ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \left(\arctan^3(x) + x^2 Li_3(-x^3)\right) dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$General\ \ problem\ \ and\ \ full\ \ solution\ :$$

$$\therefore \ln(\cos(y)) = -\ln(2) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cos(2ny),\quad |y| < \frac{\pi}{2}$$

$$\therefore \int y^m \cos(ay)\, dy = \sum_{j=0}^{\infty} j! \binom{m}{j} \frac{y^{m-j}}{a^{j+1}} \sin(ay + j\frac{\pi}{2})$$

$$For\ \ m, n, k \in Z^+ : S = \int_0^1 \left(\arctan^m(x) + x^{k-1} Li_n(ax^k)\right) dx = \int_0^1 \arctan^m(x) dx +$$

$$\int_0^1 x^{k-1} Li_n(ax^k) dx = I + J$$

$$1) I = \int_0^1 \arctan^{m+2}(x) dx \overset{x=\tan(y)}{\cong} \int_0^{\frac{\pi}{4}} y^{m+2} d(\tan(y)) = y^{m+2}\tan(y)\Big|_0^{\frac{\pi}{4}} -$$

$$(m+2)\int_0^{\frac{\pi}{4}} y^{m+1}\tan(y)\, dy = \left(\frac{\pi}{4}\right)^{m+2} - (m+2) y^{m+1} \ln(\cos(y))\Big|_0^{\frac{\pi}{4}} +$$

$$(m+2)(m+1)\underbrace{\int_0^{\frac{\pi}{4}} y^m \ln(\cos(y))\, dy}_{A} = \left(\frac{\pi}{4}\right)^{m+2} + (m+2)\left(\frac{\pi}{4}\right)^{m+1} \ln(\sqrt{2}) + (m+2)(m+1)A$$

$$a)\ A = \int_0^{\frac{\pi}{4}} y^m \ln(\cos(y))\, dy = -\int_0^{\frac{\pi}{4}} y^m (\ln(2) + \sum_{a=1}^{\infty} \frac{(-1)^a}{a}\cos(2ay)) dy = -\ln(2)\int_0^{\frac{\pi}{4}} y^m dy +$$

$$\sum_{a=1}^{\infty} \frac{(-1)^a}{a} \int_0^{\frac{\pi}{4}} y^m \cos(2ay)\, dy = -\left(\frac{\pi}{4}\right)^m \ln(2) + \sum_{a=1}^{\infty} \frac{(-1)^a}{a} \sum_{j=0}^{\infty} j! \binom{m}{j} \frac{y^{m-j}}{(2a)^{j+1}} \sin\left(ay + j\frac{\pi}{2}\right)\Bigg|_0^{\frac{\pi}{4}} =$$

$$-\left(\frac{\pi}{4}\right)^m \ln(2) + \sum_{j=0}^{0} \frac{j!}{2^{j+1}} \binom{m}{j} \left(\frac{\pi}{4}\right)^{m-j} \sum_{a=1}^{\infty} \frac{(-1)^a}{a^{j+2}} \sin(\frac{a\pi}{2} + j\frac{\pi}{2}) - \frac{m!}{2^{m+1}} \sin(\frac{m\pi}{2}) \sum_{a=1}^{\infty} \frac{(-1)^a}{a^{m+2}} =$$

$$-\left(\frac{\pi}{4}\right)^m \ln(2)+\sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\underbrace{\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\begin{pmatrix}\sin\left(\frac{a\pi}{2}\right)\cos\left(\frac{j\pi}{2}\right)\\+\cos\left(\frac{a\pi}{2}\right)\sin\left(\frac{j\pi}{2}\right)\end{pmatrix}}_{D}+$$

$$\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)\left(1-2^{-1-m}\right)\zeta(m+2)=\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)\left(1-2^{-1-m}\right)\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+$$

$$\sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}D$$

$$D=\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\left(\sin\left(\frac{a\pi}{2}\right)\cos\left(\frac{j\pi}{2}\right)+\cos\left(\frac{a\pi}{2}\right)\sin\left(\frac{j\pi}{2}\right)\right)=$$

$$\cos\left(\frac{j\pi}{2}\right)\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\sin\left(\frac{a\pi}{2}\right)+\sin\left(\frac{j\pi}{2}\right)\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\cos\left(\frac{a\pi}{2}\right)=-\cos\left(\frac{j\pi}{2}\right)\sum_{a=0}^{\infty}\frac{(-1)^a}{(2a+1)^{j+2}}+$$

$$\sin\left(\frac{j\pi}{2}\right)\sum_{a=0}^{\infty}\frac{1}{(2a)^{j+2}}=-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)$$

$$A=\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)\left(1-2^{-1-m}\right)\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+$$

$$\sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\left(\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\right)$$

