a}{\sqrt[3]{4(b^6 + c^6) + 7bc}} + \frac{b}{\sqrt[3]{4(c^6 + a^6) + 7ca}} + \frac{c}{\sqrt[3]{4(a^6 + b^6) + 7ab}} \geq \\ &\geq \frac{a}{3(b^2 + bc + c^2)} + \frac{b}{3(c^2 + ca + a^2)} + \frac{c}{3(a^2 + ab + b^2)} \quad (5) \end{aligned}$$

- *Other, by Cauchy-Schwarz we have:*

$$\begin{aligned} \frac{a}{b^2 + bc + c^2} + \frac{b}{c^2 + ca + a^2} + \frac{c}{a^2 + ab + b^2} &= \frac{a^2}{ab^2 + abc + ac^2} + \frac{b^2}{bc^2 + bca + ba^2} + \frac{c^2}{ca^2 + cab + cb^2} \geq \\ &\geq \frac{(a+b+c)^2}{(ab^2 + abc + ac^2) + (bc^2 + bca + ba^2) + (ca^2 + cab + cb^2)} \quad (6) \end{aligned}$$

$$\begin{aligned} - \text{ *That* } &\frac{(a+b+c)^2}{(ab^2 + abc + ac^2) + (bc^2 + bca + ba^2) + (ca^2 + cab + cb^2)} \\ &= \frac{(a+b+c)^2}{ab(a+b) + bc(b+c) + ca(c+a) + 3abc} = \frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)} = \frac{a+b+c}{ab+bc+ca} \quad (7) \end{aligned}$$

$$- \text{ *Then (6), (7):* } \Rightarrow \frac{a}{b^2 + bc + c^2} + \frac{b}{c^2 + ca + a^2} + \frac{c}{a^2 + ab + b^2} \geq \frac{a+b+c}{ab+bc+ca} \quad (8)$$

+ *And* $a + b + c = 3$. *Then (8):*

$$\Rightarrow \frac{a}{b^2 + bc + c^2} + \frac{b}{c^2 + ca + a^2} + \frac{c}{a^2 + ab + b^2} \geq \frac{3}{ab+bc+ca} \quad (9)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

- Then (5), (9) $\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)+7bc}} + \frac{b}{\sqrt[3]{4(c^6+a^6)+7ca}} + \frac{c}{\sqrt[3]{4(a^6+b^6)+7ab}} \geq \frac{1}{ab+bc+ca}$ (10)

- By AM-GM for five positive real numbers:

$$\sqrt[3]{a} + \sqrt[3]{a} + \sqrt[3]{a} + a^2 + a^2 \geq 5 \sqrt[5]{\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} \cdot a^2 \cdot a^2} = 5 \sqrt[5]{a^5} = 5a$$

$$\Leftrightarrow 3 \cdot \sqrt[3]{a} + 2a^2 \geq 5a \Leftrightarrow 3\sqrt[3]{a} \geq 5a - 2a^2 \quad (11)$$

$$+ \text{ Similar: } \sqrt[3]{b} \geq 5b - 2b^2; \sqrt[3]{c} \geq 5c - 2c^2 \quad (12)$$

- Then (11), (12) $\Rightarrow 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}) \geq 5(a + b + c) - 2(a^2 + b^2 + c^2)$

$$\Leftrightarrow 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}) \geq 15 - 2(a^2 + b^2 + c^2) \quad (a + b + c = 3)$$

$$\Leftrightarrow 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + 1) \geq 18 - 2(a^2 + b^2 + c^2) = 2(a + b + c)^2 - 2(a^2 + b^2 + c^2)$$

(Because $a + b + c = 3 \Rightarrow 2(a + b + c)^2 = 18$)

$$\Leftrightarrow 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + 1) \geq 2(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) - 2(a^2 + b^2 + c^2)$$

$$\Leftrightarrow 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + 1) \geq 4(ab + bc + ca) \Leftrightarrow 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}) \geq 4(ab + bc + ca) - 3$$

$$\Leftrightarrow \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{4(ab+bc+ca)-3}{36} \Leftrightarrow \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{ab+bc+ca}{9} - \frac{1}{2} \quad (13)$$

- Then (10), (13):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)+7bc}} + \frac{b}{\sqrt[3]{4(c^6+a^6)+7ca}} + \frac{c}{\sqrt[3]{4(a^6+b^6)+7ab}} + \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{1}{ab+bc+ca} + \frac{ab+bc+ca}{9} - \frac{1}{12} \quad (14)$$

- By AM-GM we have:

$$\frac{1}{ab+bc+ca} + \frac{ab+bc+ca}{9} \geq 2 \cdot \sqrt{\frac{1}{ab+bc+ca} \cdot \frac{ab+bc+ca}{9}} = 2 \sqrt{\frac{1}{9}} = \frac{2}{3}$$

$$\Rightarrow \frac{1}{ab+bc+ca} + \frac{ab+bc+ca}{9} - \frac{1}{12} \geq \frac{2}{3} - \frac{1}{12} = \frac{7}{12} \Leftrightarrow \frac{1}{ab+bc+ca} + \frac{ab+bc+ca}{9} - \frac{1}{12} \geq \frac{7}{12} \quad (15)$$

- Then (14), (15):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)+7bc}} + \frac{b}{\sqrt[3]{4(c^6+a^6)+7ca}} + \frac{c}{\sqrt[3]{4(a^6+b^6)+7ab}} + \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{7}{12}$$

\Rightarrow Inequality (1) True and we get the result

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$+ \text{Equality occurs if: } \begin{cases} a, b, c > 0; a + b + c = 3 \\ a = b = c \\ \frac{1}{b^2+bc+c^2} = \frac{1}{c^2+ca+a^2} = \frac{1}{a^2+ab+b^2} \Leftrightarrow a = b = c = 1. \\ \sqrt[3]{a} = a^2; \sqrt[3]{b} = b^2; \sqrt[3]{c} = c^2 \\ \frac{1}{ab+bc+ca} = \frac{ab+bc+ca}{9} \end{cases}$$

SP.138. Let a, b, c be positive real numbers such that: $a + b + c = 3$.

Prove that:

$$\frac{a^2}{\sqrt{5(b^4+4)}} + \frac{b^2}{\sqrt{5(c^4+4)}} + \frac{c^2}{\sqrt{5(a^4+4)}} \geq \frac{3}{5}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Heikichi Ezakiya-Jakarta-Indonesia

$$\text{Let: } \varphi = \frac{a^2}{\sqrt{5(b^4+4)}} + \frac{b^2}{\sqrt{5(a^4+4)}} + \frac{c^2}{\sqrt{5(a^4+4)}} = \frac{1}{\sqrt{5}} \left(\frac{a^2}{\sqrt{b^4+4}} + \frac{b^2}{\sqrt{c^4+4}} + \frac{c^2}{\sqrt{a^4+4}} \right)$$

$$\text{Using CBS: } \varphi \geq \frac{1}{\sqrt{5}} \cdot \frac{(a+b+c)^2}{(\sqrt{a^4+4} + \sqrt{b^4+4} + \sqrt{c^4+4})} = \varphi^{(1)}$$

Using QM-AM for $(\sqrt{a^4+4} + \sqrt{b^4+4} + \sqrt{c^4+4})$:

$$\sqrt{\frac{a^4+b^4+c^4+12}{3}} \geq \frac{\sqrt{a^4+4} + \sqrt{b^4+4} + \sqrt{c^4+4}}{3} \Leftrightarrow \frac{1}{(\sqrt{a^4+4} + \sqrt{b^4+4} + \sqrt{c^4+4})} \geq \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{a^4+b^4+c^4+12}} \text{ so,}$$

$$\varphi^{(1)} \geq \frac{1}{\sqrt{15}} \cdot \frac{1}{\sqrt{a^4+b^4+c^4+12}} = \varphi^{(2)} \quad (\#)$$

Using QM-AM for $a^2 + b^2 + c^2$

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} \Leftrightarrow a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3}$$

because $a + b + c$, then: $a^2 + b^2 + c^2 \geq 3$ (1)

Using QM-AM for $a^4 + b^4 + c^4$

$$\sqrt{\frac{a^4+b^4+c^4}{3}} \geq \frac{a^2+b^2+c^2}{3} \Leftrightarrow a^4 + b^4 + c^4 \geq \frac{(a^2+b^2+c^2)^2}{3} \quad (2)$$

$$\text{From (1) \& (2): } a^4 + b^4 + c^4 \geq \frac{(3)^2}{3} = 3$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

From (#), if we choose $a^4 + b^4 + c^4 = 3$, # becomes equal, then

$$\varphi^{(1)} \geq \varphi^{(2)} = \frac{1}{\sqrt{15}} \cdot \frac{(a+b+c)^2}{\sqrt{3+12}} = \frac{(a+b+c)^2}{15}$$

$$\text{Because } a+b+c=3, \text{ then: } \varphi \geq \varphi^{(1)} \geq \varphi^{(2)} = \frac{(3)^2}{15} = \frac{3}{5}$$

SP.139. In ABC triangle the lengths of sides BC, CA, AB are a, b, c . Let h_a, h_b, h_c be the distances from A, B, C to BC, CA, AB ; l_a, l_b, l_c are the lengths of the bisectors A, B, C .

Prove that:

$$\frac{l_a l_b}{l_c} + \frac{l_b l_c}{l_a} + \frac{l_c l_a}{l_b} \geq \frac{h_a h_b}{h_c} + \frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

In any ΔABC , $\sum \frac{w_b w_c}{w_a} \geq \sum \frac{h_b h_c}{h_a}$. Firstly,

$$\prod (a+b) = 2abc + \sum ab(2s-c) = 2s(s^2 + 4Rr + r^2) - 4Rrs \stackrel{(1)}{=} 2s(s^2 + 2Rr + r^2)$$

$$\text{Also, } \sum (s-b)(s-c) = \sum (s^2 - s(b+c) + bc) = 3s^2 - 4s^2 + \sum ab \stackrel{(2)}{=} 4Rr + r^2$$

$$\text{Also, } \prod w_a = \prod \left(\frac{2bc}{b+c} \cos \frac{A}{2} \right) = \frac{8(16R^2 r^2 s^2)}{\prod (b+c)} \left(\frac{s}{4R} \right)^{\text{by (1)}} = \frac{128R^2 r^2 s^2}{2s(s^2 + 2Rr + r^2)} \left(\frac{s}{4R} \right)^{\text{(3)}} = \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2}$$

$$\text{Now, } \sum \frac{h_b h_c}{h_a} = \sum \frac{(h_b h_c)^2}{h_a h_b h_c} = \left(\frac{8R^3}{a^2 b^2 c^2} \right) \sum \left(\frac{ca}{2R} \cdot \frac{ab}{2R} \right)^2 = \left(\frac{8R^3}{a^2 b^2 c^2} \right) \left(\frac{a^2 b^2 c^2}{16R^4} \right) \sum a^2 \stackrel{(4)}{=} \frac{\sum a^2}{2R}$$

$$\text{Now, } \sum \frac{w_b w_c}{w_a} = \left(\frac{1}{\prod w_a} \right) \sum w_b^2 w_a^2 \stackrel{\text{by (3)}}{=} \left(\frac{s^2 + 2Rr + r^2}{16Rr^2 s^2} \right) \sum \left[\frac{4c^2 a^2}{(c+a)^2} \cdot \frac{s(s-b)}{ca} \cdot \frac{4a^2 b^2}{(a+b)^2} \cdot \frac{s(s-c)}{ab} \right]$$

$$= \left(\frac{s^2 + 2Rr + r^2}{16Rr^2 s^2} \right) \cdot \frac{16 \cdot 4Rrs}{(\prod (a+b))^2} \left[\sum a(s-b)(s-c)(b+c)^2 \right]$$

$$= \left(\frac{s^2 + 2Rr + r^2}{16Rr^2 s^2} \right) \cdot \frac{64Rrs \cdot r^2 s}{4s^2 (s^2 + 2Rr + r^2)^2} \left[\sum \frac{a(b+c)^2}{s-a} \right] \stackrel{(4)}{=} \left(\frac{r}{s^2 + 2Rr + r^2} \right) \left[\sum \frac{a(b+c)^2}{s-a} \right]$$

$$\text{Now, } \sum \frac{a(b+c)^2}{s-a} = \sum \frac{a(s+s-a)^2}{s-a} = \sum \frac{as^2 + a(s-a)^2 + 2as(s-a)}{s-a} = s^2 \sum \frac{a-s+s}{s-a} + \sum a(s-a) + 2s(2s)$$

$$= s^2 \left(-3 + 4 + \frac{s}{r^2 s} \sum (s-b)(s-c) \right) + s(2s) - 2(s^2 - 4Rr - r^2)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\stackrel{\text{by (2)}}{=} s^2 \left(1 + \frac{4R+r}{r} \right) + 2(4Rr+r^2) \stackrel{(5)}{=} \frac{s^2(4R+2r) + 2r^2(4R+r)}{r}$$

$$(4),(5) \Rightarrow \sum \frac{w_b w_c}{w_a} \stackrel{(b)}{=} \frac{s^2(4R+2r) + 2r^2(4R+r)}{s^2 + 2Rr + r^2}$$

$$(a), (b) \Rightarrow \text{given inequality} \Leftrightarrow \frac{s^2(4R+2r) + 2r^2(4R+r)}{s^2 + 2Rr + r^2} \geq \frac{\sum a^2}{2R} = \frac{s^2 - 4Rr - r^2}{R}$$

$$\Leftrightarrow s^2(4R^2 + 2Rr) + 2Rr^2(4R+r) \geq (s^2 - 4Rr - r^2)(s^2 + 2Rr + r^2)$$

$$\Leftrightarrow s^2(4R^2 + 2Rr) + 2Rr^2(4R+r) \geq s^4 - 2Rrs^2 - r^2(4R+r)(2R+r)$$

$$\Leftrightarrow s^2(4R^2 + 4Rr) + r^2(4R+r)^2 \stackrel{(c)}{\geq} s^4$$

$$\text{Now, RHS of (4)} \stackrel{\text{Gerretsen}}{\leq} s^2(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} s^2(4R^2 + 4Rr) + r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R+r)^2 \geq 3s^2 \rightarrow \text{true (Trucht) (Proved)}$$

SP.140. Let a, b, c be positive real numbers. Prove that:

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Rade Krenkov-Sturmica-Macedonia

From Cauchy – Schwarz inequality we have:

$$\left(\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} \right) (b^3a + c^3a + c^3b + a^3b + a^3c + b^3c) \geq 4(a^2 + b^2 + c^2)^2 \quad (1)$$

$$\left(\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} \right) (abc^2 + acb^2 + bca^2 + bac^2 + cab^2 + cba^2) \geq 4(ab + bc + ca)^2$$

$$\left(\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} \right) (abc^2 + bca^2 + cab^2) \geq 2(ab + bc + ca)^2 \quad (2)$$

From (1) and (2) we get:

$$\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right) (ab + bc + ca)(a^2 + b^2 + c^2) \geq 4(a^2 + b^2 + c^2)^2 + 2(ab + bc + ca)^2$$

Now, we have that:

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2 + b^2 + c^2)}{ab + bc + ca} + \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Soumava Chakraborty-Kolkata-India

$\sum \frac{b+c}{a} \geq \frac{4\sum a^2}{\sum ab} + \frac{2\sum ab}{\sum a^2}$. Given inequality transforms into:

$$\begin{aligned} & (\sum ab)(\sum a^2)\{\sum ab(a+b)\} - abc\left\{4(\sum a^2)^2 + 2(\sum ab)^2\right\} \geq 0 \Leftrightarrow \\ & \Leftrightarrow \sum a^5b^2 + \sum a^2b^5 + \sum a^4b^3 + \sum a^3b^4 + 2abc(\sum a^3b + \sum ab^3) \geq \\ & \stackrel{(a)}{\geq} 6abc(\sum a^2b^2) + 2abc(\sum a^4), \text{ we have } 2abc(\sum a^3b + \sum ab^3) \stackrel{A-G}{\geq} \underset{(1)}{2abc(\sum ab(2ab))} = \\ & = 4abc(\sum a^2b^2). \text{ Also, } \sum a^4b^3 + \sum a^3b^4 = \sum c^4(a^3 + b^3) \geq \sum c^4ab(a+b) = \\ & = abc(\sum a^3b + \sum ab^3) = abc(\sum ab(a^2 + b^2)) \stackrel{A-G}{\geq} \underset{(2)}{2abc(\sum a^2b^2)}. \text{ Moreover,} \\ & \sum a^5b^2 + \sum a^2b^5 = \sum c^5(a^2 + b^2) \stackrel{A-G}{\geq} \underset{(3)}{\sum c^5(2ab)} = 2abc(\sum a^4) \\ & \quad (1)+(2)+(3) \Rightarrow (a) \text{ is true (proved)} \end{aligned}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ we have:

$$a^3b^2 + a^2b^3 + b^3c^2 + b^2c^3 + c^3a^2 + c^2a^3 \geq 2(a^2bc + ab^3c + abc^3)$$

$$\text{Hence } (a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2)(ab + bc + ca)$$

$$\begin{aligned} & \geq 4abc(a^2 + b^2 + c^2) + 2(abc)(ab + bc + ca) \Rightarrow \frac{(a^2b + ab^2)}{abc} + \frac{(b^2c + bc^2)}{abc} + \frac{(c^2a + ca^2)}{abc} \geq \\ & \geq \frac{4(a^2+b^2+c^2)}{(ab+bc+ca)} + \frac{2(ab+bc+ca)}{(ab+bc+ca)} \Rightarrow \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq \frac{4(a^2+b^2+c^2)}{(ab+bc+ca)} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}. \end{aligned}$$

Therefore it is to be true.

