ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\sum_{k=1}^{\infty}\frac{k^2}{(2k+1)^3\binom{2k}{2k+1}}$$

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$$\Omega=\sum_{k=1}^{\infty}\frac{k^2}{(2k+1)^3\left({}^{2k}C_{k+1}\right)}=\sum_{k=0}^{\infty}\frac{k^2\,(k-1)!\,(k+1)!}{(2k+1)^3\,(2k)!}=\sum_{k=0}^{\infty}\frac{k(k+1)\,k!\,k!}{(2k+1)^2\,(2k+1)!}=$$

$$=\sum_{k=0}^{\infty}\frac{k(k+1)\,\Gamma(k+1)\Gamma(k+1)}{(2k+1)^2\,\Gamma(2k+2)}=\sum_{k=0}^{\infty}\frac{k(k+1)}{(2k+1)^2}B(k+1,k+1)$$

$$Now,\quad \frac{k(k+1)}{(2k+1)^2}=\frac{1}{4}-\frac{1}{4(2k+1)^2}\qquad and\quad B(k+1,k+1)=\int_0^1 t^k(1-t)^k\,dt$$

$$\text{Therefore,}\ \ \Omega=\int_0^1\sum_{k=0}^{\infty}\left(\frac{1}{4}-\frac{1}{4(2k+1)^2}\right)t^k(1-t)^k dt$$

$$=\frac{1}{4}\int_0^1\sum_{k=0}^{\infty}t^k(1-t)^k dt-\frac{1}{4}\int_0^1\sum_{k=0}^{\infty}\frac{t^k(1-t)^k}{(2k+1)^2}dt\qquad Now\ use,\frac{1}{(2k+1)^2}$$

$$=-\int_0^1(\ln x)\,x^{2k}dx$$

$$\Rightarrow\ \Omega=\frac{1}{4}\int_0^1\frac{dt}{1-t(1-t)}+\frac{1}{4}\int_0^1\int_0^1\sum_{k=0}^{\infty}(\ln x)\,x^{2k}t^k(1-t)^k dxdt$$

$$=\frac{1}{4}\int_0^1\frac{dt}{\frac{3}{4}+\left(t-\frac{1}{2}\right)^2}+\frac{1}{4}\int_0^1\int_0^1\frac{\ln x}{1-x^2t(1-t)}dtdx$$

$$use\quad \int_0^1\frac{dt}{\frac{3}{4}+\left(t-\frac{1}{2}\right)^2}=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\right]_0^1=\frac{2\pi}{3\sqrt{3}}$$

$$\int_0^1\frac{dt}{1-x^2t(1-t)}=\frac{1}{x^2}\int_0^1\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{x^2}-\frac{1}{4}\right)}=\frac{1}{x^2}\frac{1}{\sqrt{\frac{1}{x^2}-\frac{1}{4}}}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{\frac{1}{x^2}-\frac{1}{4}}}\right)\right]_0^1$$

$$= \frac{2}{x^2\sqrt{\frac{1}{x^2}-\frac{1}{4}}} tan^{-1}\left(\frac{\frac{1}{2}}{\sqrt{\frac{1}{x^2}-\frac{1}{4}}}\right) = \frac{4}{x\sqrt{4-x^2}} tan^{-1}\left(\frac{x}{\sqrt{4-x^2}}\right) = \frac{4}{x\sqrt{4-x^2}} sin^{-1}\left(\frac{x}{2}\right)$$

Therefore, $\Omega = \frac{\pi}{6\sqrt{3}} + \int_0^1 \frac{lnx}{x\sqrt{4-x^2}} sin^{-1}\left(\frac{x}{2}\right) dx$ $substitute\ \theta = sin^{-1}\left(\frac{x}{2}\right)\ or\ x = 2sin\theta$

$$= \frac{\pi}{6\sqrt{3}} + \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{\theta\ln(2sin\theta)}{\sin\theta} d\theta = \frac{\pi}{6\sqrt{3}} + \frac{ln2}{2}\int_0^{\frac{\pi}{6}} \frac{\theta}{\sin\theta} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{\theta\ln(sin\theta)}{\sin\theta} d\theta =$$

$$= \frac{\pi}{6\sqrt{3}} + \frac{ln2}{2}P + \frac{1}{2}C$$

$$Now,\quad P = \int_0^{\frac{\pi}{6}} \frac{\theta}{\sin\theta} d\theta = \int_0^{\frac{\pi}{6}} \theta d\left[\ln\left(tan\frac{\theta}{2}\right)\right] \underbrace{=}_{IBP} \left[\theta\ln\left(tan\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \ln\left(tan\frac{\theta}{2}\right) d\theta$$

$$= \frac{\pi}{6}\ln\left(tan\frac{\pi}{12}\right) + \int_0^{\frac{\pi}{6}} \ln\left(cot\frac{\theta}{2}\right) d\theta$$

$$\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3} \qquad and \qquad \ln\left(cot\frac{\theta}{2}\right) = 2\sum_{n=odd>0} \frac{\cos(n\theta)}{n}$$

$$P = \frac{\pi}{6}\ln(2-\sqrt{3}) + 2\sum_{n=odd>0}\int_0^{\frac{\pi}{6}} \frac{\cos(n\theta)}{n} d\theta = \frac{\pi}{6}\ln(2-\sqrt{3}) + \sum_{n=odd>0} \frac{2sin\left(\frac{n\pi}{6}\right)}{n^2}$$

$$Now, \sum_{n=odd>0} \frac{2sin\left(\frac{n\pi}{6}\right)}{n^2} = \left\{\frac{1}{1^2} + \frac{2}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \frac{2}{9^2} - \frac{1}{11^2} + \ldots\ldots\ldots\ldots\ldots\right\}$$

$$= \left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \ldots\ldots\ldots\ldots\ldots\right\} + \left\{\frac{3}{3^2} - \frac{3}{9^2} + \ldots\ldots\ldots\ldots\ldots\right\} = G + \frac{3}{3^2}G = \frac{4G}{3}$$

$$Hence, P = \frac{\pi}{6}\ln(2-\sqrt{3}) + \frac{4G}{3}. \text{ Therefore, } \Omega = \frac{\pi}{6\sqrt{3}} + \frac{ln2}{2}\left\{\frac{\pi}{6}\ln(2-\sqrt{3}) + \frac{4G}{3}\right\} + \frac{1}{2}C$$

$$\Omega = \frac{\pi}{6\sqrt{3}} + \frac{\pi\ln(2)\ln(2-\sqrt{3})}{12} + \frac{2Gln2}{3} + \frac{1}{2}C$$

$here\quad G = catalan's\ constant$

$$C = \int_0^{\frac{\pi}{6}} \frac{\theta\ln(sin\theta)}{\sin\theta} d\theta = some\ numerical\ constant$$