ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{x^2 ln^3(x)}{(1+x)^3}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Kartick Chandra Betal-India

$$\int_0^1 \frac{x^2 ln^3(x)}{(1+x)^3}dx = \sum_{n=1}^{\infty}(-1)^{n-1}.\frac{n(n+1)}{2}\int_0^1 x^{n+1}ln^3(x)dx =$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}.\frac{n(n+1)}{2}\left[\frac{x^{n+2}ln^3(x)}{n+2}-\frac{3x^{n+2}ln^2(x)}{(n+2)^2}+\frac{6x^{n+2}\ln(x)}{(n+2)^3}-\frac{6x^{n+2}}{(n+2)^4}\right]_0^1 =$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}.\frac{n(n+1)}{2}\left[-\frac{6}{(n+2)^4}\right] = 3\sum_{n=1}^{\infty}(-1)^n.\frac{n(n+1)}{(n+2)^4} =$$

$$3\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{(n+2)^2}-\frac{3}{(n+2)^3}+\frac{2}{(n+2)^4}\right\} = 3\sum_{n=3}^{\infty}(-1)^{n-1}\left\{-\frac{1}{n^2}+\frac{3}{n^3}-\frac{2}{n^4}\right\} =$$

$$3\left[\sum_{n=1}^{\infty}(-1)^{n-1}\left\{-\frac{1}{n^2}+\frac{3}{n^3}-\frac{2}{n^4}\right\}\right] = 3\{-\eta(2)+3\eta(3)-2\eta(4)\} =$$

$$3\left\{-\frac{\pi^2}{12}+3.\frac{3}{4}\zeta(3)-2.\frac{7}{8}.\zeta(4)\right\} = 3\left(-\frac{\pi^2}{12}+\frac{9}{4}\zeta(3)-\frac{7}{4}.\frac{\pi^4}{90}\right) =$$

$$\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

$$\int_0^1 \frac{x^2 ln^3(x)}{(1+x)^3}dx = \frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

Solution 2 by Pham Duc Nam-Vietnam

$$* \, n,m \in N : I(m,n) = \int_0^1 \frac{ln^n(x)}{(1+x)^m}dx = \frac{(-1)^{m-1}}{(m-1)!}.\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\int_0^1 \frac{ln^n(x)}{a+x}dx =$$

$$\frac{(-1)^{m-1}}{(m-1)!}.\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\sum_{k=0}^{\infty}\frac{(-1)^k}{a^{k+1}}\int_0^1 x^k ln^n(x)dx =$$

$$\frac{(-1)^{m-1}(-1)^n n!}{(m-1)!}\cdot\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\sum_{k=0}^{\infty}\frac{(-1)^k}{a^{k+1}(1+k)^{n+1}}=\frac{(-1)^{m-1}(-1)^n n!}{(m-1)!}\cdot\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\left(-Li_{n+1}\left(-\frac{1}{a}\right)\right)$$

$$*\, I=\int_0^1\frac{x^2 ln^3(x)}{(1+x)^3}dx=\int_0^1\left(\frac{1}{(1+x)^3}-\frac{2}{(1+x)^2}+\frac{1}{1+x}\right)ln^3(x)dx=I(3,3)-2I(3,2)+I(3,1)$$

$$I(3,3)=\frac{(-1)^2(-1)^3 3!}{(3-1)!}\cdot\frac{\partial^2}{\partial a^2}\bigg|_{a=1}\left(-Li_4\left(-\frac{1}{a}\right)\right)=-\frac{9}{4}\zeta(3)-\frac{3}{2}\zeta(2)$$

$$I(3,2)=\frac{(-1)^1(-1)^3 3!}{(1)!}\cdot\frac{\partial^1}{\partial a^1}\bigg|_{a=1}\left(-Li_3\left(-\frac{1}{a}\right)\right)=-\frac{9}{2}\zeta(3)$$

$$I(3,1)=\frac{(-1)^0(-1)^3 3!}{(0)!}\left(-Li_4(-1)\right)=-\frac{21}{4}\zeta(4)$$

$$*\, I=I(3,3)-2I(3,2)+I(3,1)=-\frac{9}{4}\zeta(3)-\frac{3}{2}\zeta(2)+2.\frac{9}{2}\zeta(3)-\frac{21}{4}\zeta(4)$$

$$\boxed{\int_0^1\frac{x^2 ln^3(x)}{(1+x)^3}dx=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}}$$