SP.141. Let $a, b, c > 0$ such that: $a + b + c = 3$. Prove that:

$$\frac{a^4}{b^4(2ab - \sqrt{c} + 2)} + \frac{b^4}{c^4(2bc - \sqrt{a} + 2)} + \frac{c^4}{a^4(2ca - \sqrt{b} + 2)} \geq \frac{a^2 + b^2 + c^2}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Michael Sterghiou-Greece

$$\sum_{cyc} \frac{a^4}{b^4(2ab - \sqrt{c} + 2)} \geq \frac{\sum_{cyc} a^2}{3} \quad (1)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Let $(\sum_{cyc} a, \sum_{cyc} ab, abc) = (p, q, r)$. $p = 3$. $\sum_{cyc} a^3 = 9 - 2q$

$$LHS \text{ of (1)} \geq \frac{(\sum_{cyc} \frac{a^2}{b^2})^2}{\sum_{cyc} (2ab - \sqrt{c} + 2)} [BCS] \geq \frac{(\sum_{cyc} \frac{a}{b})^4}{9[2q+6-\sum_{cyc} \sqrt{a}]} [again] BCS \quad (2)$$

It suffices that $(2) \geq \frac{9-2q}{3}$. But it holds that

$$\sum_{cyc} \frac{a}{b} \geq \frac{p}{r^{\frac{1}{3}}} \text{ (AM-GM) and } \sum_{cyc} \sqrt{a} \geq q \text{ (as } p = 3)$$

The last one: as $\sum_{cyc} a^2 + 2 \sum_{cyc} ab = (\sum_{cyc} a)^2 = 9$ it suffices that

$$\sum_{cyc} a^2 + 2 \sum_{cyc} \sqrt{a} \geq 9. \text{ But}$$

$$\sum_{cyc} (a^2 + \sqrt{a} + \sqrt{a}) \stackrel{AM-GM}{\geq} 3 \cdot \sum_{cyc} a = 9$$

Therefore we have to show that $\frac{81}{9r^{\frac{4}{3}}(q+6)} \geq \frac{9-2q}{3}$ or

$$f(q) = \left(\frac{a}{3}\right)^2 (q+6)(9-2q) - 27 \leq 0 \text{ because this stronger inequality arises from}$$

the fact that $r \leq \left(\frac{q}{3}\right)^{\frac{3}{2}}$. But $f(q) = \frac{1}{9}(3-q)(2q^3 + 9q^2 - 27q - 81)$ and $q \leq 3$,

$$9(q) = q(2q^2 + 9q - 27) \leq 81 \text{ as } q \leq 3 \text{ and } 2q^2 + 9q - 27 \leq 27$$

The proof is complete.

Solution 2 by proposer

* By Inequality Cauchy- Schwarz. We have:

$$\sum \frac{a^4}{b^4(2ab - \sqrt{c} + 2)} = \sum \frac{\left(\frac{a^2}{b^2}\right)^2}{(2ab - \sqrt{c} + 2)} \geq \frac{(\sum \frac{a^2}{b^2})^2}{\sum (2ab - \sqrt{c} + 2)} = \frac{(\sum \frac{a^2}{b^2})^2}{2 \sum ab - \sum \sqrt{a} + 6} \quad (2)$$

- Other, by AM-GM:

$$\sum \frac{a^2}{b^2} = \sum \frac{\frac{a^2}{b^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2}}{3} \geq \sum \frac{3 \sqrt[3]{\frac{a^2}{b^2} \cdot \frac{a^2}{b^2} \cdot \frac{b^2}{c^2}}}{3} = \sum \sqrt[3]{\frac{a^4}{b^2 c^2}} = \frac{\sum a^2}{\sqrt[3]{a^2 b^2 c^2}} \quad (3)$$

$$3 = a + b + c \geq 3 \sqrt[3]{abc} \Rightarrow \sqrt[3]{abc} \leq 1 \Rightarrow \sqrt[3]{a^2 b^2 c^2} \leq 1. \text{ Let (3): } \Rightarrow \sum \frac{a^2}{b^2} \geq \sum a^2$$

$$\text{- Let (2): } \Rightarrow \sum \frac{a^4}{b^4(2ab - \sqrt{c} + 2)} \geq \frac{(\sum a^2)^2}{2 \sum ab - \sum \sqrt{a} + 6} \quad (4)$$

- By AM-GM and $a + b + c = 3$. We have:

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$2\sum\sqrt{a} + \sum a^2 = \sum(\sqrt{a} + \sqrt{a} + a^2) > \sum 3 \cdot \sqrt[3]{\sqrt{a} \cdot \sqrt{a} \cdot a^2} = 3 \sum a = 9 = (\sum a)^2 \Rightarrow \sum \sqrt{a} \geq \sum ab$$

- Let (4):

$$\Rightarrow \sum \frac{a^4}{b^4(2ab-\sqrt{c}+2)} \geq \frac{(\sum a^2)^2}{2\sum ab-\sum ab+6} = \frac{(\sum a^2)^2}{\sum ab+6} \geq \frac{(\sum a^2)^2}{\sum a^2+6} \quad (\text{because } \sum ab \leq \sum a^2) \quad (5)$$

$$* \text{ We will prove that: } \frac{(\sum a^2)^2}{\sum a^2+6} \geq \frac{\sum a^2}{3} \Leftrightarrow \frac{\sum a^2}{\sum a^2+6} \geq \frac{1}{3} \Leftrightarrow 3 \sum a^2 \geq \sum a^2 + 6 \Leftrightarrow \sum a^2 \geq 3$$

$$(\text{True because by AM-GM: } \sum a^2 \geq \frac{(\sum a)^2}{3} = \frac{3^2}{3} = 3)$$

$$\text{- Therefore, let (5): } \Rightarrow \sum \frac{a^4}{b^4(2ab-\sqrt{c}+2)} \geq \frac{\sum a^2}{3}$$

$$\Leftrightarrow \frac{a^4}{b^4(2ab-\sqrt{c}+2)} + \frac{b^4}{c^4(2bc-\sqrt{a}+2)} + \frac{c^4}{a^4(2ca-\sqrt{b}+2)} \geq \frac{a^2+b^2+c^2}{3} \Rightarrow \text{Q.E.D.}$$

SP.142. Let a, b, c be positive real numbers such that: $abc = 1$. Prove that:

$$\frac{a^2b^2}{a^2-2a+b^2+2} + \frac{b^2c^2}{b^4-2b+c^2+2} + \frac{c^2a^2}{c^4-2c+a^2+2} \leq \frac{a^2+b^2+c^2+3}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursarescu-Romania

$$a^4 - 2a + b^2 + 2 = a^4 - 2a^2 + 1 + 2a^2 + b^2 + 2^{-2a} = (a^2 - 1)^2 + a^2 - 2a + 1 + a^2 + b^2 = (a^2 - 1)^2 + (a - 1)^2 + a^2 + b^2 \geq a^2 + b^2 \geq 2ab, \text{ with equality for } a = b = 1.$$

$$\text{Inequality becomes: } \frac{a^2b^2}{2ab} + \frac{b^2c^2}{2bc} + \frac{a^2c^2}{2ac} \leq \frac{a^2+b^2+c^2+3}{4} \Leftrightarrow$$

$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ac), \forall a, b, c > 0 \text{ with } abc = 1 \quad (1)$$

$$a^2 + b^2 \geq 2ab \quad (2); c^2 + 1 \geq 2c \quad (3); 2 + 2c \geq 2ac + 2bc \quad (4) \Leftrightarrow 1 + \frac{1}{ab} \geq \frac{1}{b} + \frac{1}{a} \Leftrightarrow$$

$$\Leftrightarrow ab + 1 \geq a + b \Leftrightarrow (a - 1)(b - 1) \geq 0, \text{ true because we can choose two numbers}$$

$$\text{so that } a, b \geq 1 \text{ or } a, b \leq 1. \text{ From (2)+(3)+(4)} \Rightarrow a^2 + b^2 + c^2 + 1 + 2 + 2c \geq$$

$$\geq 2ab + 2c + 2ac + 2bc \Rightarrow a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ac) \Rightarrow \text{then (1) its true.}$$

Solution 2 by Michael Sterghiou-Greece

$$a, b, c > 0 \wedge abc = 1 \rightarrow \sum_{cyc} \frac{a^2b^2}{a^4-2a+b^2+2} \leq \frac{(\sum_{cyc} a^2)+3}{4} \quad (1)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{a^2b^2}{a^4 + 2a + b^2 + 2} = \frac{a^2b^2}{a^4 - a^2 - 2a + b^2 + a^2 + 2} \leq \frac{a^2b^2}{\underbrace{(a-1)^2(a^2 + 2a + 2) + 2ab}_{>0}} \leq \frac{ab}{2}$$

Applying this in a cyclic manner it suffices to prove

$a^2 + b^2 + c^2 + 3 - 2(ab + bc + ca) \geq 0$ (2). From the numbers $a - 1, b - 1, c - 1$ two will have the same sign (pigeonhole principle). Let WLOG these be $a - 1, b - 1$

$$(2) \rightarrow a^2 + b^2 + c^2 + 1 + 2abc - 2ab - 2bc - 2ca = (a-1)^2 + (b-1)^2 + 2a(1-b)(1-c) \text{ which is clearly } \geq 0. \text{ Done.}$$

Solution 3 by Rade Krenkov-Sturmica-Macedonia

$$\begin{aligned} \text{Using AM-GM we get: } & \frac{a^2b^2}{a^4 - 2a + b^2 + 2} + \frac{b^2c^2}{b^4 - 2b + c^2 + 2} + \frac{c^2a^2}{c^4 - 2c + a^2 + 2} \leq \\ & \leq \frac{a^2b^2}{a^4 - a^2 + b^2 + 4} + \frac{b^2c^2}{b^4 - b^2 + c^2 + 4} + \frac{c^2a^2}{c^4 - c^2 + a^2 + 4} \\ & \frac{a^2b^2}{a^4 - 2a + b^2 + 2} + \frac{b^2c^2}{b^4 - 2b + c^2 + 2} + \frac{c^2a^2}{c^4 - 2c + a^2 + 2} \leq \frac{a^2b^2}{a^2 + b^2} + \frac{b^2c^2}{b^2 + c^2} + \frac{c^2a^2}{c^2 + a^2} \\ & \frac{a^2b^2}{a^4 - 2a + b^2 + 2} + \frac{b^2c^2}{b^4 + 2b + c^2 + 2} + \frac{c^2a^2}{c^4 - 2c + a^2 + 2} \leq \\ & \leq \frac{1}{2} \left[\frac{ab \cdot 2ab}{a^2 + b^2} + \frac{bc \cdot 2bc}{b^2 + c^2} + \frac{ca(2ca)}{c^2 + a^2} \right] \\ & \frac{a^2b^2}{a^4 - 2a + b^2 + 2} + \frac{b^2c^2}{b^4 - 2b + c^2 + 2} + \frac{c^2a^2}{c^4 - 2c + a^2 + 2} \leq \frac{1}{2} (ab + bc + ca) \end{aligned}$$

We have to prove that: $a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca)$

We will prove that: $(a + b + c)(a^2 + b^2 + c^2 + 3) \geq 2(ab + bc + ca)(a + b + c)$

Note that: $a + b + c \geq 3\sqrt[3]{abc} = 3\sqrt[3]{(abc)^3} = 3abc$. Now,

$$(a + b + c)(a^2 + b^2 + c^2 + 3) = a^3 + b^3 + c^3 + \sum a^2b + \sum ab^2 + 3(a + b + c)$$

$$(a + b + c)(a^2 + b^2 + c^2 + 3) \geq a^3 + b^3 + c^3 + \sum a^2b + \sum ab^2 + 6abc + 3abc$$

Using Schur's inequality we get:

$$(a + b + c)(a^2 + b^2 + c^2 + 3) \geq 2 \left(\sum a^2b + \sum ab^2 + 3abc \right)$$

$$(a + b + c)(a^2 + b^2 + c^2 + 3) \geq 2(a + b + c)(ab + bc + ca) \text{ (Done)}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ and $abc = 1$, give $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ and since $\frac{(ab)^2}{a^4 - 2a + b^2 + z} \leq \frac{ab}{2}$

$$\frac{(bc)^2}{b^4 - 2b + c^2 + 2} \leq \frac{bc}{2}, \frac{(ca)^2}{c^4 - 2c + a^2 + 2} \leq \frac{ca}{2} \text{ and } \frac{ab}{2} + \frac{bc}{2} + \frac{ca}{2} \leq \frac{a^2 + b^2 + c^2 + 3}{4}$$

$$\text{Because } ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{y}{x} + \frac{z}{y} + \frac{x}{z}$$

$$a^2 + b^2 + c^2 = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 \text{ and since}$$

$$x^4 z^2 + y^4 x^2 + z^4 y^2 + 3(xyz)^2 \geq 2(xy^3 z^2 + x^3 y^2 z + x^2 yz^3) \Rightarrow$$

$$\Rightarrow \frac{x^2}{y^2} + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 + 3 \geq 2\left(\frac{y}{x} + \frac{x}{z} + \frac{z}{y}\right) \Rightarrow a^2 + b^2 + c^2 + 3 \geq 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \Rightarrow$$

$$\Rightarrow \frac{a^2 + b^2 + c^2 + 3}{4} \geq \frac{ab + bc + ca}{2}. \text{ Therefore it is to be true.}$$

SP.143. Let x, y, z be non-negative real numbers. Prove that:

$$x\sqrt{3x^2 + yz} + y\sqrt{3y^2 + zx} + z\sqrt{3z^2 + xy} \geq x^2 + y^2 + z^2 + xy + yz + zx$$

Proposed by Do Quoc Chinh – Ho Chi Minh – Vietnam

Solution by Rade Krenkov-Strumica-Macedonia

Using Cauchy – Schwarz inequality we have: $(3x^2 + yz)(3x^2 + x^2) \geq (3x^2 + x\sqrt{yz})^2$.

$$2x\sqrt{3x^2 + yz} \geq 3x^2 + \sqrt{yz} \quad (1)$$

$$\text{Now, we get: } 2y\sqrt{3y^2 + zx} \geq 3y^2 + \sqrt{zx} \quad (2)$$

$$2z\sqrt{3z^2 + xy} \geq 3z^2 + \sqrt{xy} \quad (3)$$

From (1), (2) and (3) we get: $2(x\sqrt{3x^2 + yz} + y\sqrt{3y^2 + zx} + z\sqrt{3z^2 + xy}) \geq$

$$\geq 2(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 + \sqrt{xy} + \sqrt{yz} + \sqrt{zx}).$$

It is enough to prove that:

$$x^2 + y^2 + z^2 + x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy} \geq 2(xy + yz + zx). \text{ Introducing substitution}$$

$$x = a^2, y = b^2, z = c^2$$

$$\text{we get: } a^4 + b^4 + c^4 + abc(a + b + c) \geq 2(a^2 b^2 + b^2 c^2 + c^2 a^2).$$

Using Schur's inequality we have:

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{cyc} a^4 + abc \sum_{cyc} a = \left(\sum_{cyc} a^3 + 3abc \right) \cdot \sum_{cyc} a - \left(\sum_{cyc} a^3 b + \sum_{cyc} ab^3 \right)$$

$$\sum_{cyc} a^4 + abc \sum_{cyc} a \geq \left(\sum_{cyc} a^2 b + \sum_{cyc} ab^2 \right) \sum_{cyc} a - 2abc \sum_{cyc} a - \left(\sum_{cyc} a^3 b + \sum_{cyc} ab^3 \right)$$

$$\sum_{cyc} a^4 + abc \sum_{cyc} a \geq 2 \sum_{cyc} a^2 b^2$$

SP.144. Let A, B, C be the corners in a triangle ABC . Prove that:

$$\left(\frac{\sin \frac{A}{2}}{\tan \frac{B}{2}} \right)^2 + \left(\frac{\sin \frac{B}{2}}{\tan \frac{C}{2}} \right)^2 + \left(\frac{\sin \frac{C}{2}}{\tan \frac{A}{2}} \right)^2 \geq \frac{9}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursarescu-Romania

$$s = \frac{a + b + c}{2}$$

$$\left(\frac{\sin \frac{A}{2}}{\tan \frac{B}{2}} \right)^2 = \frac{(s-b)(s-c)}{\frac{bc}{s-a}} = \frac{s(s-b)^2}{bc(s-a)} = 1 \text{ we must show: } \sum \frac{s(s-b)^2}{bc(s-a)} \geq \frac{9}{4} \Leftrightarrow \sum \frac{as(s-b)^2}{abc(s-a)} \geq \frac{9}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{a(s-b)^2}{s-a} \geq \frac{9}{4} \cdot \frac{abc}{s} \Leftrightarrow \sum \frac{a^2(s-b)^2}{a(s-a)} \geq \frac{9abc}{4s} \quad (1)$$

$$\text{From Bergstrom inequality we have: } \sum \frac{a(s-b)^2}{a(s-a)} \geq \frac{(\sum a(s-b))^2}{\sum a(s-a)} \quad (2)$$

$$\text{From (1)+(2) we must show this: } \frac{(2s^2 - (ab+bc+ac))^2}{2s^2 - (a^2+b^2+c^2)} \geq \frac{9abc}{4s} \quad (3)$$

$$\text{But we have } abc = 4sRr \quad (4)$$

$$ab + bc + ac = s^2 + r^2 + 4Rr \quad (5)$$

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \quad (6)$$

$$\text{From (3)+(4)+(5)+(6) we must show: } \frac{(2s^2 - s^2 - r^2 - 4Rr)^2}{2s^2 - 2s^2 + 2r^2 + 8Rr} \geq \frac{9 \cdot 4sRr}{4s} \Leftrightarrow \frac{(s^2 - r^2 - 4Rr)^2}{2r(4R+r)} \geq 9Rr \Leftrightarrow$$

$$(s^2 - r^2 - 4Rr)^2 \geq 18Rr^2(4R + r) \quad (7)$$

$$\text{From Gerretsen inequality we have: } s^2 \geq 16Rr - 5r^2 \quad (8)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

