ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_1^\infty \frac{xln^2(x)}{(1+x)^2(3+x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Shobhit Jain-India

$$\Omega=\int_1^\infty \frac{xln^2(x)}{(1+x)^2(3+x)}dx=\int_0^1 \frac{ln^2(x)}{(1+x)^2(3x+1)}dx \;\; put\; x=\frac{1}{x}$$

$$\int_0^1 ln^2(x)\left(\frac{\frac{9}{4}}{1+3x}-\frac{\frac{3}{4}}{1+x}-\frac{\frac{1}{2}}{(1+x)^2}\right)dx$$

$$=\frac{9}{4}\int_0^1 \frac{ln^2(x)}{1+3x}dx-\frac{3}{4}\int_0^1 \frac{ln^2(x)}{1+x}dx-\frac{1}{2}\int_0^1 \frac{ln^2(x)}{(1+x)^2}=$$

$$\frac{9}{4}A-\frac{3}{4}B-\frac{1}{2}C$$

$$B=\int_0^1 ln^2(x)(1-x+x^2-x^3+\cdots)dx=\Gamma 3\left(\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\cdots\right)=2\eta(3)=\frac{3}{2}\zeta(3)$$

$$C=\int_0^1 ln^2(x)(1-2x+3x^2-4x^3+\cdots)dx=\Gamma 3\left(\frac{1}{1^3}-\frac{2}{2^3}+\frac{3}{3^3}-\frac{4}{4^3}+\cdots\right)=2\eta(2)=\zeta(2)$$

$$A=\int_0^1 \frac{ln^2(x)}{1+3x}dx\overset{x=\frac{x}{3}}{=}\frac{1}{3}\int_0^3 \frac{(\ln(x)-\ln(3))^2}{1+x}dx=\frac{1}{3}\int_0^3 \frac{ln^2(x)}{1+x}dx-\frac{2}{3}\ln(3)\int_0^3 \frac{ln(x)}{1+x}dx+$$

$$\frac{ln^2(3)}{3}\int_0^3 \frac{dx}{1+x}=\frac{1}{3}\int_0^3 \frac{ln^2(x)}{1+x}dx-\frac{2}{3}\ln(3)\int_0^3 \frac{ln(x)}{1+x}dx+\frac{2}{3}ln^2(3)\ln(2)$$

$$Therefore:\;\; \Omega=\frac{3}{4}\int_0^3 \frac{ln^2(x)}{1+x}dx-\frac{3}{2}\ln(3)\int_0^3 \frac{ln(x)}{1+x}dx+\frac{3}{2}ln^2(3)\ln(2)-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)$$

$$Now:$$

$$\int_0^3 \frac{ln(x)}{1+x}dx=\ln(x)\ln(1+x)\Big|_0^3-\int_0^3 \frac{\ln(1+x)}{x}dx=2\ln(2)\ln(3)+Li_2(-3)$$

$$\int_0^3 \frac{ln^2(x)}{1+x}dx=ln^2(x)\ln(1+x)\Big|_0^3-2\int_0^3 \frac{\ln(x)\ln(1+x)}{x}dx=2\ln(2)ln^2(3)+$$

$$2\int_0^3 \ln(x)\,d(Li_2(-x))=2\ln(2)ln^2(3)+2ln(x)Li_2(-x)\Big|_0^3-\int_0^3 \frac{Li_2(-x)}{x}dx=$$

$$2\ln(2)ln^2(3)+2\ln(3)\,Li_2(-3)-2Li_3(-3)$$

$$Hence\ :\ \Omega=\frac{3}{4}\{2\ln(2)ln^2(3)+2\ln(3)\,Li_2(-3)-2Li_3(-3)\}-$$

$$\frac{2}{3}\ln(3)\{2\ln(2)\ln(3)+Li_2(-3)\}+\frac{3}{2}ln^2(3)\ln(2)-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)$$

$$\int_1^\infty\frac{xln^2(x)}{(1+x)^2(3+x)}dx=-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)-\frac{3}{2}Li_3(-3)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\psi=\int_1^\infty\frac{xln^2(x)}{(1+x)^2(3+x)}dx$$

$$\psi_n=\int_1^\infty\frac{xln^n(x)}{(1+x)^2(3+x)}dx\overset{x=\frac{1}{x}}{\cong}\int_0^1\frac{\frac{1}{x}ln^n(\frac{1}{x})}{\left(\frac{1}{x}+1\right)^2\left(\frac{1}{x}+3\right)}\frac{dx}{x^2}=(-1)^n\int_0^1\frac{ln^n(x)}{(1+x)^2(3x+1)}dx=$$

