ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Shobhit Jain-India

$$\Omega=\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}\to$$

$$\Omega+1=\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}$$

$$=\sum_{k=0}^{\infty}(-1)^k\{\frac{12}{k+1}+\frac{4}{(k+1)^2}+\frac{1}{(k+1)^3}-\frac{24}{2k+1}+\frac{8}{(2k+1)^2}\}\to$$

$$\Omega+1=12\left\{\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots\right\}+4\left\{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right\}+\left\{\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots\right\}-$$

$$24\left\{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\cdots\right\}+8\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right\}=12\ln(2)+4\eta(2)+\eta(3)-$$

$$-24\beta(1)+8\beta(2)=12\ln(2)+2\zeta(2)+\frac{3}{4}\zeta(3)-24.\frac{\pi}{4}+8G$$

$$\Omega=\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}=8G+12\ln(2)+2\zeta(2)+\frac{3}{4}\zeta(3)-6\pi-1$$

Solution 2 by Kartick Chandra Betal-India

$$\Omega=\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}=\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k^3(2k-1)^2}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3(2k-1)^2}-1=$$

$$=8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{(2k)^2(2k-1)^2}-\frac{1}{(2k)^3(2k-1)}\right\}-1=$$

$$=8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{2k(2k-1)^2}-\frac{1}{(2k)^2(2k-1)}-\frac{1}{(2k)^2(2k-1)}+\frac{1}{(2k)^3}\right\}-1=$$

$$= 8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{(2k-1)^2}-\frac{3}{2k(2k-1)}+\frac{2}{(2k)^2}+\frac{1}{(2k)^3}\right\}-1=$$

$$= 8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{(2k-1)^2}-\frac{3}{(2k-1)}+\frac{3}{2k}+\frac{1}{2k^2}+\frac{1}{(2k)^3}\right\}-1=$$

$$= 8\left\{\beta(2)-3.\frac{\pi}{4}+\frac{3}{2}\ln(2)+\frac{1}{2}.\frac{\pi^2}{12}+\frac{1}{8}.\frac{3}{4}\zeta(3)\right\}-1=$$

$$= 8\left\{G-\frac{3\pi}{4}+\frac{3}{2}ln(2)+\frac{\pi^2}{24}+\frac{3}{32}\zeta(3)\right\}-1=8G+12\ln(2)+2\zeta(2)+\frac{3}{4}\zeta(3)-6\pi-1$$