From (7)+(8) we must show: $(12Rr - 6r^2)^2 \geq 18Rr^2(4R + r) \Leftrightarrow$
 $\Leftrightarrow 36r^2(2R - r)^2 \geq 18Rr^2(4Rr + r) \Leftrightarrow 2(2R - r)^2 \geq R(4R + r) \Leftrightarrow$
 $\Leftrightarrow 8R^2 - 8Rr + 2r^2 \geq 4R^2 + Rr \Leftrightarrow 4R^2 - 9Rr + 2r^2 \geq 0 \Leftrightarrow (R - 2r)(4R - r) \geq 0$
true, because from Euler $R \geq 2r$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} LHS &= \frac{(s-b)(s-c)s(s-b)}{bc(s-c)(s-a)} + \frac{(s-c)(s-a)s(s-c)}{ca(s-a)(s-b)} + \frac{(s-a)(s-b)s(s-a)}{ab(s-b)(s-c)} = \\ &= \frac{s}{4Rrs} \left[\frac{a(s-b)^2}{(s-a)} + \frac{b(s-c)^2}{(s-b)} + \frac{c(s-a)^2}{(s-c)} \right] = \\ &= \frac{1}{4Rr} \left[\frac{\{s - (s-a)\}(s-b)^2}{s-a} + \frac{\{s - (s-b)\}(s-c)^2}{s-b} + \frac{\{s - (s-c)\}(s-a)^2}{s-c} \right] = \\ &= \frac{s}{4Rr} \left[\frac{(s-b)^3}{(s-a)(s-b)} + \frac{(s-c)^3}{(s-b)(s-c)} + \frac{(s-a)^3}{(s-c)(s-a)} \right] - \frac{1}{4Rr} \left[\sum (s-b)^2 \right] \geq \\ &\stackrel{\text{Hölder}}{\geq} \frac{s}{4Rr} \cdot \frac{(3s-2s)^3}{3 \sum (s-a)(s-b)} - \frac{1}{4Rr} \sum (s^2 - 2bs + b^2) = \\ &= \frac{s^4}{12Rr \sum \{s^2 - s(a+b) + ab\}} - \frac{1}{4Rr} \{3s^2 - 4s^2 + 2(s^2 - 4Rr - r^2)\} \\ &= \frac{s^4}{1Rr(3s^2 - 4s^2 + s^2 + 4Rr + r^2)} - \frac{s^2 - 8Rr - 2r^2}{4Rr} = \\ &= \frac{s^4 - 3(4Rr + r^2)(s^2 - 8Rr - 2r^2)}{12Rr(4Rr + r^2)} \stackrel{\text{Gerretsen}}{\geq} \frac{s^2(16Rr - 5r^2 - 12Rr - 3r^2) + 6r^2(4R + r)^2}{12R(4R + r)r^2} \\ &\stackrel{\text{Gerretsen}}{\geq} \frac{(16Rr - 5r^2)(4Rr - 8r^2) + 6r^2(4R + r)^2}{12R(4R + r)r^2} \stackrel{?}{\geq} \frac{9}{4} \Leftrightarrow 52R^2 - 127Rr + 46r^2 \stackrel{?}{\geq} 0 \\ &\Leftrightarrow (R - 2r)(52R - 23r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \text{ (proved)} \end{aligned}$$

Solution 3 by Rade Krenkov-Strumica-Macedonia

$$\begin{aligned} \text{From } \sin\left(\frac{A}{2} + \frac{B}{2}\right) &= \cos\frac{C}{2} \Leftrightarrow \frac{\sin\frac{A}{2}}{\tan\frac{B}{2}} + \cos\frac{A}{2} = \frac{\cos\frac{C}{2}}{\sin\frac{B}{2}} \Big|^2 \\ 2\left(\frac{\sin^2\frac{A}{2}}{\tan^2\frac{B}{2}} + \cos^2\frac{A}{2}\right) &\geq \left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}} + \cos\frac{A}{2}\right)^2 = \frac{\cos^2\frac{C}{2}}{\sin^2\frac{B}{2}}. \text{ Now,} \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$2LHS + 2 \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right) \geq \frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{C}{2}} + \frac{\cos^2 \frac{B}{2}}{\sin^2 \frac{A}{2}} + \frac{\cos^2 \frac{C}{2}}{\sin^2 \frac{B}{2}} \quad (1)$$

Using equality $\tan \frac{A}{2} \cdot \tan \frac{B}{2} + \tan \frac{B}{2} \cdot \tan \frac{C}{2} + \tan \frac{C}{2} \cdot \tan \frac{A}{2} = 1$ and Cauchy-Schwarz

$$\begin{aligned} \text{inequality we get: } & \left(\frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{C}{2}} + \frac{\cos^2 \frac{B}{2}}{\sin^2 \frac{A}{2}} + \frac{\cos^2 \frac{C}{2}}{\sin^2 \frac{B}{2}} \right) \left(\frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}} \cdot \frac{\sin^2 \frac{C}{2}}{\cos^2 \frac{C}{2}} + \frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}} \cdot \frac{\sin^2 \frac{B}{2}}{\cos^2 \frac{B}{2}} + \frac{\sin^2 \frac{B}{2}}{\cos^2 \frac{B}{2}} \cdot \frac{\sin^2 \frac{C}{2}}{\cos^2 \frac{C}{2}} \right) \geq \\ & \geq \left(\sqrt{\frac{\sin A}{\sin C}} + \sqrt{\frac{\sin B}{\sin A}} + \sqrt{\frac{\sin C}{\sin B}} \right)^2 \end{aligned}$$

$$\text{We have that } \frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{C}{2}} + \frac{\cos^2 \frac{B}{2}}{\sin^2 \frac{A}{2}} + \frac{\cos^2 \frac{C}{2}}{\sin^2 \frac{B}{2}} \geq 9 \quad (2)$$

From Jensen's inequality on a function $f(x) = \cos^2 \frac{x}{2}$ which is concave (\cap) we have:

$$\begin{aligned} 3 \cos^2 \left(\frac{A+B+C}{6} \right) & \geq \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \\ \frac{9}{4} & \geq \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \quad (3) \end{aligned}$$

Using (1), (2) and (3) we get: $2LHS + 2 \cdot \frac{9}{4} \geq 9$; $LHS \geq \frac{9}{4}$

Solution 4 by Lahiru Samarakoon-Sri Lanka

$$\text{For } \triangle ABC, \frac{\sin^2 \frac{A}{2}}{\tan^2 \frac{B}{2}} + \frac{\sin^2 \frac{B}{2}}{\tan^2 \frac{C}{2}} + \frac{\sin^2 \frac{C}{2}}{\tan^2 \frac{A}{2}} \geq \frac{9}{4}$$

$$LHS = \sum \frac{\sin^2 \frac{A}{2} \cdot \cos^2 \frac{B}{2}}{\sin^2 \frac{B}{2}} = \sum \frac{\sin^2 \frac{A}{2}}{\sin^2 \frac{B}{2}} \left(1 - \sin^2 \frac{B}{2} \right) = \sum \left(\frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} \right)^2 - \sum \sin^2 \frac{A}{2}$$

Let's consider, $\left(\frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} \right)^2 + \left(\frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} \right)^2 + \left(\frac{\sin \frac{B}{2}}{\sin \frac{C}{2}} \right)^2$. Using AM-GM

$$2 \left(\frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} \right)^2 + \left(\frac{\sin \frac{B}{2}}{\sin \frac{C}{2}} \right)^2 \geq 3 \sqrt[3]{\frac{\sin^4 \frac{A}{2} \cdot \sin^2 \frac{B}{2}}{\sin^4 \frac{B}{2} \cdot \sin^2 \frac{C}{2}}} = \frac{3 \sin^2 \frac{A}{2}}{\sqrt[3]{\sin^2 \frac{A}{2} \cdot \sin^2 \frac{B}{2} \cdot \sin^2 \frac{C}{2}}} \geq 12 \sin^2 \frac{A}{2} \quad (1) \quad (\because \prod \sin \frac{A}{2} \leq \frac{1}{8})$$

$$\text{Similarly, } 2 \left(\frac{\cos^2 \frac{B}{2}}{\sin^2 \frac{C}{2}} \right) + \left(\frac{\sin^2 \frac{C}{2}}{\cos^2 \frac{A}{2}} \right) \geq 12 \sin^2 \frac{B}{2} \quad (2)$$

$$2 \left(\frac{\sin^2 \frac{C}{2}}{\cos^2 \frac{A}{2}} \right) + \left(\frac{\sin^2 \frac{A}{2}}{\sin^2 \frac{B}{2}} \right) \geq 12 \sin^2 \frac{C}{2} \quad (3)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$(1)+(2)+(3), \sum \left(\frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} \right)^2 \geq 4 \sum \sin^2 \frac{A}{2}$$

$$\therefore LHS \geq 4 \sum \sin^2 \frac{A}{2} - \sum \cos^2 \frac{A}{2} = 3 \sum \sin^2 \frac{A}{2}$$

$$\text{But, } \sum \sin^2 \frac{A}{2} \geq \frac{3}{4}. \text{ So, } LHS \geq 3 \times \frac{3}{4} = \frac{9}{4} \text{ (proved)}$$

SP.145. If $1 < a \leq b$ then:

$$\int_a^b \int_a^b \int_a^b \frac{dx dy dz}{1 + \sqrt[3]{xyz}} \leq \log \left(\sqrt[3]{\frac{b+1}{a+1}} \right)^{(b-a)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Let $x = e^m, y = e^n$ and $e^p = z$ where $m, n, p > 0$

$$\text{Let } f(m) = \frac{1}{1+e^m} \text{ for all } m > 0, f'(m) = -\frac{e^m}{(1+e^m)^2}, f''(m) = \frac{e^m(e^m-1)}{(1+e^m)^3} > 0$$

$$\text{hence } f \text{ is convex function, } \therefore \sum_{cyc} \frac{1}{1+e^m} \geq \frac{3}{1+e^{\frac{m+n+p}{3}}}$$

$$\Rightarrow \sum_{cyc} \frac{1}{1+x} \geq \frac{3}{1+\sqrt[3]{xyz}} \Rightarrow \frac{1}{3} \sum_{cyc} \int_a^b \int_a^b \int_a^b \frac{1}{1+x} dx \geq \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{1+\sqrt[3]{xyz}}$$

$$\Rightarrow \frac{(a-b)^2}{3} \sum_{cyc} [\log(x+1)]_{x=a}^{x=b} \geq \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{1+\sqrt[3]{xyz}}$$

$$\Rightarrow \log \left(\frac{b+1}{a+1} \right)^{(a-b)^2} \geq \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{1+\sqrt[3]{xyz}}$$

SP.146. Let be $A, B \in M_3(\mathbb{R})$ such that:

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Find: $\det((BA)^2 - 3I_3)$

Proposed by Marian Ursărescu – Romania

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by Ravi Prakash-New Delhi-India

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\det(AB) = 2 \Rightarrow \det(A) \det(B) = 2 \neq 0 \Rightarrow \det(A) \neq 0, \det(B) \neq 0$$

$$\begin{aligned} \therefore A^{-1}, B^{-1} \text{ both exist. Now, } (BA)^2 - 3I_3 &= A^{-1}A((BA)^2 - 3I_3)BB^{-1} = \\ &= A^{-1}[A(BA)^2B - 3AB]B^{-1} = A^{-1}[(AB)^3 - 3(AB)]B^{-1} \quad (1) \end{aligned}$$

Characteristic equation of AB

$$\begin{vmatrix} 2-t & 1 & 1 \\ 0 & -1-t & 1 \\ 0 & 0 & -1-t \end{vmatrix} = 0 \Rightarrow (1+t)^2(2-t) = 0 \Rightarrow (1+2t+t^2)(2-t) = 0 \Rightarrow$$

$$\Rightarrow 2 + 4t + 2t^2 - t - 2t^2 - t^3 = 0 \text{ or } 2 + 3t - t^3 = 0. \text{ As } AB \text{ satisfies this equation}$$

$$2I_3 = (AB)^3 - (3AB) \quad (2)$$

$$\text{From (1), (2): } (BA)^2 - 3I_3 = A^{-1}(2I_3)B^{-1} = 2A^{-1}B^{-1}$$

$$\det((BA)^2 - 3I_3) = 8 \det(A^{-1}) \det(B^{-1}) = \frac{8}{\det(A) \det(B)} = \frac{8}{\det(BA)} = \frac{8}{2} = 4$$

SP.147. Find all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ having the property:

$$f(x) + 2f(2x) + f(4x) = 25x^2 + 9x + 4, \forall x \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan

$$\begin{aligned} f(x) + 2f(2x) + f(4x) = 25x^2 + 9x + 4 \quad (1) &\Rightarrow [f(x) - (x^2 + x + 1)] + \\ + 2[f(2x) - (4x^2 + 2x + 1)] + [f(4x) - (16x^2 + 4x + 1)] &= 0. \text{ Let } g(x) = f(x) - \\ -(x^2 + x + 1). \text{ Then we have: } g(x) + 2g(2x) + g(4x) &= 0 \quad (2) \end{aligned}$$

$$(2) \stackrel{x=0}{\Rightarrow} g(0) = 0 \quad (3)$$

$$(2) \Rightarrow g(4x) + g(2x) = -(g(2x) + g(x)) \Rightarrow$$

$$g(2x) + g(x) = -(g(2^{1-n}x) + g(2^{-n}x))$$

$$\stackrel{n \rightarrow +\infty}{\Rightarrow} \text{ using continuity } \Rightarrow g(2x) + g(2x) = 0 \Rightarrow g(2x) = -g(x) \text{ and so}$$

$$g(x) = -g(2^{-n}x); g(x) = -g(2^{-n}x) \stackrel{n \rightarrow \infty}{\Rightarrow} \text{ and using continuity, we have}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$g(x) = -g(0) \stackrel{(3)}{=} \mathbf{0}. \text{ Hence } g(x) = \mathbf{0} \Rightarrow f(x) - (x^2 + x + 1) = \mathbf{0} \Rightarrow f(x) = x^2 + x + 1.$$

Solution 2 by Sagar Kumar-Kolkata-India

$$\begin{aligned} f(x) + 2f(2x) + f(4x) &= (x^2 + 2x + 1) + 2((2x)^2 + (2x) + 1) + ((4x)^2 + (4x) + 1) \\ &\Rightarrow (f(x) - (x^2 + x + 1)) + 2(f(2x) - ((2x)^2 + 2x + 1)) + f(4x) - \\ &\quad - ((4x)^2 + 4x + 1) = \mathbf{0} \text{ on comparing } f(x) = x^2 + x + 1 \end{aligned}$$

Solution 3 by Tran Hong-Vietnam

$$\begin{aligned} \text{Let } g(x) &= f(x) - [x^2 + x + 1]; \forall x \in \mathbb{R} \text{ since } f \text{ continuous} \Rightarrow g \text{ continuous on } \mathbb{R} \Rightarrow \\ \Rightarrow g(x) + 2g(2x) + g(4x) &= \mathbf{0}, \forall x \in \mathbb{R} \Rightarrow g\left(\frac{x}{2^{n+2}}\right) + 2g\left(\frac{x}{2^{n+1}}\right) + g\left(\frac{x}{2^n}\right) = \mathbf{0}, \forall x \in \mathbb{R} \\ \Rightarrow \lim_{n \rightarrow \infty} \left[g\left(\frac{x}{2^{n+2}}\right) + 2g\left(\frac{x}{2^{n+1}}\right) + g\left(\frac{x}{2^n}\right) \right] &= \mathbf{0}; (\forall) x \in \mathbb{R} \Rightarrow g(\mathbf{0}) + 2g(\mathbf{0}) + g(\mathbf{0}) = \mathbf{0} \Rightarrow \\ &\Rightarrow 3g(\mathbf{0}) = \mathbf{0} \Rightarrow g(\mathbf{0}) = \mathbf{0}; [g(x) + g(2x)] + [g(2x) + g(x)] = \mathbf{0}; \end{aligned}$$

$$\text{Let } h(x) = g(x) = g(x); h(\mathbf{0}) = \mathbf{0} \Rightarrow h(x) + h(2x) = \mathbf{0} \Rightarrow$$

$$h(2x) = (-1)^n h\left(\frac{x}{2^n}\right); \forall n \in \mathbb{N}$$

$$\Rightarrow h(x) = \lim_{n \rightarrow \infty} (-1)^n h\left(\frac{x}{2^n}\right) = \mathbf{0} \Rightarrow g(2x) = \lim_{n \rightarrow \infty} (-1)^n g\left(\frac{x}{2^n}\right) = \mathbf{0}$$

$$f(x) = x^2 + x + 1$$

Solution 4 by Chris Kyriazis-Greece

Set $g(x) = f(x) - x^2 - x - 1, \forall x \in \mathbb{R}$. Then, the given functional equation becomes:

$$g(x) + 2g(2x) + g(4x) = \mathbf{0} \quad (1)$$

If we put $x = \mathbf{0}$, then $g(\mathbf{0}) = \mathbf{0}$. Setting $h(x) = g(x) + g(2x) \quad (1)$

Becomes $h(x) + h(2x) = \mathbf{0}, \forall x \in \mathbb{R}$ or (putting $x \rightarrow x/2$): $h(x) = -h\left(\frac{x}{2}\right)$

Once more and $h(x) = h\left(\frac{x}{4}\right) = h\left(\frac{x}{2^2}\right)$. By induction, we can show that:

$h(x) = (-1)^n h\left(\frac{x}{2^n}\right)$. Using continuity of h and taking limits, we have that:

$$\lim_{n \rightarrow +\infty} h(x) = \lim_{n \rightarrow \infty} (-1)^n h\left(\frac{x}{2^n}\right) = \mathbf{0} \Rightarrow h(x) = \mathbf{0}, \forall x \in \mathbb{R}.$$

This means $g(x) + g(2x) = \mathbf{0}$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$\forall x \in \mathbb{R}$. Working the same way, $g(x) = 0, \forall x \in \mathbb{R}$. So, $f(x) = x^2 + x + 1, \forall x \in \mathbb{R}$
which satisfies the given equation.

SP.148. Let be $x_0 > 0$ and $x_{n+1} = \arctan \frac{x_n}{1+x_n}, \forall n \in \mathbb{N}$.

Find:

$$\Omega = \lim_{n \rightarrow \infty} (n \cdot x_n)$$

Proposed by Marian Ursărescu – Romania

Solution by Remus Florin Stanca-Romania

We prove by using Mathematical induction that $x_n > 0, \forall n \in \mathbb{N}$.

