ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{x^2\log(x)}{x^2+x+1}dx = -\frac{\pi^2}{54} - 1 + \frac{1}{9}\varphi^{(1)}\left(\frac{1}{3}\right)$$

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Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_0^1 \frac{x^2\log(x)}{x^2+x+1}dx = \int_0^1 \frac{(x^2-x^3)\log(x)}{1-x^3}dx = \int_0^1 \frac{x^2\log(x)}{1-x^3}dx - \int_0^1 \frac{x^3\log(x)}{1-x^3}dx$$

$$= \int_0^1 \frac{x^2\log(x)}{1-x^3}dx + \int_0^1 \log(x)\,dx - \int_0^1 \frac{\log(x)}{1-x^3}dx$$

$$= \frac{1}{9}\int_0^1 \frac{\log(y)}{1-y}dy - 1 - \frac{1}{9}\int_0^1 \frac{y^{-\frac{2}{3}}\log(y)}{1-y}dy = -\frac{\pi^2}{54} - 1 + \frac{1}{9}\varphi^{(1)}\left(\frac{1}{3}\right)$$

$$NOTE: \int_0^1 \frac{x^{n-1}}{1-x}\log^m(x)\,dx = -\varphi^{(m)}(n)$$