ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$\int_0^\infty \frac{\tan^{-1}(x)}{x(1+x)^2}dx = G + \frac{\pi}{4}\log(2) - \frac{\pi}{4}$$

where G denotes *Catalan's constant*

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$$\int_0^\infty \frac{\tan^{-1}(x)}{x(1+x)^2}dx = \int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^2}dx + \int_1^\infty \frac{\tan^{-1}(x)}{x(1+x)^2}dx = \int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^2}dx + \int_0^1 \frac{x\tan^{-1}\left(\frac{1}{x}\right)}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^2}dx + \int_0^1 \frac{x\left(\frac{\pi}{2} - \tan^{-1}(x)\right)}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x) + \frac{\pi}{2}x^2 - x^2\tan^{-1}(x)}{x(1+x)^2}dx$$

$$= \int_0^1 \frac{(1-x^2)\tan^{-1}(x)}{x(1+x)^2}dx + \frac{\pi}{2}\int_0^1 \frac{x}{(1+x)^2}dx$$

$$= \int_0^1 \frac{(1-x)\tan^{-1}(x)}{x(1+x)}dx + \frac{\pi}{2}\int_0^1 \frac{x}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x)}{x}dx - 2\int_0^1 \frac{\tan^{-1}(x)}{1+x}dx + \frac{\pi}{2}\int_0^1 \frac{x}{(1+x)^2}dx$$

$$= G - \frac{\pi}{4}\log(2) + \frac{\pi}{2}\left(\log(2) - \frac{1}{2}\right) = G + \frac{\pi}{4}\log(2) - \frac{\pi}{4}$$

Note: $\int_0^1 \frac{\tan^{-1}(x)}{x}dx = G$ and $\int_0^1 \frac{\tan^{-1}(x)}{1+x}dx = \frac{\pi}{8}\log(2)$