- 1) We prove that $P(0): x_0 > 0$ is true (true)
- 2) We suppose that $P(n): x_n > 0$ is true.
- 3) We prove by using $P(n)$ that $P(n+1): x_{n+1} > 0$ is true.

$$x_n > 0 \Rightarrow x_{n+1} \in \left(0; \frac{\pi}{2}\right] \Rightarrow x_{n+1} > 0 \Rightarrow x_n > 0; \forall n \in \mathbb{N}$$

$$x_{n+1} = \arctan \frac{x_n}{x_n+1} \text{ and because } x_n > 0; \forall n \in \mathbb{N} \Rightarrow x_n \in \left(0; \frac{\pi}{2}\right]$$

$$\text{We study the sign of } x_1 - x_0 = \arctan \frac{x_0}{x_0+1} - x_0$$

$$\text{Let } f: \left(0; \frac{\pi}{2}\right] \rightarrow \mathbb{R} \text{ such that } f(x) = \arctan \frac{x}{x+1} - x$$

$$\Rightarrow f'(x) = \frac{1}{1 + \frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2} - 1 = \frac{1}{2x^2 + 2x + 1} - 1 \Rightarrow f'(x) < 0$$

$$\Rightarrow f(x) \text{ is a decreasing function } f(0) = 0 \Rightarrow f(x) < 0 \text{ for } x \in \left(0; \frac{\pi}{2}\right] \Rightarrow x_1 < x_0$$

We prove by using the Mathematical induction that $x_n > x_{n+1}$

- 1) We proved that $P(0): x_0 > x_1$ is true
- 2) We suppose that $P(n): x_n > x_{n+1}$ is true
- 3) We prove that $P(n+1): x_{n+1} > x_{n+2}$ is true by using $P(n)$

$$x_{n+1} - x_{n+2} = \arctan \frac{x_n}{x_n+1} - \arctan \frac{x_{n+1}}{x_{n+1}+1}$$

We prove that the function $f: \left(0; \frac{\pi}{2}\right] \rightarrow \mathbb{R}, f(x) = \arctan \frac{x}{x+1}$ is an increasing function.

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f(x) = \frac{1}{1+\frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2} > 0 \Rightarrow \text{true so } x_{n+1} - x_{n+2} > 0$$

$$> x_{n+1} > x_{n+2} > x_n > x_{n+1} \text{ for } n \in \mathbb{N}.$$

$x_n \in \left(0; \frac{\pi}{2}\right]$ and x_n is a decreasing sequence $\Rightarrow -l = \lim_{n \rightarrow \infty} x_n$ such that $l \in \mathbb{R}$

$\Rightarrow l = \arctan \frac{l}{l+1}, l \in \left(0; \frac{\pi}{2}\right]$ the function $f(l) = \arctan \frac{l}{l+1} - l$ is a decreasing function

so $l = 0$ is the unique solution.

$$\Rightarrow \lim_{n \rightarrow \infty} x_n = 0 \Rightarrow \lim_{n \rightarrow \infty} nx_n = \lim_{n \rightarrow \infty} \frac{n}{\frac{1}{x_n}} \stackrel{\text{Stolz-Cesaro}}{=} \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} =$$

$$= \lim_{n \rightarrow \infty} \frac{x_{n+1}x_n}{x_n - x_{n+1}} = \lim_{n \rightarrow \infty} \frac{\arctan \frac{x_n}{x_n+1} x_n}{x_n - \arctan \frac{x_n}{x_n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{\arctan \frac{x}{x+1} \cdot x}{x - \arctan \frac{x}{x+1}} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\arctan \frac{x}{x+1} + x \cdot \frac{1}{1 + \frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2}}{1 - \frac{1}{1 + \frac{x^2}{(x+1)^2} \cdot \frac{1}{(x+1)^2}}} =$$

$$= \lim_{x \rightarrow 0} \frac{(\arctan \frac{x}{x+1})(2x^2 + 2x + 1) - x}{2x^2 + 2x} = \lim_{x \rightarrow 0} \frac{\arctan \frac{x}{x+1}}{\frac{x}{x+1} \cdot (x+1)(2x+2)} (2x^2 + 2x + 1) =$$

$$= \frac{1}{2x+2} = \frac{1}{2} - \frac{1}{2} = 0$$

SP.149. Let be the sequence $(x_n)_{n \in \mathbb{N}}: x_0 > 1$ and $x_{n+1} = 1 + \ln \left(\frac{2x_n}{1+x_n} \right)$,

$\forall n \in \mathbb{N}$. Find:

$$\lim_{n \rightarrow \infty} (n \ln x_n)$$

Proposed by Marian Ursărescu – Romania

Solution by Remus Florin Stanca-Romania

We prove that $x_n > 1, \forall n \in \mathbb{N}$ by using the Mathematical induction:

1) we prove $P(0): x_0 > 1$ (true)

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

2) we suppose that $P(n): x_n > 1$ is true

3) we prove $P(n+1): x_{n+1} > 1$ by using $P(n)$

$$x_n > 1 \Rightarrow \frac{2x_n}{x_n + 1} > 1 \Rightarrow \ln\left(\frac{2x_n}{x_n + 1}\right) + 1 > 1 \Rightarrow x_{n+1} > 1 \Rightarrow x_n > 1 \forall n \in \mathbb{N}$$

We study the sign of $x_1 - x_0 = 1 + \ln\left(\frac{2x_0}{1+x_0}\right) - x_0$

Let $f: (1; +\infty) \rightarrow \mathbb{R}; f(x) = 1 + \ln\left(\frac{2x}{1+x}\right) - x$

$$f'(x) = \frac{1+x}{2x} \cdot \frac{2}{(x+1)^2} - 1 = \frac{1}{x(x+1)} - 1 < 0 \Rightarrow f \text{ is a decreasing function}$$

$$f(1) = 0 > f(x) < 0 \text{ for } x > 1 > x_1 < x_0$$

$g(x) = \frac{2x}{2x+1}$ is an increasing function so $x_{n+1} < x_n$

$$x_{n+1} < x_n \text{ and } x_n > 1 > l = \lim_{n \rightarrow \infty} x_n \in \mathbb{R}$$

$$l = 1 + \ln\left(\frac{2l}{l+1}\right) \Rightarrow f(l) = 1 + \ln\left(\frac{2l}{l+1}\right) - f \text{ is a decreasing function} \Rightarrow$$

$$l = 1 \text{ is an unique solution} \Rightarrow \lim_{n \rightarrow \infty} x_n = 1$$

$$\lim_{n \rightarrow \infty} n \ln x_n = \lim_{n \rightarrow \infty} \frac{n}{\frac{1}{\ln x_n}} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{\ln x_{n+1}} - \frac{1}{\ln x_n}} =$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{\ln\left(1 + \ln\frac{2x_n}{x_n+1}\right)} - \frac{1}{\ln x_n}} = \lim_{n \rightarrow \infty} \frac{\ln x_n \ln\left(1 + \ln\frac{2x_n}{x_n+1}\right)}{\ln x_n - \ln\left(1 + \ln\frac{2x_n}{x_n+1}\right)}$$

$$\lim_{x \rightarrow 1} \frac{\ln x \ln\left(1 + \ln\frac{2x}{x+1}\right)}{\ln x - \ln\left(1 + \ln\frac{2x}{x+1}\right)} = \lim_{x \rightarrow 1} \frac{\frac{\ln x}{x-1} \cdot (x-1) \cdot \frac{\ln\left(1 + \ln\frac{2x}{x+1}\right)}{\ln\frac{2x}{x+1}} \cdot \ln\frac{2x}{x+1}}{\ln\left(\frac{x}{1 + \ln\frac{2x}{x+1}} - 1 + 1\right)}$$

$$\frac{x-1 - \ln\frac{2x}{x+1}}{1 + \ln\frac{2x}{x+1}} \cdot \frac{1 + \ln\frac{2x}{x+1}}{x-1 - \ln\frac{2x}{x+1}} = \lim_{x \rightarrow 1} \frac{(x-1) \ln\frac{2x}{x+1}}{x-1 - \ln\frac{2x}{x+1}} = \lim_{x \rightarrow 1} \frac{(x-1) \ln\left(\frac{x-1}{x+1} + 1\right)}{x-1 - \ln\frac{2x}{x+1}}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \lim_{x \rightarrow 1} \frac{(x-1) \cdot \frac{\ln\left(\frac{x-1}{x+1} + 1\right)}{\frac{x-1}{x+1}}}{x+1 - \frac{\ln\left(\frac{x-1}{x+1} + 1\right)}{\frac{x-1}{x+1}}} = \lim_{x \rightarrow 1} \frac{x-1}{x} = 0 \Rightarrow \lim_{n \rightarrow \infty} n \ln x_n = 0$$

SP.150. Let be $f \in \mathbb{Z}$, $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, such that $a_1, a_2, \dots, a_n \in \{\pm 1, \pm 2, \dots, \pm n\}$. If a_0 is a prime number, $a_0 > n^2$ then f is irreducible over \mathbb{Z} .

Proposed by Marian Ursărescu – Romania

Solution by proposer

Suppose $\exists g, h \in \mathbb{Z}$ such that $f = g \cdot h$, grade $f, h \geq 1$. $f(0) = g(0) = h(0) \Rightarrow \Rightarrow a_0 = g(0) \cdot h(0)$. But a_0 being prime $\Rightarrow g(0) = 1$ or $h(0) = 1$.

Suppose $g(0) = 1 \Rightarrow g(x) = b_k x^k + \dots + b_1 x + 1$

Let be x_1, x_2, \dots, x_k the roots of f . From the last Viète relationship \Rightarrow

$$\Rightarrow |x_1 x_2 \dots x_n| = \left| \frac{(-1)^k}{b_k} \right| = \frac{1}{|b_k|} \leq 1, \text{ because } b_k \in \mathbb{Z}$$

$$|x_1 x_2 \dots x_k| \leq 1 \Rightarrow \exists p \in \{1, 2, \dots, k\} \text{ such that } |x_p| \leq 1.$$

But x_p is root and for $f \Rightarrow a_n x_p^n + a_{n-1} x_p^{n-1} + \dots + a_1 x_p + a_0 = 0 \Rightarrow$

$$\begin{aligned} \Rightarrow |a_0| &= |a_n x_p^n + a_{n-1} x_p^{n-1} + \dots + a_1 x_p| \leq |a_n| |x_p|^n + \dots + |a_1| |x_p| \leq \\ &\leq |a_1| + |a_2| + \dots + |a_n| \Rightarrow |a_0| \leq |a_1| + |a_2| + \dots + |a_n| \leq n^2 \Rightarrow a_0 \leq n^2 \text{ false} \Rightarrow \\ &\Rightarrow f \text{ is irreducible over } \mathbb{Z} \end{aligned}$$

UP.136. Prove that:

$$\sum_{k=0}^n T_{4k}(x) = \frac{1}{4} \left[\frac{2 + U_{4n+2}(x)}{x\sqrt{1-x^2}} \right]$$

where, $T_n(x)$ and $U_n(x)$ denotes the Chebyshev Polynomials of First and Second Kind.

Proposed by Shivam Sharma-New Delhi-India

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by proposer

$$S = \sum_{k=0}^n T_{4k}(x)$$

Let, $x = \cos(\theta)$. Then, by definition, we have,

$$\begin{aligned} &\Rightarrow \sum_{k=0}^n \cos(4k \cos^{-1}(x)) \Rightarrow \sum_{k=0}^n \cos(4k \cos^{-1}(\cos \theta)) \Rightarrow \sum_{k=0}^n \cos(4k\theta) \Rightarrow \\ &\Rightarrow \text{Real Part } \sum_{k=0}^n e^{4ki\theta} \Rightarrow 1 + e^{4i\theta} + e^{8i\theta} + \dots + (n+1) \text{ terms} \Rightarrow \frac{1 - (e^{4i\theta})^{n+1}}{1 - e^{4i\theta}} \Rightarrow \\ &\Rightarrow \frac{(1 - e^{i(4n+4)\theta})(1 - e^{-4i\theta})}{(1 - e^{4i\theta})(1 - e^{-4i\theta})} \Rightarrow \frac{(1 - e^{i(4n+4)\theta})(1 - e^{-4i\theta})}{2(e^{4i\theta} + e^{-4i\theta})} \Rightarrow \\ &\Rightarrow \frac{(1 - \cos(4n+4)\theta) - i \sin((4n+4)\theta) (1 - (\cos(4\theta)) + i \sin(4\theta))}{2 - 2 \cos(4\theta)} \Rightarrow \\ &\Rightarrow \frac{1 - \cos((4n+4)\theta) - \cos(4\theta) + \cos((4n+4)\theta) \cos(4\theta) + \sin((4n+4)\theta) \sin(4\theta)}{2(1 - \cos(4\theta))} \\ &\Rightarrow \frac{(1 - \cos(4\theta)) - \cos((4n+4)\theta) (1 - \cos(4\theta)) + \sin((4n+4)\theta) \sin(4\theta)}{2(1 - \cos(4\theta))} \\ &\Rightarrow \frac{1}{2} \left[1 - \cos((4n+4)\theta) + \frac{\sin((4n+4)\theta) \sin(4\theta)}{1 - \cos(4\theta)} \right] \\ &\Rightarrow \frac{1}{2} \left[1 - \cos((4n+4)\theta) + \frac{\sin((4n+4)\theta) 2 \sin(2\theta) \cos(2\theta)}{2 \sin^2(2\theta)} \right] \\ &\Rightarrow \frac{1}{2} \left[\frac{1 + \sin((4n+4)\theta) \cos(2\theta) - \cos((4n+4)\theta) \sin(2\theta)}{\sin(2\theta)} \right] \\ &\Rightarrow \frac{1}{2} \left[1 + \frac{\sin(((4n+4)\theta) - 2\theta)}{\sin(2\theta)} \right] \Rightarrow \frac{1}{2} \left[1 + \frac{\sin((4n+2)\theta)}{\sin(2\theta)} \right] \Rightarrow \\ &\Rightarrow \frac{1}{2} \left[1 + \frac{\sin((4n+2)\theta)}{2[\cos(\theta) \sqrt{1 - \cos^2 \theta}]} \right] \end{aligned}$$

As, $x = \cos \theta \Rightarrow U_{4n+2}(x) = \sin((4n+2) \cos^{-1}(\cos \theta))$, so,

$U_{4n+2}(x) = \sin((4n+2)\theta)$. Using above, we get,

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$S = \frac{1}{4} \left[\frac{2+U_{4n+2}(x)}{x\sqrt{1-x^2}} \right] \text{ (Answer)}$$

UP.137. Let $f, g: \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be functions such that:

$$\lim_{x \rightarrow \infty} (f(x+1) - f(x)) = a \in \mathbb{R}_+^*, \lim_{x \rightarrow \infty} \frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^*$$

and exists $\lim_{x \rightarrow \infty} \frac{f(x)}{x}$ and $\lim_{x \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x}$. For $t \in \mathbb{R}$ calculate the limit:

$$\lim_{x \rightarrow \infty} (f(x))^{\cos^2 t} \left((g(x))^{\frac{\sin^2 t}{x+1}} - g(x)^{\frac{\sin^2 t}{x}} \right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\lim_{n \rightarrow \infty} \frac{f(x)}{x} = \lim_{n \rightarrow \infty} (f(x+1) - f(x)) = a; \lim_{n \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x} = \lim_{n \rightarrow \infty} \frac{g(x+1)x^x}{g(x) \cdot (x+1)^{x+1}} = \frac{b}{e}$$

$$\text{Now, } \lim_{n \rightarrow \infty} (f(x))^{\cos^2 t} \left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}} \right) =$$

$$= \lim_{n \rightarrow \infty} \left(\frac{f(x)}{x} \right)^{\cos^2 t} \cdot x^{\cos^2 t} \left(\frac{(g(x))^{\frac{1}{x}}}{x} \right)^{\sin^2 t} \cdot x^{\sin^2 t} \left((g(x))^{\frac{\sin^2 t}{x(x+1)}} - 1 \right) =$$

$$= a^{\cos^2 t} \cdot \left(\frac{b}{e} \right)^{\sin^2 t} \cdot \lim_{n \rightarrow \infty} \left(\frac{e^{-\frac{\sin^2 t}{x(x+1)} \ln(g(x))} - 1}{-\frac{\sin^2 t}{x(x+1)} \ln(g(x))} \cdot \ln(g(x))^{\frac{\sin^2 t}{x+1}} \right) = a^{\cos^2 t} \cdot \left(\frac{b}{e} \right)^{\sin^2 t} \cdot 1 \cdot \ln 0$$

$$\therefore \lim_{n \rightarrow \infty} (f(x))^{\cos^2 t} \left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}} \right) = -\infty$$

$$\text{Note: } \lim_{n \rightarrow \infty} \frac{\sin^2 t}{x(x+1)} \ln(g(x)) = \lim_{n \rightarrow \infty} \frac{\sin^2 t}{x+1} \ln\left(\frac{bx}{e}\right) = 0 \text{ and } \lim_{n \rightarrow \infty} (g(x))^{\frac{\sin^2 t}{x+1}} =$$

$$\lim_{n \rightarrow \infty} \left(\frac{bx}{e} \right)^{\frac{x \sin^2 t}{x+1}} = 0$$

$$\lim_{n \rightarrow \infty} \frac{f(x)}{x} = \lim_{n \rightarrow \infty} (f(x+1) - f(x)) = a; \lim_{n \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x} = \lim_{n \rightarrow \infty} \frac{g(x+1)x^x}{g(x) \cdot (x+1)^{x+1}} = \frac{b}{e}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Now,

$$\begin{aligned} \lim_{n \rightarrow \infty} (f(x))^{\cos^2 t} \left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}} \right) &= \lim_{n \rightarrow \infty} \left(\frac{f(x)}{x} \right)^{\cos^2 t} \cdot x^{\cos^2 t} \cdot \left(\frac{(g(x))^{\frac{1}{x}}}{x} \right)^{\sin^2 t} \\ &\cdot x^{\sin^2 t} \left((g(x))^{\frac{\sin^2 t}{x(x+1)}} - 1 \right) = a^{\cos^2 t} \left(\frac{b}{e} \right)^{\sin^2 t} \cdot x \cdot \lim_{n \rightarrow \infty} \left(\left(\frac{bx}{e} \right)^{\frac{\sin^2 t}{x+1}} - 1 \right) = \\ &= a^{\cos^2 t} \cdot \left(\frac{b}{e} \right)^{\sin^2 t} \lim_{n \rightarrow \infty} \left(-\frac{x \sin^2 t}{x+1} \ln \left(\frac{bx}{e} \right) + 0 \left(\frac{\ln^2 \left(\frac{bx}{e} \right)}{x+1} \right) \right) = -a^{\cos^2 t} \cdot \left(\frac{b}{e} \right)^{\sin^2 t} \lim_{n \rightarrow \infty} \frac{x \sin^2 t}{x+1} \ln \left(\frac{bx}{e} \right) \\ &\therefore \lim_{n \rightarrow \infty} (f(x))^{\cos^2 t} \left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}} \right) = -\infty \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{f(x)}{x} = \lim_{n \rightarrow \infty} (f(x+1) - f(x)) = a; \quad \lim_{n \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x} = \lim_{n \rightarrow \infty} \frac{g(x+1) \cdot x^x}{g(x)(x+1)^{x+1}} = \frac{b}{e}$$

$$\begin{aligned} \text{Now, } \lim_{n \rightarrow \infty} (g(x))^{\cos^2 t} \left((g(x+1))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}} \right) &= \\ &= \lim_{n \rightarrow \infty} \left(\frac{f(x)}{x} \right)^{\cos^2 t} \cdot \left((x+1)^{\sin^2 t} \left(\frac{g(x+1)}{(x+1)^{x+1}} \right)^{\frac{\sin^2 t}{x+1}} - x^{\sin^2 t} \left(\frac{g(x)}{x^x} \right)^{\frac{\sin^2 t}{x}} \right) = \\ &= a^{\cos^2 t} \cdot \sin^2 t \cdot \lim_{n \rightarrow \infty} \left(\frac{\frac{g(x+1)}{(x+1)^{x+1}}}{\frac{g(x)}{x^x}} \right)^{\sin^2 t} \\ &\therefore \lim_{n \rightarrow \infty} (f(x))^{\cos^2 t} \left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}} \right) = a^{\cos^2 t} \cdot \sin^2 t \left(\frac{b}{e} \right)^{\sin^2 t} \end{aligned}$$