$$Then\ : I=\left(\frac{\pi}{4}\right)^{m+2}+(m+2)\left(\frac{\pi}{4}\right)^{m+1}\ln\left(\sqrt{2}\right)+$$

$$(m+2)(m+1)\left\{\begin{matrix}\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)\left(1-2^{-1-m}\right)\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+\\ \sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\left(\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\right)\end{matrix}\right\}$$

$$2)J=\int_0^1 x^{k-1}Li_n\left(ax^k\right)dx\overset{ax^k\to x}{\cong}\frac{1}{ka}\int_0^a Li_n(x)d)=\frac{1}{ka}K_n$$

$$K_n=\int_0^a Li_n(x)dx=xLi_n(x)\Big|_0^a-\int_0^a x\frac{\partial Li_n(x)}{\partial x}dx=aLi_n(a)-\int_0^a Li_{n-1}(x)\,dx=aLi_n(a)-K_{n-1}$$

$$=(-1)^0 aLi_n(a)+(-1)^1 aLi_{n-1}(a)+(-1)^2 aLi_{n-2}(a)+(-1)^3K_{n-3}=$$

$$a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)+(-1)^{n-1}K_{n-1}=a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)+(-1)^{n-1}\int_0^a Li_1(x)dx=$$

$$a\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)+(-1)^{n-1}\int_0^a \ln(1-x)d(1-x)=a\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)+$$

$$(-1)^{n-1}\{(1-x)\ln(1-x)+x\}\Big|_0^a+a\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)+(-1)^{n-1}\{(1-a)\ln(1-a)+a\}$$

Then :

$$J=\frac{\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)}{k}+(-1)^{n-1}\frac{\{(1-a)\ln(1-a)+a\}}{ka}$$

Therefore : $S=I+J$

$$S=\left(\frac{\pi}{4}\right)^{m+2}+(m+2)\left(\frac{\pi}{4}\right)^{m+1}\ln(\sqrt{2})+$$

$$(m+2)(m+1)\left\{\begin{array}{c}\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)(1-2^{-1-m})\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+\\ \sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\left(\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\right)\end{array}\right\}+$$

$$\frac{\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)}{k}+(-1)^{n-1}\frac{\{(1-a)\ln(1-a)+a\}}{ka}$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$\Omega=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=\int_0^1 arctan^3(x)dx+\int_0^1 x^2Li_3(-x^3)dx=I_1+I_2$$

$$I_1=\int_0^1 arctan^3(x)dx\ \ \left\{x=\tan(y)\ ,\qquad dx=sec^2(y)\ \left[0,\frac{\pi}{4}\right]\right\}$$

$$I_1=\int_0^{\frac{\pi}{4}} arctan^3(\tan(y)).sec^2(y)dy=\int_0^{\frac{\pi}{4}} y^3\,sec^2(y)=y^3\tan(y)\Big|_0^{\frac{\pi}{4}}-3\int_0^{\frac{\pi}{4}} y^2\tan(y)\,dy=$$

$$\frac{\pi^3}{64}-3\underbrace{\int_0^{\frac{\pi}{4}} y^2\tan(y)\,dy}_{A}$$

$$A=\int_0^{\frac{\pi}{4}} y^2\tan(y)\,dy=-y^2\ln(cos(y))\Big|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}} yln\big(cos(y)\big)\,dy=\frac{\pi^2}{32}\ln(2)-2\ln(2)\int_0^{\frac{\pi}{4}} ydy-$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} ycos(2ny)dy=\frac{\pi^2}{32}\ln(2)-2\ln(2)\left[\frac{1}{2}\cdot\frac{\pi^2}{16}\right]-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathcal{R}\int_0^{\frac{\pi}{4}} ye^{-2iny}dy=$$

$$\frac{\pi^2}{32}\ln(2)-\frac{\pi^2}{16}\ln(2)-2\mathcal{R}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{-1+e^{-\frac{i\pi n}{2}}+\frac{i\pi n}{2}e^{-\frac{i\pi n}{2}}}{4n^2}\right]=$$

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi\sin\left(\frac{\pi n}{2}\right)}{8n}+\frac{\cos\left(\frac{\pi n}{2}\right)}{4n^2}-\frac{1}{4n^2}\right]=-\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\frac{\pi^2}{32}\ln(2)$$

$$A=\frac{\pi}{4}G+\frac{3}{64}\zeta(3)-\frac{3}{8}\zeta(3)-\frac{\pi^2}{32}\ln(2)$$

$$I_1=\frac{\pi^3}{64}-3A=\frac{\pi^3}{64}-\frac{3\pi}{4}G+\frac{63}{64}\zeta(3)+\frac{3\pi^2}{32}\ln(2)$$

$$I_2=\int_0^1 x^2Li_3(-x^3)dx\ \left\{u=x^3,\frac{du}{dx}=3x^2,dx=\frac{du}{3x^2}\right\}$$