Solution 3 by Shobhit Jain-India

$$\Omega=\int_0^1\frac{x^2 ln^3(x)}{(1+x)^3}dx$$

$$\textit{Now}\ ,\qquad (1+x)^{-3}=\sum_{n=0}{}^{n+2}_{\ \ 2}C(-1)^n x^n$$

$$\Omega=\sum_{n=0}{}^{n+2}_{\ \ 2}C(-1)^n\int_0^1 x^{n+2}ln^3(x)dx=-\Gamma 4\sum_{n=0}\frac{{}^{n+2}_{\ \ 2}C(-1)^n}{(n+3)^4}=$$

$$-6\sum_{n=2}\frac{{}^{n}_{2}C(-1)^n}{(n+1)^4}=-3\sum_{n=2}(-1)^n\frac{n(n-1)}{(n+1)^4}=-3\sum_{n=0}(-1)^n\frac{n(n-1)}{(n+1)^4}=$$

$$-3\sum_{n=1}(-1)^{n-1}\frac{(n-1)(n-2)}{n^4}=-3\sum_{n=1}(-1)^{n-1}\left\{\frac{1}{n^2}-\frac{3}{n^3}+\frac{2}{n^4}\right\}=$$

$$-3\{\eta(2)-3\eta(3)+2\eta(4)\}=-3\eta(2)+9\eta(3)-6\eta(4)=$$

$$-\frac{3}{2}\zeta(2)+\frac{27}{4}\zeta(3)-\frac{21}{4}\zeta(4)=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

$$\int_0^1 \frac{x^2 \ln^3(x)}{(1+x)^3}dx = \frac{27}{4}\zeta(3) - \frac{\pi^2}{4} - \frac{7\pi^4}{120}$$

Solution 4 by Cosghun Memmedov-Azerbaijan

$$\Omega = \int_0^1 \frac{x^2 \ln^3(x)}{(1+x)^3}dx = \int_0^1 \frac{\ln^3(x)}{1+x}dx - 2\int_0^1 \frac{\ln^3(x)}{(1+x)^2}dx + \int_0^1 \frac{\ln^3(x)}{(1+x)^3}dx = \Omega_1 - 2\Omega_2 + \Omega_3$$

$$\Omega_1 = \int_0^1 \frac{\ln^3(x)}{1+x}dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^n \ln^3(x)dx = -6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4} =$$

$$-6\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = -6\eta(4) = -6.\frac{7\pi^4}{720} = -\frac{7\pi^4}{120}$$

$$\Omega_2 = \int_0^1 \frac{\ln^3(x)}{(1+x)^2}dx = -\sum_{n=1}^{\infty}(-1)^n n \int_0^1 x^{n-1}\ln^3(x)dx = 6\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^4} = -6\eta(3) =$$

$$-6.\frac{3}{4}\zeta(3) = -\frac{9}{2}\zeta(3)$$

$$\Omega_3 = \int_0^1 \frac{\ln^3(x)}{(1+x)^3}dx = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)\int_0^1 x^{n-2}\ln^3(x)dx = -3\sum_{n=2}^{\infty}\frac{(-1)^n n(n-1)}{(n-1)^4} =$$

$$-3\sum_{n=2}^{\infty}\frac{(-1)^n n}{(n-1)^3} = -3\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)^2} - 3\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)^3} = -3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} - 3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} =$$

$$-3\eta(2) - 3\eta(3) = -3.\frac{\pi^2}{12} - 3.\frac{3}{4}\zeta(3) = -\frac{\pi^2}{4} - \frac{9}{4}\zeta(3)$$

$$\Omega = \Omega_1 - 2\Omega_2 + \Omega_3 = -\frac{7\pi^4}{120} + 9\zeta(3) - \frac{\pi^2}{4} - \frac{9}{4}\zeta(3)$$

$$\int_0^1 \frac{x^2 \ln^3(x)}{(1+x)^3}dx = \frac{27}{4}\zeta(3) - \frac{\pi^2}{4} - \frac{7\pi^4}{120}$$

Solution 5 by Quadri Faruk Temitope-Nigeria

$$I = \int_0^1 \frac{x^2 \ln^3(x)}{(1+x)^3}dx$$

$$I = \int_0^1 \frac{\ln^3(x)}{1+x}dx - 2\int_0^1 \frac{\ln^3(x)}{(1+x)^2}dx + \int_0^1 \frac{\ln^3(x)}{(1+x)^3}dx$$

$$I=\sum_{n=0}^{\infty}\binom{-3}{n}\int_0^1 x^n ln^3(x)dx-2\sum_{n=0}^{\infty}\binom{-2}{n}\int_0^1 x^n\, ln^3(x)dx+\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n ln^3(x)dx$$

$$I=\sum_{n=0}^{\infty}\binom{-3}{n}\left[-\frac{6}{(n+1)^4}\right]-2\sum_{n=0}^{\infty}\binom{-2}{n}\left[-\frac{6}{(n+1)^4}\right]+\sum_{n=0}^{\infty}(-1)^n\left[-\frac{6}{(n+1)^4}\right]$$

$$I=-6\sum_{n=0}^{\infty}\frac{\binom{-3}{n}}{(n+1)^4}+12\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{(n+1)^4}-6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}$$

$$I=-6\left[\frac{3}{8}\zeta(3)+\frac{1}{4}\zeta(2)\right]+12\left[\frac{3}{4}\zeta(3)\right]-6\left[\frac{7}{8}\zeta(4)\right]$$

$$\int_0^1\frac{x^2 ln^3(x)}{(1+x)^3}dx=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$