UP.138. Let $f, g: \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ such that: $\lim_{x \rightarrow \infty} (f(x+1) - f(x)) = a \in \mathbb{R}_+^*$, $\lim_{x \rightarrow \infty} \frac{g(x+1)}{xg(x)} =$

$b \in \mathbb{R}_+^*$ and there is $\lim_{x \rightarrow \infty} \frac{f(x)}{x}$, $\lim_{x \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x}$. For $t \in \mathbb{R}$, calculate:

$$\lim_{x \rightarrow \infty} (f(x))^{\sin^2 t} \left((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}} \right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by proposers

By Cesaro – Stolz theorem we have:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{\substack{x \rightarrow \infty \\ n \in \mathbb{N}^*}} \frac{f(n)}{n} \stackrel{c-s}{=} \lim_{x \rightarrow \infty} \frac{f(n+1) - f(n)}{(n+1) - n} = \lim_{x \rightarrow \infty} (f(n+1) - f(n)) = a,$$

and by Cauchy-D'Alembert theorem we deduce that:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x} &= \lim_{\substack{x \rightarrow \infty \\ n \in \mathbb{N}^*}} \frac{(g(n))^{\frac{1}{n}}}{n} = \lim_{x \rightarrow \infty} \sqrt[n]{\frac{g(n)}{n^2}} \stackrel{c-D'A}{=} \lim_{n \rightarrow \infty} \left(\frac{g(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{g(n)} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{g(n+1)}{ng(n)} \left(\frac{n}{n+1} \right)^{n+1} \right) = \frac{b}{e}. \text{ So, } \lim_{x \rightarrow \infty} (f(x))^{\sin^2 t} \left((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}} \right) = \end{aligned}$$

$$= \lim_{x \rightarrow \infty} \left(\frac{f(x)}{x} \right)^{\sin^2 t} \lim_{x \rightarrow \infty} \left(\frac{(g(x))^{\frac{1}{x}}}{x} \right)^{\cos^2 t} (u(x) - 1)x^{\sin^2 t + \cos^2 t} =$$

$$= a^{\sin^2 t} \cdot \frac{b^{\cos^2 t}}{e^{\cos^2 t}} \cdot \lim_{x \rightarrow \infty} \left(\frac{u(x) - 1}{\ln u(x)} \cdot \ln(u(x))^x \right),$$

where $u(x) = \left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(g(x))^{\frac{1}{x}}} \right)^{\cos^2 t}$ with $\lim_{x \rightarrow \infty} u(x) = 1$, then $\lim_{x \rightarrow \infty} \frac{u(x)-1}{\ln u(x)} = 1$. We

have:

$$\lim_{x \rightarrow \infty} (u(x))^x = \lim_{x \rightarrow \infty} \left(\frac{(g(x+1))^{\frac{x}{x+1}}}{g(x)} \right)^{\cos^2 t} = \lim_{x \rightarrow \infty} \left(\frac{g(x+1)}{g(x)} \cdot \frac{1}{(g(x+1))^{\frac{1}{x+1}}} \right)^{\cos^2 t} =$$

$$= \lim_{x \rightarrow \infty} \left(\frac{g(x+1)}{xg(x)} \cdot \frac{x+1}{(g(x+1))^{\frac{1}{x+1}}} \cdot \frac{x}{x+1} \right)^{\cos^2 t} = \left(b \cdot \frac{e}{b} \cdot 1 \right)^{\cos^2 t} = e^{\cos^2 t}$$

Therefore: $\lim_{x \rightarrow \infty} (f(x))^{\sin^2 t} \left((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}} \right) = \frac{a^{\sin^2 t} \cdot b^{\cos^2 t}}{e^{\cos^2 t}} \cdot \ln e^{\cos^2 t} =$

$$= \frac{a^{\sin^2 t} \cdot b^{\cos^2 t}}{e^{\cos^2 t}} \cdot \cos^2 t, \text{ and we are done.}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

UP.139. Calculate: $\lim_{x \rightarrow \infty} \left(x^{\cosh^2 t} \left((\Gamma(x+1))^{\frac{-\sinh^2 t}{x}} - ((\Gamma(x+2)))^{\frac{-\sinh^2 t}{x+1}} \right) \right),$

Where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by proposers

$$\begin{aligned} \text{We have: } \lim_{x \rightarrow \infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} &= \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}^*}} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} \stackrel{C-D'A}{=} \\ &= \lim_{n \rightarrow \infty} \left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right) = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e}. \text{ Let} \end{aligned}$$

$$f(x) = x^{\cosh^2 t} \left((\Gamma(x+1))^{\frac{-\sinh^2 t}{x}} - (\Gamma(x+2))^{\frac{-\sinh^2 t}{x+1}} \right) = -x^{\cosh^2 t} (\Gamma(x+1))^{\frac{-\sinh^2 t}{x}} (u(x) - 1),$$

$$\text{where } u: \mathbb{R}_+^* \rightarrow \mathbb{R}, u(x) = \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}} \right)^{-\sinh^2 t}. \text{ We deduce that:}$$

$$\lim_{n \rightarrow \infty} u(x) = \lim_{n \rightarrow \infty} \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1} \cdot \frac{x}{(\Gamma(x+1))^{\frac{1}{x}}} \cdot \frac{x+1}{x} \right)^{-\sinh^2 t} = \left(\frac{1}{e} \cdot e \cdot 1 \right)^{-\sinh^2 t} = 1$$

We have, $\lim_{n \rightarrow \infty} \frac{u(x)-1}{\ln u(x)} = 1$. Also, we have:

$$\begin{aligned} \lim_{x \rightarrow \infty} (u(x))^x &= \lim_{x \rightarrow \infty} \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}} \right)^{-x \sin^2 t} = \lim_{x \rightarrow \infty} \left(\frac{\Gamma(x+2)}{\Gamma(x+1)} \cdot \frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}} \right)^{-\sin^2 t} = \\ &= \lim_{x \rightarrow \infty} \left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}} \right)^{-\sin^2 t} = e^{-\sin^2 t}. \text{ Therefore:} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= - \lim_{x \rightarrow \infty} \left(x^{\cosh^2 t} \left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \cdot x \right)^{-\sin^2 t} \cdot \frac{u(x) - 1}{\ln u(x)} \cdot \ln u(x) \right) = \\ &= - \lim_{x \rightarrow \infty} \left(\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \right)^{-\sin^2 t} \cdot x^{\cosh^2 t - \sinh^2 t} \cdot \frac{u(x) - 1}{\ln u(x)} \cdot \ln u(x) \right) = \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= -\left(\frac{1}{e}\right)^{-\sinh^2 t} \lim_{x \rightarrow \infty} \frac{u(x) - 1}{\ln u(x)} \cdot \ln \left(\lim_{x \rightarrow \infty} (u(x))^x \right) = -e^{\sinh^2 t} \cdot 1 \cdot \ln e^{-\sinh^2 t} = e^{\sinh^2 t} \cdot \sinh^2 t$$

UP.140. Calculate: $\lim_{x \rightarrow \infty} \left(x^{\sin^2 t} \left((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - (\Gamma(x+1))^{\frac{\cos^2 t}{x}} \right) \right)$, where $t \in \mathbb{R}$ and

Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by proposers

$$\begin{aligned} \text{We have: } \lim_{x \rightarrow \infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} &= \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}^*}} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n \rightarrow \infty} \frac{n^{\sqrt[n]{n!}}}{n} = \lim_{x \rightarrow \infty} \frac{n!}{\sqrt[n]{n!}} \stackrel{C-D'A}{=} \\ &= \lim_{x \rightarrow \infty} \left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n!}{n!} \right) = \lim_{x \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e}. \text{ We denote:} \end{aligned}$$

$$f(x) = x^{\sin^2 t} \left((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - (\Gamma(x+1))^{\frac{\cos^2 t}{x}} \right) = x^{\sin^2 t} (\Gamma(x+1))^{\frac{\cos^2 t}{x}} (u(x) - 1),$$

$$\text{where } u: \mathbb{R}_+^* \rightarrow \mathbb{R}, u(x) = \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}} \right)^{\cos^2 t}. \text{ We deduce that:}$$

$$\lim_{x \rightarrow \infty} u(x) = \lim_{x \rightarrow \infty} \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1} \cdot \frac{x}{(\Gamma(x+1))^{\frac{1}{x}}} \cdot \frac{x+1}{x} \right)^{\cos^2 t} = \left(\frac{1}{e} \cdot e \cdot 1 \right)^{\cos^2 t} = 1.$$

We have, $\lim_{x \rightarrow \infty} \frac{u(x)-1}{\ln u(x)} = 1$ and also we have:

$$\begin{aligned} \lim_{x \rightarrow \infty} (u(x))^x &= \lim_{x \rightarrow \infty} \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}} \right)^{x \cos^2 t} = \lim_{x \rightarrow \infty} \left(\frac{\Gamma(x+2)}{\Gamma(x+1)} \cdot \frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}} \right)^{\cos^2 t} = \\ &= \lim_{x \rightarrow \infty} \left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}} \right)^{\cos^2 t} = e^{\cos^2 t}. \text{ Therefore:} \end{aligned}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(x^{\sin^2 t} \left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \cdot x \right) \cdot \frac{u(x) - 1}{\ln u(x)} \cdot \ln u(x) \right) =$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= -\lim_{x \rightarrow \infty} \left(\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \right)^{\cos^2 t} \cdot x^{\sin^2 t + \cos^2 t} \cdot \frac{u(x) - 1}{\ln u(x)} \cdot \ln u(x) \right) =$$

$$= e^{\cos^2 t} \lim_{x \rightarrow \infty} \frac{u(x) - 1}{\ln u(x)} \cdot \ln \left(\lim_{x \rightarrow \infty} (u(x))^x \right) = e^{\cos^2 t} \cdot 1 \cdot \ln e^{\cos^2 t} = e^{\cos^2 t} \cdot \cos^2 t$$

UP.141. For $\{a_n\}_{n \geq 0}$, $a_n = \frac{(n+2)^{n+1}}{(n+1)^n}$, $x \in (-\infty, \infty)$, $\{b_n(x)\}_{n \geq 1}$,

$b_n(x) = n^{\sin^2 x} (a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x})$, find $\lim_{n \rightarrow \infty} b_n(x)$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$L = \lim_{n \rightarrow \infty} n^{\sin^2 x} \left[\left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}} \right)^{\cos^2 x} - \left(\frac{(n+2)^{n+1}}{(n+1)^n} \right)^{\cos^2 x} \right]$$

$$\text{Let } f: [n, n+1] \rightarrow \mathbb{R}, f(t) = \left(\frac{(t+2)^{t+1}}{(t+1)^t} \right)^{\cos^2 x}$$

From Lagrange's theorem we have: $\exists c \in (n, n+1)$ such that $f(n+1) - f(n) = f'(c) \Rightarrow$

$$L = \lim_{n \rightarrow \infty} n^{\sin^2 x} (f(n+1) - f(n)) = \lim_{n \rightarrow \infty} n^{\sin^2 x} f'(c) \quad (1)$$

$$f(t) = \left((t+2) \left(\frac{t+2}{t+1} \right)^t \right)^{\cos^2 x} = (t+2)^{\cos^2 x} \left(1 + \frac{1}{t+1} \right)^{t \cos^2 x}$$

$$f'(t) = \cos^2 x (t+2)^{\cos^2 x - 1} \left(1 + \frac{1}{t+1} \right)^{t \cos^2 x} + (t+2)^{\cos^2 x} \cdot$$

$$\cdot \left(\left(1 + \left(\frac{1}{t+1} \right) \right)^{t \cos^2 x} \left(\cos^2 x \cdot \ln \left(1 + \frac{1}{t+1} \right) \right) + \tan^2 x \cdot \frac{-\frac{1}{t+1}}{1 + \frac{1}{t+1}} \right) \Rightarrow$$

$$f'(t) = \cos^2 x \left(1 + \frac{1}{t+1} \right)^{t \cos^2 x} \left[(t+2)^{-\sin^2 x} + (t+2)^{\cos^2 x} \left(\ln \left(1 + \frac{1}{t+1} \right) - \frac{t}{(t+1)(t+2)} \right) \right] \quad (2)$$

From (1)+(2) and because $c \in (n, n+1)$ we must calculate:

$$L = \lim_{n \rightarrow \infty} n^{\sin^2 x} \cdot \cos^2 x \left(1 + \frac{1}{n+1} \right)^{\cos^2 x} \left[(n+2)^{-\sin^2 x} + (n+2)^{\cos^2 x} \left(\ln \left(1 + \frac{1}{n+1} \right) - \frac{n}{(n+1)(n+2)} \right) \right] \quad (3)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\lim_{n \rightarrow \infty} \cos^2 x \left(1 + \frac{1}{n+1}\right)^{n \cos^2 x} = \cos^2 x \cdot e^{\cos^2 x} \quad (4)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{n+2}{n}\right)^{\sin^2 x} + (n+2)^{\cos^2 x - 1} \cdot n \cdot n^{\sin^2 x} \left(\ln\left(1 + \frac{1}{n+1}\right) - \frac{n}{(n+1)(n+2)}\right) \\ = 1 + \lim_{n \rightarrow \infty} \left(\frac{n}{n+2}\right)^{\sin^2 x} \cdot \left(\frac{\ln\left(1 + \frac{1}{n+1}\right)}{\frac{1}{n}} - \frac{n}{(n+1)(n+2)}\right) = \\ 1 + 1(1-1) = 1 \quad (5) \end{aligned}$$

$$\text{From (3)+(4)+(5)} \Rightarrow L = \cos^2 x e^{\cos^2 x}$$

UP.142. Let $(x_n)_{n \geq 1}$ be a sequence which satisfy:

$$-\ln(mn + x_n) + \sum_{k=1}^{mn} \frac{1}{k} = \gamma$$

where m is positive integer and γ is Euler – Mascheroni's constant. Compute:

$$\lim_{n \rightarrow \infty} x_n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\begin{aligned} -\ln(mn + x_n) + \sum_{k=1}^{mn} \frac{1}{k} = \gamma \Rightarrow \lim_{n \rightarrow \infty} x_n = \lim_{x \rightarrow \infty} \left(e^{\left(\sum_{k=1}^{mn} \frac{1}{k} - \gamma\right)} - mn\right) = \\ = \lim_{n \rightarrow \infty} \left(e^{\left(\sum_{k=1}^{mn} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k} + \ln n\right)} - mn\right) = \lim_{n \rightarrow \infty} n \left(e^{\left(\sum_{k=n+1}^{mn} \frac{1}{k}\right)} - m\right) = \\ = \lim_{n \rightarrow \infty} \left(e^{\int_1^{md} \frac{1}{x} dx} - m\right) = \lim_{n \rightarrow \infty} n(e^{\ln m} - m) \therefore \lim_{n \rightarrow \infty} x_n = 0 \end{aligned}$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$\begin{aligned} -\ln(mn + x_n) + \sum_{k=1}^{k=mn} \frac{1}{k} = \gamma. \text{ Find } \lim_{n \rightarrow \infty} (x_n) \Rightarrow x_n + mn = \\ = e^{-\gamma + \sum_{k=1}^{k=mn} \frac{1}{k}} \left(\sum_{k=1}^{k=mn} \frac{1}{k}\right) - \ln(mn) + \ln(mn) - \gamma \end{aligned}$$

$$\therefore x_n + mn = (mn) e^{\left(\sum_{k=1}^{k=mn} \frac{1}{k}\right) - \ln(mn) - \gamma} \therefore x_n = (mn) \left(e^{\left(\sum_{k=1}^{k=mn} \frac{1}{k}\right) - \ln(mn) - \gamma} - 1\right)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} (x_n) &= \lim_{n \rightarrow \infty} \left(n \left(e^{\left(\sum_{k=1}^n \frac{1}{k} \right) - \ln(n) - \gamma} - 1 \right) \right) \therefore \lim_{n \rightarrow \infty} (x_n) = \lim_{n \rightarrow \infty} \left(n \left(e^{-\gamma + \int_0^1 \left(\frac{1-x^n}{1-x} + \frac{1-x^{n-1}}{\ln(x)} \right) dx} - 1 \right) \right) \\ &= \lim_{p \rightarrow 0} \frac{-1+e^{-\gamma + \int_0^1 \left(\frac{1-x^p}{1-x} + \frac{1-x^{p-1}}{\ln(x)} \right) dx}}{p}, \text{ using Lop-rulle} \\ \lim_{n \rightarrow \infty} (x_n) &= \lim_{p \rightarrow 0} \left(e^{-\gamma} \cdot e^{\int_0^1 \left(\frac{1-x^p}{1-x} + \frac{1-x^{p-1}}{\ln(x)} \right) dx} \cdot \int_0^1 \left(\frac{x^p \ln(x)}{p^2(1-x)} + \frac{x^{p-1}}{p^2} \right) dx \right) \\ &= \lim_{p \rightarrow 0} \int_0^1 \frac{x^p}{p^2} \left(\frac{\ln(x)}{1-x} + \frac{1}{x} \right) dx = \lim_{n \rightarrow \infty} \int_0^1 n^2 x^n \cdot \frac{\ln(x)}{1-x} dx + \lim_{n \rightarrow \infty} \int_0^1 n^2 \cdot x^{n-1} dx \\ &\text{Since } \lim_{n \rightarrow \infty} (n^2 x^n) = 0, \text{ and } \lim_{n \rightarrow \infty} (n^2 x^{n-1}) = 0, 1 > x > 0 \\ &\therefore \lim_{n \rightarrow \infty} (x_n) = 0 + 0 = 0 \Rightarrow \lim_{n \rightarrow \infty} (x_n) = 0 \end{aligned}$$