$$I_2=\frac{1}{3}\int_0^1 Li_3(-u)du\overset{u=x}{\cong}\frac{1}{3}\int_0^1 Li_3(-x)dx\ \left\{u=Li_3(-x),\quad \frac{du}{dx}=\frac{Li_2(-x)}{x},du=dx\ v=x\right\}$$

$$I_2=\frac{1}{3}\left[xLi_3(-x)\Big|_0^1-\int_0^1 Li_2(-x)dx\right]=\frac{1}{3}\left(-\frac{3}{4}\zeta(3)-\left(-\frac{\zeta(2)}{2}+2\ln(2)-1\right)\right)=$$

$$I_2=-\frac{1}{4}\zeta(3)+\frac{\zeta(2)}{6}-\frac{2}{3}\ln(2)+\frac{1}{3}$$

$$\Omega=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=\frac{\pi^3}{64}-\frac{3\pi}{4}G+\frac{47}{64}\zeta(3)+\frac{3\pi^2}{32}\ln(2)+\frac{\zeta(2)}{6}-\frac{2}{3}\ln(2)+\frac{1}{3}$$

Solution 3 by Shobhit Jain-India

$$I=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=M+K$$

$$M=\int_0^1 arctan^3(x)dx=\int_0^{\frac{\pi}{4}}\theta^3 d(\tan(\theta))=\theta^3\tan(\theta)\Big|_0^{\frac{\pi}{4}}-3\int_0^{\frac{\pi}{4}}\theta^2\tan(\theta)\,d(\theta)=\frac{\pi^3}{64}+$$

$$3\int_0^{\frac{\pi}{4}}\theta^2 d(\ln(2\cos(\theta))=\frac{\pi^3}{64}+3.\frac{\pi^2}{16}.\ln\left(\frac{2}{\sqrt{2}}\right)-3\int_0^{\frac{\pi}{4}}\ln(2\cos(\theta))=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)-$$

$$\frac{3}{2}\int_0^{\frac{\pi}{2}}\theta\ln\left(2\cos\left(\frac{\theta}{2}\right)\right)d\theta\ \ by\ replacing\ \theta\ with\ \theta/2$$

$$Now\ :\ \ln\left(2\cos\left(\frac{\theta}{2}\right)\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(n\theta)}{n}$$

$$\int_0^{\frac{\pi}{2}}\theta\ln\left(2\cos\left(\frac{\theta}{2}\right)\right)d\theta=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}}\theta\cos(n\theta)\,d\theta=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left\{\frac{\theta\sin(n\theta)}{n}+\frac{\cos(n\theta)}{n^2}\right\}\Bigg|_0^{\frac{\pi}{2}}=$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left\{\frac{\frac{\pi}{2}\sin\left(\frac{n\pi}{2}\right)}{n}+\frac{\cos\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{n^2}\right\}=\frac{\pi}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi}{2}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right\}+\left\{\frac{1}{2^3}-\frac{1}{4^3}+\frac{1}{6^3}-\cdots\right\}-\left\{\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots\right\}=$$

$$\frac{\pi}{2}G+\frac{\eta(3)}{8}-\eta(3)=\frac{\pi}{2}G-\frac{21}{32}\zeta(3)$$

$$M=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)-\frac{3}{2}\left\{\frac{\pi}{2}G-\frac{21}{32}\zeta(3)\right\}=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)+\frac{63}{64}\zeta(3)-\frac{3\pi}{4}G$$

Now

$$K=\int_0^1 x^2Li_3(-x^3)dx=\frac{1}{3}\int_0^1 Li_3(-t)dt\ \ \textit{By substituting}\ \ t=x^3$$

$$Li_3(-t)=\sum_{n=1}^{\infty}(-1)^n\frac{t^n}{n^3}$$

$$\textit{Therefore}\ \ K=\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)}=\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n^3}-\frac{1}{n^2}+\frac{1}{n}-\frac{1}{n+1}\right\}=$$

$$\frac{1}{3}\{-\eta(3)+\eta(2)-\ln(2)+1-\ln(2)\}=\frac{1}{3}\left\{-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)+1-2\ln(2)\right\}=$$

$$\frac{\pi^2}{36}-\frac{1}{4}\zeta(3)-\frac{2}{3}\ln(2)+\frac{1}{3}$$

$$I=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=M+K=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)+\frac{63}{64}\zeta(3)-\frac{3\pi}{4}G+$$

$$\frac{\pi^2}{36}-\frac{1}{4}\zeta(3)-\frac{2}{3}\ln(2)+\frac{1}{3}$$

$$I=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=\frac{\pi^3}{64}-\frac{3\pi}{4}G+\frac{47}{64}\zeta(3)+\frac{3\pi^2}{32}\ln(2)+\frac{\zeta(2)}{6}-\frac{2}{3}\ln(2)+\frac{1}{3}$$