$$\frac{(-1)^n}{4}\left\{3.3\underbrace{\int_0^1\frac{ln^n(x)}{3x+1}dx}_{A}-3\underbrace{\int_0^1\frac{ln^n(x)}{1+x}dx}_{B}-2\underbrace{\int_0^1\frac{ln^n(x)}{(1+x)^2}dx}_{C}\right\}=\frac{(-1)^n}{4}(3A-3B-2C)$$

$$A=3\int_0^1\frac{ln^n(x)}{3x+1}dx=\int_0^1\frac{ln^n(x)}{x+\frac{1}{3}}dx=(-1)^{n+1}n!\,Li_{n+1}(-3)$$

$$B=\int_0^1\frac{ln^n(x)}{1+x}dx=(-1)^{n+1}n!\,Li_{n+1}(-1)=(-1)^n n!\,(1-2^{-n})\zeta(n+1)$$

$$C=\int_0^1\frac{ln^n(x)}{(1+x)^2}dx=\int_0^1 ln^n(x)d\left(\frac{x}{x+1}\right)=\frac{xln^n(x)}{x+1}\bigg|_0^1-n\int_0^1\frac{ln^{n-1}(x)}{x+1}dx=(-1)^{n+1}n!\,Li_n(-1)$$

$$\psi_n=\frac{(-1)^n}{4}(3A-3B-2C)=$$

$$(-1)^n\frac{3(-1)^{n+1}n!\,Li_{n+1}(-3)-3(-1)^n n!\,(1-2^{-n})\zeta(n+1)-2(-1)^{n+1}n!\,Li_n(-1)}{4}=$$

$$-\frac{n!}{4}\big(3Li_{n+1}(-3)+3(1-2^{-n})\zeta(n+1)+2\eta(n)\big)$$

$$If\ :\ n=2$$

$$\psi=\int_1^\infty\frac{xln^2(x)}{(1+x)^2(3+x)}dx=-\frac{3}{4}.4\left(3Li_3(-3)+3(1-2^{-2})\zeta(2+1)+2\eta(2)\right)$$

$$\int_1^\infty\frac{xln^2(x)}{(1+x)^2(3+x)}dx=-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)-\frac{3}{2}Li_3(-3)$$

Solution 3 by Obiajunwa Januarius-Nigeria

$$\int_1^{\infty} \frac{x\ln^2(x)}{(1+x)^2(3+x)}dx$$

$$\int_1^{\infty} \frac{x\ln^2(x)}{(1+x)^2(3+x)}dx \overset{x\to\frac{1}{x}}{\cong} \int_0^1 \frac{\ln^2(x)}{(1+x)^2(3x+1)}dx = \frac{9}{4}\int_0^1 \frac{\ln^2(x)}{1+3x}dx -$$

$$\frac{3}{4}\int_0^1 \frac{\ln^2(x)}{1+x}dx - \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{(1+x)^2}dx = \frac{9}{4}\sum_{n=0}^{\infty}(-1)^n 3^n \int_0^1 x^n \ln^2(x)dx -$$

$$\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n \int_0^1 x^n \ln^2(x)dx - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1} n \int_0^1 x^{n-1}\ln^2(x)dx =$$

$$\frac{9}{4}\sum_{n=0}^{\infty}(-1)^n 3^n \frac{\partial^2}{\partial n^2}\left[\int_0^1 x^n dx\right] - \frac{3}{4}\sum_{n=0}^{\infty}(-1)^n \frac{\partial^2}{\partial n^2}\left[\int_0^1 x^n dx\right] -$$

$$\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1} n \frac{\partial^2}{\partial n^2}\left[\int_0^1 x^{n-1} dx\right] = \frac{9}{4}\sum_{n=0}^{\infty}(-1)^n 3^n \frac{\partial^2}{\partial n^2}\left[\frac{1}{n+1}\right] -$$

$$\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n \frac{\partial^2}{\partial n^2}\left[\frac{1}{n+1}\right] - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1} n \frac{\partial^2}{\partial n^2}\left[\frac{1}{n}\right] = \frac{9}{4}(-1)^2 2! \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{(n+1)^3} -$$

$$\frac{3}{4}(-1)^2 2! \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} - \frac{1}{2}(-1)^2 2! \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} \overset{n+1=m}{\cong} \frac{9}{2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}3^{m-1}}{m^3} -$$

$$\frac{3}{2}\eta(3) - \eta(2) = -\frac{3}{2}Li_3(-3) - \frac{3}{2}.\frac{3}{4}\zeta(3) - \frac{1}{2}\zeta(2)$$

$$\boxed{\int_1^{\infty} \frac{x\ln^2(x)}{(1+x)^2(3+x)}dx = -\frac{9}{8}\zeta(3) - \frac{1}{2}\zeta(2) - \frac{3}{2}Li_3(-3)}$$