UP.143. Let $a, b \in \mathbb{R}_+, \gamma_n(a, b) = -\ln(n+a) + \sum_{k=1}^n \frac{1}{k+b}$ with

$$\lim_{n \rightarrow \infty} \gamma_n(a, b) = \gamma(a, b) \in \mathbb{R}$$

Calculate:

$$\lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \lim_{n \rightarrow \infty} n(\gamma_n(a, b) - \gamma(a, b)) &= \lim_{n \rightarrow \infty} \frac{\gamma_n(a, b) - \gamma(a, b)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\gamma_{n+1}(a+b) - \gamma_n(a, b)}{\frac{1}{n+1} - \frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{n+1} \frac{1}{k+b} - \ln(n+1+a) - \sum_{k=1}^n \frac{1}{k+b} + \ln(n+a)}{\frac{1}{n+1} - \frac{1}{n}} \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{n+a}\right) - \frac{1}{n+1+b}}{\frac{1}{(n+1)}} = \lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1}}{\frac{1}{n(n+1)}} = \\
 &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right) \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n}}{\frac{1}{n^2}} = \\
 &= \lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{(1+x) \ln(1+x) - x}{x^2} \stackrel{\text{L'HOSPITAL}}{=} \lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\ln(1+x) + 1 - 1}{2x} = \frac{1}{2} \lim_{\substack{x \rightarrow 0 \\ x > 0}} \ln \sqrt[3]{1+x} = \\
 &= \frac{\ln e}{2} = \frac{1}{2}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n = e^{b-a+\frac{1}{2}}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\begin{aligned}
 \text{Let } \Omega &= \lim_{n \rightarrow \infty} \left(\ln \left(\frac{e}{n+a} \right) + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n \Rightarrow \\
 \Rightarrow \ln \Omega &= \lim_{n \rightarrow \infty} n \ln \left(1 + \sum_{k=1}^n \frac{1}{k+b} - \ln(n+a) - \gamma(a, b) \right) \\
 &= \lim_{n \rightarrow \infty} \frac{\left(\sum_{k=1}^n \frac{1}{k+b} - \ln(n+a) - \gamma(a, b) \right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n+b+1} - \ln\left(1 + \frac{1}{n+a}\right)}{\frac{1}{n+1} - \frac{1}{n}} \right) = \\
 &= \lim_{n \rightarrow \infty} n(n+1) \left(\frac{1}{n+a} - \frac{1}{2(n+a)^2} + 0 \left(\frac{1}{(n+a)^3} \right) - \frac{1}{n+b+1} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{(2b-2a+1)n^2(n+1) + (2ab-2a^2+2a-b-1)n(n+1)}{2(n+a)^2(n+b+1)} \right) = b - a + \frac{1}{2} \\
 \therefore \Omega &= \lim_{n \rightarrow \infty} \left(\ln \left(\frac{e}{n+a} \right) + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n = e^{b-a+\frac{1}{2}}
 \end{aligned}$$

Solution 3 by Remus Florin Stanca-Romania

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Let $a, b \in \mathbb{R}_+$ $\gamma_n(a, b) = -\ln(a+n) + \sum_{k=1}^n \frac{1}{k+b}$ with $\lim_{n \rightarrow \infty} \gamma_n(a, b) = \gamma(a, b) \in \mathbb{R}$,

$$\begin{aligned}
 & \text{calculate: } \lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n \\
 & \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k+b} - \ln(n+a) \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{b+1} + \dots + \frac{1}{b+n} - \ln(n+a) \right) \\
 & = \lim_{n \rightarrow \infty} \left(1 + \dots + \frac{1}{b+n} - \ln(b+n) + \ln(b+n) - \ln(n+a) - \left(1 + \dots + \frac{1}{b} \right) \right) = \\
 & \quad \gamma - \left(1 + \dots + \frac{1}{b} \right) + \lim_{n \rightarrow \infty} \ln \frac{b+n}{n+a} = \gamma - 1 - \dots - \frac{1}{b} = \gamma(a, b) \\
 & \lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^n \frac{1}{k+b} - \gamma + 1 + \dots + \frac{1}{b} \right) = \\
 & = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^{b+n} \frac{1}{k} - \ln(b+n) + \ln(b+n) + \ln \frac{e}{n+a} - \gamma \right) = \lim_{n \rightarrow \infty} \ln \frac{en+be}{n+a} = 1 \\
 & \lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma \right)^n = \lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma - 1 + 1 \right)^n \\
 & = \lim_{n \rightarrow \infty} \left(\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma - 1 + 1 \right)^{\frac{1}{\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma - 1} \left(\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma - 1 \right) n} = \\
 & \quad = \lim_{n \rightarrow \infty} e^{n \left(\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma - 1 \right)} \\
 & \lim_{n \rightarrow \infty} \frac{\ln \frac{e}{n+a} + \sum_{k=1}^{b+n} \frac{1}{k} - \gamma - 1}{\frac{1}{n}} \stackrel{\text{Stolz Cesoro}}{=} \lim_{n \rightarrow \infty} \frac{\ln \frac{n+a}{n+a+1} + \frac{1}{b+n+1}}{-\frac{1}{n(n+1)}}
 \end{aligned}$$

Let $f: \mathbb{R} \setminus \{-1; 0; (-a-1); (b-1-1)\}$. Such that $f(x) = \frac{\ln \left(\frac{x+a}{x+a+1} \right) + \frac{1}{b+x+1}}{\frac{-1}{x(x+1)}}$

$$\lim_{n \rightarrow \infty} f(x) \stackrel{0}{=} \lim_{n \rightarrow \infty} \frac{\frac{x+a+1}{x+a} \cdot \frac{1}{(x+a+1)^2} - \frac{1}{(b+x+1)^2}}{\frac{1}{(x^2+x)^2} (2x+1)} = \lim_{n \rightarrow \infty} \frac{x^4 + 2x^3 + x^2}{2x+1}.$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{b^2 + x^2 + 1 + 2bx + 2b + 2x - x^2 - a^2 - 2ax - x - a}{b^2x^2 + b^2a^2 + 2ab^2x + b^2x + b^2a + x^4 + x^2a^2 + 2ax^3 + x^3 + x^2a + x^2 + a^2 + 2ax + x + a + 2bx^3 + 2bxa^2 + 4abx^2 + 2bx^2 + 2abx + 2bx^2 + 2ba^2 + 4abx + 2bx + 2ab + 2x^3 + 2xa^2 + 4ax^2 + 2x^2 + 2ax} = \frac{2b - 2a + 1}{2} \Rightarrow L = e^{\frac{2b-2a+1}{2}}$$

UP.144. If $x, y, z \geq 0$ then:

$$\cosh^2 x \cosh^2 y \cosh^2 z \geq 2(1 + \cosh(x - y) + \cosh(y - z) + \cosh(z - x)) \cdot \sinh \frac{x + y}{2} \sinh \frac{y + z}{2} \sinh \frac{z + x}{2}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

If $a, b \geq 0$ then: $(a^2 + 1)(b^2 + 1) = (a^2b^2 + 1) + a^2 + b^2 \geq a^2 + 2ab + b^2 = (a + b)^2$

$\prod(a^2 + 1)^2 = \prod(a^2 + 1)(b^2 + 1) \geq \prod(a + b)^2$ therefore $\prod(a^2 + 1) \geq \prod(a + b)$

We take $a = \sinh x, b = \sinh y, c = \sinh z \Rightarrow \prod \cosh^2 x = \prod(1 + \sinh^2 x) \geq$

$$\geq \prod(\sinh x + \sinh y) = \prod 2 \sinh \frac{x + y}{2} \cosh \frac{x - y}{2} =$$

$$= 2 \prod \sinh \frac{x + y}{2} \cdot (4 \prod \sinh \frac{x - y}{2}) = 2 \prod \sinh \frac{x + y}{2} (1 + \sum \cosh(x - y))$$

UP.145. Let be $(x_n)_{n \geq 1}, x_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$, such that exists

$\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = x \in \mathbb{R}_+^*$. Find:

$$\lim_{n \rightarrow \infty} \left(\frac{(n + 1)x_{n+1}}{\sqrt[n+1]{(2n + 1)!!}} - \frac{nx_n}{\sqrt[n]{(2n - 1)!!}} \right)$$

Proposed by Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Ruanghaw Chaokha-Chiangrai-Thailand

$$\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = x \Rightarrow L = \lim_{n \rightarrow \infty} \left(\frac{(n + 1)x_{n+1}}{\sqrt[n+1]{(2n + 1)!!}} - \frac{nx_n}{\sqrt[n]{(2n - 1)!!}} \right) = ??$$

Stolz-Cesaro; $\lim_{n \rightarrow \infty} (a_n - a_{n-1}) = \lim_{n \rightarrow \infty} \frac{a_n}{n}; a_n = \frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}$

$$L = \lim_{n \rightarrow \infty} \frac{(n + 1)x_{n+1}}{n \cdot \sqrt[n+1]{(2n + 1)!!}} = \lim_{n \rightarrow \infty} \frac{x_{n+1}}{\sqrt[n+1]{(2n + 1)!!}} \stackrel{n \rightarrow n-1}{=} \lim_{n \rightarrow \infty} \frac{x_n}{\sqrt[n]{(2n - 1)!!}}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Again, Stolz - Cesaro; $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$; $y_n = \sqrt[n]{(2n-1)!!}$

$$L = \lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{\sqrt[n]{(2n-1)!!} - \sqrt[n-1]{(2n-3)!!}} = \frac{\lim_{n \rightarrow \infty} (x_n - x_{n-1})}{\lim_{n \rightarrow \infty} (\sqrt[n]{(2n-1)!!} - \sqrt[n-1]{(2n-3)!!})} = \frac{x}{K} \rightarrow *$$

$$a_n = \sqrt[n]{(2n-1)!!} \Rightarrow K = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n^n}$$

And Stolz-Cesaro again; $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}}$; $a_n = \frac{(2n-1)!!}{n^n}$

$$\begin{aligned} K &= \lim_{n \rightarrow \infty} \frac{(2n-1)!!}{n^n} \cdot \frac{(n-1)^{n-1}}{(2n-3)!!} = \lim_{n \rightarrow \infty} \frac{(2n-1)}{(n-1)} \cdot \left(\frac{n-1}{n}\right)^n = \\ &= \lim_{n \rightarrow \infty} \frac{(2n-1)}{(n-1)} \cdot \left(1 - \frac{1}{n}\right)^n = 2e^{-1}; *; \therefore L = \frac{xe}{2} \end{aligned}$$

Solution 2 by Nassim Nicholas Taleb-USA

$$f = \frac{(n+1)}{n+1\sqrt{(2n+1)!!}} x_{n+1} - \frac{n}{n\sqrt{(2n-1)!!}} x_n$$

We write at the limit, with $a_n = \frac{(n+1)}{n+1\sqrt{(2n+1)!!}}$ and $a_{n+1} = \frac{n}{n\sqrt{(2n-1)!!}}$

$$\text{for } n \text{ large, } f \rightarrow f' = a_{n+1}x + (a_{n+1} - a_n)x_n$$

$$\lim_{n \rightarrow \infty} f' = \lim_{n \rightarrow \infty} a_{n+1}x + \lim_{n \rightarrow \infty} (a_{n+1} - a_n)x_n$$

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$$

$$\lim_{n \rightarrow \infty} f = x \lim_{n \rightarrow \infty} \frac{n}{(2n-1)!!} \frac{1}{n} = \frac{x}{2} e$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n\sqrt{(2n-1)!!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} \stackrel{\text{CAUCHY}}{\text{D'ALEMBERT}} = \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n^n} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{2^{n+1} \cdot (n+1)!} \cdot \frac{(2n)!}{2^n \cdot n!} \cdot \left(1 + \frac{1}{n}\right)^n \right) = e \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2 + \frac{1}{n}} = \frac{e}{2} \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} \frac{x_n}{n} &= \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{n+1 - n} = x, \lim_{n \rightarrow \infty} \left(\frac{x_n + 1}{x_n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{x_{n+1} - x_n}{x_n} \right)^n \\ &= \lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{x_{n+1} - x_n}{x_n} \right)^{\frac{x_n}{x_{n+1} - x_n}} \right\}^{\frac{x_{n+1} - x_n}{x_n}} = e^{\frac{1}{a} \cdot a} = e \end{aligned}$$

$$\text{Let } u_n = \frac{(n+1)x_{n+1}}{n^{n+1}\sqrt{(2n+1)!!}} \cdot \frac{n^{\sqrt{(2n-1)!!}}}{nx_n} \text{ for all } n \in \mathbb{N}$$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n^{n+1}\sqrt{(2n+1)!!}} \cdot \frac{x_{n+1}}{n+1} \cdot \frac{n^{\sqrt{(2n-1)!!}}}{n} \cdot \frac{n}{x_n} \cdot \left(1 + \frac{1}{n} \right) \right) = 1$$

$$\text{so, } \frac{u_n - 1}{\ln u_n} \rightarrow a \text{ as } n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{1}{n} \right)^n \cdot \left(\frac{x_{n+1}}{x_n} \right)^n \cdot \frac{n+1}{2n+1} \cdot \frac{n^{\sqrt{(2n+1)!!}}}{n+1} \right) = e$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{(n+1)x_{n+1}}{n^{n+1}\sqrt{(2n+1)!!}} - \frac{nx_n}{n^{\sqrt{(2n-1)!!}}} \right) = \lim_{n \rightarrow \infty} \left(\frac{n}{n^{\sqrt{(2n-1)!!}}} \cdot \frac{x_n}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n \right) = \frac{ex}{2}$$

UP.146. Let $f: (0, \infty) \rightarrow (0, \infty)$ be a function with:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = a \in (0, \infty) \text{ and } t \in \mathbb{R}. \text{ Find:}$$

$$\lim_{n \rightarrow \infty} \left((n+1)^{\sin^2 t} \cdot \sqrt[n+1]{(f(1)f(2) \dots f(n)f(n+1))^{\cos^2 t}} - n^{\sin^2 t} \cdot \sqrt[n]{(f(1)f(2) \dots f(n))^{\cos^2 t}} \right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = a \Rightarrow \lim_{n \in \mathbb{N}} \frac{f(n)}{n} = a \text{ let } u_n = \frac{(n+1)^{\sin^2 t} \sqrt[n+1]{(\prod_{k=1}^{n+1} f(k))^{\cos^2 t}}}{n^{\sin^2 t} \sqrt[n]{(\prod_{k=1}^n f(k))^{\cos^2 t}}} \text{ for all } n \in \mathbb{N}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{(\prod_{k=1}^n f(k))^{\cos^2 t}}}{n^{\cos^2 t}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(\prod_{k=1}^n f(k))^{\cos^2 t}}}{n^{n \cos^2 t}} =$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\stackrel{\text{CAUCHY}}{=} \stackrel{\text{D'ALEMBERT}}{=} \lim_{n \rightarrow \infty} \left(\left(\frac{f(n+1)}{n+1} \right)^{\cos^2 t} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^{n \cos^2 t}} \right) = \left(\frac{a}{e} \right)^{\cos^2 t}$$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\frac{n+1 \sqrt{\left(\prod_{k=1}^{n+1} f(k)\right)^{\cos^2 t}}}{(n+1)^{\cos^2 t}}}{\frac{n \sqrt{\left(\prod_{k=1}^n f(k)\right)^{\cos^2 t}}}{n^{\cos^2 t}}} \left(1 + \frac{1}{n}\right) = 1 \text{ so, } \frac{u_n - 1}{\ln u_n} \rightarrow 1 \text{ as } n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\left(\frac{f(n+1)}{n+1} \right)^{\cos^2 t} \cdot \frac{(n+1)^{\cos^2 t}}{\sqrt[n+1]{\left(\prod_{k=1}^{n+1} f(k)\right)^{\cos^2 t}}} \right) = e^{\cos^2 t}$$

$$\therefore \lim_{n \rightarrow \infty} \left((n+1)^{\sin^2 t} \sqrt[n+1]{\left(\prod_{k=1}^{n+1} f(k)\right)^{\cos^2 t}} - n^{\sin^2 t} \sqrt[n]{\left(\prod_{k=1}^n f(k)\right)^{\cos^2 t}} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\left(\frac{n \sqrt{\left(\prod_{k=1}^n f(k)\right)^{\cos^2 t}}}{n^{\cos^2 t}} \right) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n \right) = \left(\frac{a}{e} \right)^{\cos^2 t}$$

Solution 2 by Remus Florin Stanca-Romania

$$\begin{aligned} l &= \lim_{n \rightarrow \infty} n^{\sin^2 t} \sqrt[n]{\left(f(1) \cdot \dots \cdot f(n)\right)^{\cos^2 t}} \left(\left(\frac{n+1}{n}\right)^{\sin^2 t} \cdot \left(\frac{\sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)}}{\sqrt[n]{f(1) \cdot \dots \cdot f(n)}} \right) - 1 \right) \\ &= \lim_{n \rightarrow \infty} n \cdot \sqrt[n]{\left(\frac{f(1) \cdot \dots \cdot f(n)}{n^n}\right)^{\cos^2 t}} \cdot \left(\left(\frac{n+1}{n}\right)^{\tan^2 t} \cdot \frac{\sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)}}{\sqrt[n]{f(1) \cdot \dots \cdot f(n)}} \right)^{\cos^2 t} - 1 \end{aligned}$$

We know that $\lim_{n \rightarrow \infty} \left(\sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)} - \sqrt[n]{f(1) \cdot \dots \cdot f(n)} \right) = \frac{a}{e}$ and

$$\lim_{x \rightarrow 1} \frac{x^a - 1}{x - 1} = a$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{f(1) \cdot \dots \cdot f(n)}{n^n}} = \lim_{n \rightarrow \infty} e^{\frac{\ln f(1) \cdot \dots \cdot f(n)}{n^n}} = \lim_{n \rightarrow \infty} e^{\frac{\ln f(n+1)}{n+1}} = \frac{a}{e}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \cdot \left(\left(\frac{n+1}{n} \cdot \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}}{\sqrt[n]{\prod_{k=1}^n f(k)}} \right)^{\tan^2 t} - 1 \right) = \\ & \tan^2 t \cdot \lim_{n \rightarrow \infty} n \cdot \left(\frac{n+1}{n} \cdot \frac{\sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)}}{\sqrt[n]{f(1) \cdot \dots \cdot f(n)}} - 1 \right) = \\ & \tan^2 t \cdot \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1} \sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)} - n^n \sqrt[n]{f(1) \cdot \dots \cdot f(n)}}{\sqrt[n]{f(1) \cdot \dots \cdot f(n)}} = \\ & t = \tan^2 t \cdot \frac{e}{a} \cdot \left(\frac{a}{e} + \frac{a}{e} \right) = 2 \tan^2 t \\ & \lim_{n \rightarrow \infty} n \left(\left(\frac{\sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)}}{\sqrt[n]{f(1) \cdot \dots \cdot f(n)}} \right)^{\tan^2 t - 1} - 1 \right) = \\ & (\tan^2 t - 1) \lim_{n \rightarrow \infty} n \cdot \left(\frac{\sqrt[n+1]{f(1) \cdot \dots \cdot f(n+1)}}{\sqrt[n]{f(1) \cdot \dots \cdot f(n)}} - 1 \right) = (\tan^2 t - 1) \cdot \frac{e}{a} \cdot \frac{a}{e} = \tan^2 t - 1 \\ & \Rightarrow l_1 = \tan^2 t + 1 \Rightarrow l = \left(\frac{a}{e} \right)^{\cos^2 t} \cdot \cos^2 t \cdot (\tan^2 t + 1) = \left(\frac{a}{e} \right)^{\cos^2 t} \Rightarrow l = \left(\frac{a}{e} \right)^{\cos^2 t} \end{aligned}$$

Solution 3 by Nassim Nicholas Taleb – USA

We have $\sin(t)^2 = 1 - \cos^2(t) = \beta$, with $\beta \in [0, 1]$

$$g = (n+1)^\beta \left(\prod_{i=1}^{n+1} f(i) \right)^{\frac{1-\beta}{n+1}} - n^\beta \left(\prod_{i=1}^n f(i) \right)^{\frac{1-\beta}{n}}$$

Replacing $f(i)$ by a^i we have

$$g = (n+1)^\beta \left(a^{n+1} \Gamma(n+2) \right)^{\frac{1-\beta}{n+1}} - n^\beta \left(a^n \Gamma(n+1) \right)^{\frac{1-\beta}{n}}$$

Where $\Gamma(\cdot)$ is the (standard) gamma function.

By Stirling's formula, as n becomes large

$$\begin{aligned} \Gamma(n+2) & \rightarrow \sqrt{2\pi(n+1)} \left(\frac{n+1}{e} \right)^{n+1}, \quad \Gamma(n+1) \rightarrow \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \\ g & \rightarrow e^{\beta-1} a^{1-\beta} \left((n+1)^{\frac{-\beta+2n+3}{2n+2}} (2\pi)^{\frac{1-\beta}{2n+2}} - n^{\frac{-\beta+2n+1}{2n}} (2\pi)^{\frac{\beta-1}{2n}} \right) \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\left((n+1)^{\frac{-\beta+2n+3}{2n+2}} (2\pi)^{\frac{1-\beta}{2n+2}} - n^{\frac{-\beta+2n+1}{2n}} (2\pi)^{\frac{\beta-1}{2n}} \right) \rightarrow 1$$

Hence $\lim_{n \rightarrow \infty} g = e^{\beta-1} a^{1-\beta}$, with $\beta = \sin(t)^2$

Solution 4 by Aaditya Joshi-Mumbai-India

$$f: (0, \infty) \rightarrow (0, \infty); \lim_{x \rightarrow \infty} \frac{f(x)}{x} = a \text{ s.t. } a \in (0, \infty), t \in \mathbb{R}$$

$$\lim_{n \rightarrow \infty} \left\{ (n+1)^{\sin^2 t} \sqrt[n+1]{\left(\prod_{i=1}^{n+1} f(i) \right)^{\cos^2 t}} - n^{\sin^2 t} \sqrt[n]{\left(\prod_{i=1}^n f(i) \right)^{\cos^2 t}} \right\}$$

$$= \lim_{n \rightarrow \infty} \left\{ (n+1)^{\sin^2 t} \left(\prod_{i=1}^n f(i) \right)^{\frac{\cos^2 t}{n+1}} - n^{\sin^2 t} \left(\prod_{i=1}^n f(i) \right)^{\frac{\cos^2 t}{n}} \right\}$$

$$= \lim_{n \rightarrow \infty} \left\{ \underbrace{n^{\sin^2 t} \left(\prod_{i=1}^n f(i) \right)^{\frac{\cos^2 t}{n}}}_{nt_n} \left[\underbrace{\left(\frac{n+1}{n} \right)^{\sin^2 t} \left(\frac{\prod_{i=1}^{n+1} f(i)^{\frac{\cos^2 t}{n+1}}}{\prod_{i=1}^n f(i)^{\frac{\cos^2 t}{n}}} \right)}_{u_n} - 1 \right] \right\}$$

$$= \lim_{n \rightarrow \infty} \{ n \cdot t_n (u_n - 1) \}$$

$$\lim_{n \rightarrow \infty} \left\{ n^{\sin^2 t} \left(\prod_{i=1}^n f(i) \right)^{\frac{\cos^2 t}{n}} \right\} = \lim_{n \rightarrow \infty} \left\{ n^{1-\cos^2 t} \left(\prod_{i=1}^n f(i) \right)^{\frac{\cos^2 t}{n}} \right\} =$$

$$= \lim_{n \rightarrow \infty} \left\{ \underbrace{n \left(\prod_{i=1}^n \frac{f(i)}{n^n} \right)^{\frac{\cos^2 t}{n}}}_{\parallel} \right. \\ \left. \underbrace{\frac{n \sqrt[n]{\left(\prod_{i=1}^n f(i) \right)^{\cos^2 t}}}{n^n \cos^2 t}}_{\rightarrow t_n} \right\}$$

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \left(\left[\frac{f(n+1)}{n+1} \right]^{\cos^2 t} \times \frac{1}{\left(\frac{n+1}{n} \right)^{n \cos^2 t}} \right) \rightarrow \text{Cauchy - D'Alembert}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \lim_{n \rightarrow \infty} \left(\left[\frac{f(n+1)}{n+1} \right]^{\cos^2 t} \times \left(1 + \frac{1}{n} \right)^{-n \cos^2 t} \right) = \left(\frac{a}{e} \right)^{\cos^2 t}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left\{ \frac{(n+1)^{\sin^2 t} \left(\prod_{i=1}^{n+1} f(i) \right)^{\frac{\cos^2 t}{n+1}}}{n^{\sin^2 t} \left(\prod_{i=1}^n f(i) \right)^{\frac{\cos^2 t}{n}}} \right\}$$

$$= \lim_{n \rightarrow \infty} \left\{ \frac{\sqrt[n+1]{[f(1)f(2) \dots f(n+1)]^{\cos^2 t} \cdot (n+1)^{1-\cos^2 t}}}{\sqrt[n]{[f(1)f(2) \dots f(n)]^{\cos^2 t} \cdot n^{1-\cos^2 t}}} \right\}$$

$$= \lim_{n \rightarrow \infty} \left\{ \frac{\sqrt[n+1]{[f(1)f(2) \dots f(n)f(n+1)]^{\cos^2 t}}}{\sqrt[n]{[f(1)f(2) \dots f(n)]^{\cos^2 t}}} \times \frac{n^{\cos^2 t - 1}}{(n+1)^{\cos^2 t - 1}} \right\}$$

$$= \lim_{n \rightarrow \infty} \left\{ \frac{\left(\prod_{i=1}^{n+1} f(i) \right)^{\cos^2 \frac{t}{n+1}}}{\left(\prod_{i=1}^n f(i) \right)^{\cos^2 \frac{t}{n}}} \times \frac{n^{\cos^2 t}}{(n+1)^{\cos^2 t}} \times \frac{n+1}{n} \right\}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(p_n \cdot \frac{n+1}{n} \right)$$

$$\lim_{n \rightarrow \infty} (n \cdot [u_n - 1]) = \lim_{n \rightarrow \infty} \left(n \cdot \left[p_n \cdot \frac{n+1}{n} - 1 \right] \right) = \lim_{n \rightarrow \infty} (p_n(n+1) - n)$$

$$\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} \left\{ \frac{\left(\prod_{i=1}^{n+1} f(i) \right)^{\cos^2 \frac{t}{n+1}}}{\left(\prod_{i=1}^n f(i) \right)^{\cos^2 \frac{t}{n}}} \times \frac{n^{\cos^2 t}}{(n+1)^{\cos^2 t}} \right\}$$

$$\therefore \lim_{n \rightarrow \infty} p_n = 1 \quad \therefore \lim_{n \rightarrow \infty} (p_n(n+1) - n) = 1$$

$$\therefore \lim_{n \rightarrow \infty} \left\{ (n+1)^{\sin^2 t} \sqrt[n+1]{\prod_{i=1}^{n+1} f(i)^{\cos^2 t}} - n^{\sin^2 t} \sqrt[n]{\prod_{i=1}^n f(i)^{\cos^2 t}} \right\}$$

$$= \lim_{n \rightarrow \infty} \{ n \cdot t_n \cdot (u_n - 1) \} = \left(\frac{a}{e} \right)^{\cos^2 t}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

UP.147. In an ABC triangle let be a, b, c the lengths of BC, CA, AB , and r_a, r_b, r_c exradii.

Prove that:

$$\frac{r_a^2}{\tan \frac{B}{2} \tan \frac{C}{2}} + \frac{r_b^2}{\tan \frac{C}{2} \tan \frac{A}{2}} + \frac{r_c^2}{\tan \frac{A}{2} \tan \frac{B}{2}} \geq \frac{9(a^2 + b^2 + c^2)}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

* Let a, b, c be the lengths BC, CA, AB of ΔABC . We have:

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\sqrt{\frac{1 - \cos A}{2}}}{\sqrt{\frac{1 + \cos A}{2}}} = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{1 - \frac{b^2 + c^2 - a^2}{2bc}}{1 + \frac{b^2 + c^2 - a^2}{2bc}}} = \sqrt{\frac{a^2 - (b - c)^2}{(b + c)^2 - a^2}} = \sqrt{\frac{a^2 - (b - c)^2}{(b + c)^2 - a^2}}$$

$$\Leftrightarrow \tan \frac{A}{2} = \sqrt{\frac{(a-b+c)(a+b-c)}{(b+c+a)(b+c-a)}}. \text{ Similar: } \tan \frac{B}{2} = \sqrt{\frac{(b-c+a)(b+c-a)}{(c+a+b)(c+a-b)}}$$

$$\Rightarrow \tan \frac{A}{2} \tan \frac{B}{2} = \sqrt{\frac{(a-b+c)(a+b-c)}{(b+c+a)(b+c-a)}} \cdot \sqrt{\frac{(b-c+a)(b+c-a)}{(c+a+b)(c+a-b)}} = \frac{a+b-c}{a+b+c} \quad (2)$$

- Other, let $p = \frac{a+b+c}{2}$ is half circumference of ΔABC and S its area. By Heron's

formula we have: $S = \sqrt{p(p-a)(p-b)(p-c)} \Leftrightarrow 4S = \sqrt{2p(2p-2a)(2p-2b)(2p-2c)} \Leftrightarrow$

$$\Leftrightarrow 4S = \sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)} \Leftrightarrow 4S^2 = \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4}$$

- Therefore (2), by the radius formula of the circle in a triangle:

$$\Rightarrow \frac{r_c^2}{\tan \frac{A}{2} \tan \frac{B}{2}} = \frac{\left(\frac{2S}{a+b-c}\right)^2}{\frac{a+b-c}{a+b+c}} = \frac{4S^2}{(a+b-c)^2} = \frac{\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4}}{\frac{(a+b-c)^2}{a+b+c}}$$

$$\Leftrightarrow \frac{r_c^2}{\tan \frac{A}{2} \tan \frac{B}{2}} = \frac{(a+b+c)^2(b+c-a)(c+a-b)}{4(a+b-c)^2} \quad (3)$$

$$+ \text{ Let: } \begin{cases} b+c-a=2x \\ c+a-b=2y \\ a+b-c=2z \end{cases}; (x, y, z > 0) \Leftrightarrow \begin{cases} a=y+z \\ b=z+x \\ c=x+y \end{cases} \Rightarrow a+b+c=2(x+y+z)$$

$$- \text{ Let (3): } \Rightarrow \frac{r_c^2}{\tan \frac{A}{2} \tan \frac{B}{2}} = \frac{(2(x+y+z))^2 \cdot 2x \cdot 2y}{4(2z)^2} = \frac{(x+y+z)^2 \cdot xy}{z^2} \quad (4)$$

$$+ \text{ Similar: } \frac{r_b^2}{\tan \frac{C}{2} \tan \frac{A}{2}} = \frac{(x+y+z)^2 \cdot zx}{y^2}; \frac{r_a^2}{\tan \frac{B}{2} \tan \frac{C}{2}} = \frac{(x+y+z)^2 \cdot yz}{x^2} \quad (5)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

- Let (4), (5):

$$\Rightarrow \frac{r_a^2}{\tan \frac{B}{2} \tan \frac{C}{2}} + \frac{r_b^2}{\tan \frac{C}{2} \tan \frac{A}{2}} + \frac{r_c^2}{\tan \frac{A}{2} \tan \frac{B}{2}} = \frac{(x+y+z)^2 \cdot yz}{x^2} + \frac{(x+y+z)^2 \cdot zx}{y^2} + \frac{(x+y+z)^2 \cdot xy}{z^2} \quad (6)$$

- Other: $\frac{9(a^2+b^2+c^2)}{4} = \frac{9}{4}((y+z)^2 + (z+x)^2 + (x+y)^2) = \frac{9(x^2+y^2+z^2+xy+yz+zx)}{2} \quad (7)$

* Let (1), (6), (7). We need to prove that:

$$\frac{(x+y+z)^2 \cdot yz}{x^2} + \frac{(x+y+z)^2 \cdot zx}{y^2} + \frac{(x+y+z)^2 \cdot xy}{z^2} \geq \frac{9(x^2+y^2+z^2+xy+yz+zx)}{2} \quad (8)$$

- By inequality: $(m+n+p)^2 \geq 3(mn+np+pm)$. We have:

$$\begin{aligned} & \frac{(x+y+z)^2 \cdot yz}{x^2} + \frac{(x+y+z)^2 \cdot zx}{y^2} + \frac{(x+y+z)^2 \cdot xy}{z^2} \geq \\ & \geq \frac{3(xy+yz+zx) \cdot yz}{x^2} + \frac{3(xy+yz+zx) \cdot zx}{y^2} + \frac{3(xy+yz+zx) \cdot xy}{z^2} \\ \Leftrightarrow & \frac{(x+y+z)^2 \cdot yz}{x^2} + \frac{(x+y+z)^2 \cdot zx}{y^2} + \frac{(x+y+z)^2 \cdot xy}{z^2} \geq 3 \left(\frac{x^2 y^2}{z^2} + \frac{y^2 z^2}{x^2} + \frac{z^2 x^2}{y^2} + \frac{xy(x+y)}{z} + \frac{yz(y+z)}{x} + \frac{zx(z+x)}{y} \right) \quad (9) \end{aligned}$$

- By AM-GM inequality for 2 positive real numbers:

$$\begin{aligned} & \frac{x^2 y^2}{z^2} + \frac{y^2 z^2}{x^2} + \frac{z^2 x^2}{y^2} + \frac{xy(x+y)}{z} + \frac{yz(y+z)}{x} + \frac{zx(z+x)}{y} = \\ & = \frac{x^2 \left(\frac{y^2}{z^2} + \frac{z^2}{y^2} \right)}{2} + \frac{y^2 \left(\frac{z^2}{x^2} + \frac{x^2}{z^2} \right)}{2} + \frac{z^2 \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} \right)}{2} + x^2 \left(\frac{y}{z} + \frac{z}{y} \right) + y^2 \left(\frac{z}{x} + \frac{x}{z} \right) + z^2 \left(\frac{x}{y} + \frac{y}{x} \right) \geq \\ & \geq \frac{x^2 \cdot 2 \sqrt{\frac{y^2}{z^2} \cdot \frac{z^2}{y^2}}}{2} + \frac{y^2 \cdot 2 \sqrt{\frac{z^2}{x^2} \cdot \frac{x^2}{z^2}}}{2} + \frac{z^2 \cdot 2 \sqrt{\frac{x^2}{y^2} \cdot \frac{y^2}{x^2}}}{2} + x^2 \cdot 2 \sqrt{\frac{y}{z} \cdot \frac{z}{y}} + y^2 \cdot 2 \sqrt{\frac{z}{x} \cdot \frac{x}{z}} + z^2 \cdot 2 \sqrt{\frac{x}{y} \cdot \frac{y}{x}} \\ & = x^2 + y^2 + z^2 + 2x^2 + 2y^2 + 2z^2 = 3(x^2 + y^2 + z^2) \\ \Leftrightarrow & \frac{x^2 y^2}{z^2} + \frac{y^2 z^2}{x^2} + \frac{z^2 x^2}{y^2} + \frac{xy(x+y)}{zx} + \frac{yz(y+z)}{x} + \frac{zx(z+x)}{y} \geq 3(x^2 + y^2 + z^2) \quad (10) \end{aligned}$$

- Let (9), (10):

$$\Rightarrow \frac{(x+y+z)^2 \cdot xy}{z^2} + \frac{(x+y+z)^2 \cdot zx}{y^2} + \frac{(x+y+z)^2 \cdot yz}{x^2} \geq 3 \cdot 3(x^2 + y^2 + z^2) = 9(x^2 + y^2 + z^2) \quad (11)$$

- Other: $m^2 + n^2 + p^2 \geq mn + np + pm$. Therefore:

$$x^2 + y^2 + z^2 = \frac{(x^2+y^2+z^2)+(x^2+y^2+z^2)}{2} \geq \frac{x^2+y^2+z^2+xy+yz+zx}{2} \quad (12)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

- Let (11), (12):

$$\Rightarrow \frac{(x+y+z)^2 \cdot xy}{z^2} + \frac{(x+y+z)^2 \cdot zx}{y^2} + \frac{(x+y+z)^2 \cdot yz}{x^2} \geq \frac{9(x^2 + y^2 + z^2 + xy + yz + zx)}{2}$$

\Rightarrow Inequality (8) True. Therefore (1) true and we get the result.

+ Equality occurs if: $x = y = z \Leftrightarrow b + c - a = c + a - b = a + b - c \Leftrightarrow a = b = c \Leftrightarrow \Delta ABC$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{r_a^2}{\tan \frac{B}{2} \tan \frac{C}{2}} = \sum \frac{r_a^3}{s \prod \tan \frac{A}{2}} = \frac{\sum r_a^3}{s \left(\frac{r}{4R}\right) \left(\frac{4R}{s}\right)} = \frac{\sum r_a^3}{r} \geq \frac{9 \sum a^2}{4} \Leftrightarrow 4 \sum r_a^3 \stackrel{(1)}{\geq} 9r \sum a^2$$

$$\begin{aligned} \text{Now, } \sum r_a^3 &= (\sum r_a)^3 - 3 \prod (r_a + r_b) = (4R + r)^3 - 3(2r_a r_b r_c + \sum r_a r_b (\sum r_a - r_c)) = \\ &= (4R + r)^3 - 3 \left((4R + r)s^2 - rs^2 \right) \stackrel{(2)}{=} (4R + r)^3 - 12Rs^2 \end{aligned}$$

$$\text{Now, RHS of (1)} \stackrel{\text{Leibniz}}{\leq} 81R^2 r \stackrel{?}{\leq} 4 \sum r_a^3 \Leftrightarrow 4(4R + r)^3 - 48Rs^2 \stackrel{?}{\geq} 81R^2 r \text{ (by (2))}$$

$$\Leftrightarrow 4(4R + r)^3 - 81R^2 r \stackrel{?}{\geq} 48s^2. \text{ Now, RHS of (3)} \stackrel{\text{Gerretsen}}{\leq} 48R(4R^2 + 4Rr + 3r^2)$$

$$\stackrel{?}{\leq} 4(4R + r)^3 - 81R^2 r \Leftrightarrow 64t^3 - 81t^2 - 96t + 4 \stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(64t^2 + 47t - 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \text{ (Proved)}$$

UP.148. Let a, b, c be positive real numbers such that: $a + b + c = 3$.

Prove that:

$$2(a^2 + b^2 + c^2) + 3 \geq 3\sqrt{3abc(a^3b + b^3c + c^3a)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursarescu-Romania

We use Vasc's inequality: $\forall a, b, c \in \mathbb{R} \Rightarrow (a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^2a) \Rightarrow$

$$\Rightarrow \sqrt{3(a^3b + b^3c + c^3a)} \leq a^2 + b^2 + c^2 \Rightarrow \text{inequality becomes:}$$

$$2(a^2 + b^2 + c^2) + 3 \geq 3\sqrt{abc}(a^2 + b^2 + c^2) \quad (1)$$

$$\left. \begin{aligned} \sqrt{abc} &\leq \frac{a+bc}{2} \Rightarrow \sqrt{abc}(a^2 + b^2 + c^2) \leq \frac{(a+bc)}{2}(a^2 + b^2 + c^2) \\ \sqrt{abc} &\leq \frac{b+ac}{2} \Rightarrow \sqrt{abc}(a^2 + b^2 + c^2) \leq \frac{(b+ac)}{2}(a^2 + b^2 + c^2) \end{aligned} \right\} \Rightarrow$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sqrt{abc} \leq \frac{c+ab}{2} \Rightarrow \sqrt{abc}(a^2+b^2+c^2) \leq \frac{(c+ab)}{2}(a^2+b^2+c^2)$$

$$3\sqrt{abc}(a^2+b^2+c^2) \leq \frac{(a^2+b^2+c^2)}{3}(3+ab+ac+bc) \quad (2)$$

From (1)+(2) we must show: $2(a^2+b^2+c^2)+3 \geq \frac{a^2+b^2+c^2}{2}(3+ab+ac+bc) \Leftrightarrow$

$$4(a^2+b^2+c^2)+6 \geq 3(a^2+b^2+c^2)+(a^2+b^2+c^2)(ab+ac+bc) \Leftrightarrow$$

$$a^2+b^2+c^2+6 \geq (a^2+b^2+c^2)(ab+ac+bc) \quad (3)$$

Because $a, b, c > 0$ such that $a+b+c=3 \Rightarrow \exists x, y, z > 0$ such that: $a = \frac{3x}{x+y+t}$

$$b = \frac{3y}{x+y+t}, c = \frac{3z}{x+y+t}. \text{ Inequality (3) becomes: } 9 \frac{(x^2+y^2+z^2)}{(x+y+t)^2} + 6 \geq \frac{9(x^2+y^2+z^2)}{(x+y+z)^2} \cdot \frac{9(xy+xz+yt)}{(x+y+z)^2}$$

$$\Leftrightarrow \frac{3(x^2+y^2+z^2)}{(x+y+z)^2} + 2 \geq \frac{27(x^2+y^2+z^2)(xy+xz+yz)}{(x+y+z)^4} \Leftrightarrow$$

$$\Leftrightarrow 3(x^2+y^2+z^2)(x+y+z)^2 + 2(x+y+z)^4 \geq 27(x^2+y^2+z^2)(xy+xz+yz) \quad (4)$$

Now, using Cartoaje's theorem: If $f_4(x, y, z)$ is an symmetric polynomial function of degree $n = 4$ then: $f_4(x, y, z) \geq 0, \forall x, y, z \geq 0 \Leftrightarrow x = 0$ and $y = z$

(if and only if)

$$\begin{aligned} \text{In our case let } f_4(x, y, z) &= 3(x^2+y^2+z^2)(x+y+z)^2 + 2(x+y+z)^4 = \\ &= 2t(x^2+y^2+z^2)(xy+zx+yz) \end{aligned}$$

$$f_4(0, y, y) = 3 \cdot 2y^2 \cdot 4y^2 + 2 \cdot 2^4 y^4 - 27 \cdot 2y^2 \cdot y^2 =$$

$$24y^4 + 32y^4 - 54y^4 = 2y^4 \geq 0 \Rightarrow \text{inequality (4) its true.}$$

Solution 2 by Michael Sterghiou-Greece

$$2(\sum_{cyc} a^2) + 3 \geq 3(3abc \cdot \sum_{cyc} a^3 b)^{\frac{1}{2}} \quad (1)$$

$$\text{Let } (\sum_{cyc} a, \sum_{cyc} ab, abc) = (p, q), p = 3, \sum_{cyc} a^2 = 9 - 2q$$

$$r \leq \left(\frac{a}{3}\right)^{\frac{3}{2}} \rightarrow \sqrt{r} \leq \left(\frac{a}{3}\right)^{\frac{3}{4}}. \text{ From Vasc's inequality:}$$

$$(\sum_{cyc} a^2)^2 \geq 3 \sum_{cyc} a^3 b \quad (1) \rightarrow f(q) = 2^4 \sqrt{3} q^{\frac{7}{4}} - 9^4 \sqrt{3} a^{\frac{3}{4}} - 4q + 21 \geq 0 \quad (2)$$

Where in the original $21 - 4q - 3\sqrt{r}(9 - 2q) \geq 0$ we've replaced \sqrt{r} by $\left(\frac{a}{3}\right)^{\frac{3}{4}}$ to get

the stronger inequality (2). Note that $9 - 2q \geq 0, q \leq 3$. Now $f(3) = 0$,

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f'(q) = \frac{7}{2} \sqrt[4]{3} q^{\frac{3}{4}} - \frac{27 \sqrt[4]{3}}{4 \sqrt[4]{q}} - 4, f''(q) = \frac{33^{\frac{1}{4}} (14q+9)}{16q^{\frac{5}{4}}} \geq 0. \text{ So, } f'(q) \uparrow \rightarrow f'(q) < f'(3) = -\frac{1}{4} < 0 \rightarrow f(q) \downarrow \rightarrow f(q) > f(3) = 0 \text{ as } q \leq 3. \text{ Done.}$$

UP.149. Prove that:

$$\begin{aligned} & \sum_{k=-l}^l [(-1)^k \binom{2l}{l+k} \binom{2m}{m+k} \binom{2n}{n+k}] = \\ & = \frac{(l+m+n)! (2l)! (2m)! (2n)!}{(l+m)! (l+n)! (m+n)! (l)! (m)! (n)!} \end{aligned}$$

Proposed by Shivam Sharma – New Delhi – India

Solution by proposer

$$S = \sum_{k=-l}^l [(-1)^k \binom{2l}{l+k} \binom{2m}{m+k} \binom{2n}{n+k}]$$

Assuming that, $l = \min(l, m, n)$. This reduces to the series,

$$\frac{(-1)^l (2m)! (2n)!}{(m-l)! (m+l)! (n-l)! (n+l)!} {}_3F_2 \left(\begin{matrix} -2l, -m-l, -n-l \\ m-l+1, n-l+1 \end{matrix}; 1 \right)$$

Now, Applying Dixon's formula, we get $\Rightarrow {}_3F_2 \left(\begin{matrix} -2l-2\varepsilon, -m-l-\varepsilon, -n-l-\varepsilon \\ m-l-\varepsilon+1, n-l-\varepsilon+1 \end{matrix}; 1 \right)$

$$= \frac{\Gamma(1-l-\varepsilon) \Gamma(1+m-l-\varepsilon) \Gamma(1+m+n+l+\varepsilon)}{\Gamma(1-2l-2\varepsilon) \Gamma(1+m) \Gamma(1+n) \Gamma(1+m+n)}$$

Now, apply Euler's reflection formula, we get,

$$\frac{\sin \pi(2l+2\varepsilon)}{\sin \pi(l+\varepsilon)} \cdot \frac{\Gamma(2l+2\varepsilon)}{\Gamma(l+\varepsilon)} \cdot \frac{\Gamma(1+m-l-\varepsilon) \Gamma(1+n-l-\varepsilon) \Gamma(1+m+n+l+\varepsilon)}{\Gamma(1+m) \Gamma(1+n) \Gamma(1+m+n)}$$

As the limit $\varepsilon \rightarrow 0$, this expression gives $\Rightarrow 2(-1)^l \frac{(2l-1)!}{(l-1)!} \cdot \frac{(m-l)!(n-l)!(m+n+l)!}{m!n!(m+n)!}$

$$\text{(or) } S = \frac{(l+m+n)! (2l)! (2m)! (2n)!}{(l+m)! (l+n)! (m+n)! l! m! n!} \text{ (Answer)}$$

UP.150. Determine all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + xy \text{ and } f(1) = 1 \text{ for all } x, y \in \mathbb{R}.$$

Proposed by Mihály Bencze – Romania

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by proposer

If $x = y = 0$ then $f(0) = 0$. By induction holds $f(nr) = nf(r) + \frac{n(n-1)}{2}r^2$

for all $r \in \mathbb{R}$. If $r = 1$ then $f(n) = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$. If $r = -1$ then

$$0 = f(0) = f(1-1) = f(1) + f(-1) - 1 \Rightarrow f(-1) = 0 \Rightarrow f(-n) = \frac{(-n)(-n+1)}{2}$$

for all $n \in \mathbb{N}$. If $r = \frac{p}{q} \in \mathbb{Q}$ then $f(qr) = f\left(q \cdot \frac{p}{q}\right) = f(1) = qf\left(\frac{p}{q}\right) + \frac{q(q-1)}{2}\left(\frac{p}{q}\right)^2$

($p, q \in \mathbb{N}^*$) but $f(p) = \frac{p(r+1)}{2} \Rightarrow f\left(\frac{p}{q}\right) = \frac{1}{2} \cdot \frac{p}{q}\left(\frac{p}{q} + 1\right)$. Let $x_n \rightarrow r \in \mathbb{R} \setminus \mathbb{Q}, x_n \in \mathbb{Q}$

f is continuously $\Rightarrow f(x_n) = \frac{1}{2}x_n(x_{n+1})$

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \frac{1}{2}x_n(x_{n+1}) \Rightarrow f(r) = \frac{r(r+1)}{2}$$

And finally $f(x) = \frac{x(x+1)}{2}, \forall x \in \mathbb{R}$.

Solution 2 by Rovsen Pirgulyev-Sumgait-Azerbaijan

$$f(x+y) = f(x) + f(y) + xy; y = 1 \stackrel{f(1)=1}{\Rightarrow} f(x+1) - f(x) = x + 1 \quad (1)$$

$$x = 1, 2 \dots n-1 \text{ we have } \begin{cases} f(2) - f(1) = 2 \\ f(3) - f(2) = 3 \\ \dots \\ f(n) - f(n-1) = n \end{cases} \oplus \Rightarrow f(n) = \frac{n(n+1)}{2}$$

then the function $f(x) = \frac{x(x+1)}{2}$ is a solution of equation $f(x+y) = f(x) + f(y) + xy$ (2)

denote $g(x) = f(x) - f_1(x)$, then using (2), we have: $g(x+y) = f(x+y) - f_1(x+y) = f(x) + f(y) + xy - f_1(x) - f_1(y) - xy = \underbrace{(f(x) - f_1(x))}_{g(x)} + \underbrace{(f(y) - f_1(y))}_{g(y)} = g(x) + g(y)$ (3)

the function $g(x)$ is continuous, since $f(x)$ is continuous.

$$g(x+y) = g(x) + g(y) \stackrel{\text{Cauchy func. equation}}{\Rightarrow} g(x) = kx, g(x) = f(x) - f_1(x) \Rightarrow$$

$$\Rightarrow kx = f(x) - \frac{x(x+1)}{2} \Rightarrow f(x) = \frac{x^2}{2} + \frac{x}{2} + kx = \frac{x^2}{2} + \left(k + \frac{1}{2}\right)x$$

using $f(1) = 1 \Rightarrow k = 0$. Hence $f(x) = \frac{x^2+x}{2}$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 3 by Tran Hong-Vietnam

$$x = y \Rightarrow f(2x) = 2f(x) + x^2 \quad (\forall x \in \mathbb{R}) \Leftrightarrow f(2x) - \frac{2x(2x+1)}{2} = 2 \left[f(x) - \frac{x(x+1)}{2} \right]$$

$$\text{Let } g(x) = f(x) - \frac{x(x+1)}{2} \Rightarrow g(1) = f(1) - 1 = 0. \text{ We have: } g(2x) = 2g(x) \quad (\forall x \in \mathbb{R})$$

$$* \forall x \neq 0 \Rightarrow \frac{g(2x)}{2x} = \frac{g(x)}{x}. \text{ Let } h(x) = \frac{g(x)}{x} \quad (\forall x \neq 0)$$

$$\Rightarrow h(2x) = h(x) \Rightarrow h(x) = h\left(\frac{x}{2}\right) = \dots = h\left(\frac{x}{2^n}\right) \quad (n \in \mathbb{N})$$

$$\Rightarrow h(x) = \lim_{n \rightarrow \infty} h(x) = h\left(\lim_{n \rightarrow \infty} \frac{x}{2^n}\right) = h(0) \Rightarrow g(x) = h(0)x \quad (\forall x \neq 0)$$

$$\text{More } g(1) = 0 \Rightarrow h(0) \Rightarrow g(x) = 0 \quad (\forall x \neq 0) \quad (1)$$

$$* x = 0 \Rightarrow g(2 \cdot 0) = 2g(0) \Rightarrow g(0) = 2 \quad (2)$$

$$\text{From (1) and (2)} \Rightarrow g(x) = 0 \quad \forall x \in \mathbb{R} \Rightarrow f(x) = \frac{x(x+1)}{2} \quad (\forall x \in \mathbb{R})$$