ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Shohbit Jain-India

$$\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx =$$

$$= \frac{1}{5}\int_0^1 \ln\left(\frac{1}{x}\right)\left(\frac{x+2}{1+x^2} - \frac{2}{1+2x}\right)dx = \frac{1}{5}\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{x^2+1}dx + \frac{2}{5}\int_0^1 \frac{\ln\left(\frac{1}{x}\right)}{x^2+1}dx -$$

$$-\frac{1}{5}\int_0^2 \frac{\ln\left(\frac{2}{u}\right)}{1+u}du = \frac{1}{5}\int_0^1 \ln\left(\frac{1}{x}\right)\{x - x^3 + x^5 - x^7 + \cdots\}dx +$$

$$+\frac{2}{5}\int_0^1 \ln\left(\frac{1}{x}\right)\{1 - x^2 + x^4 - x^6 + \cdots\}dx - \frac{1}{5}\int_0^2 \frac{\ln(2)}{1+u}du + \frac{1}{5}\int_0^2 \frac{\ln(u)}{1+u}du$$

$$= \frac{1}{5}\Gamma 2\left\{\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \cdots\right\} + \frac{2}{5}\Gamma 2\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\} -$$

$$\frac{\ln(2)\ln(3)}{5} + \frac{1}{5}\int_0^1 \frac{\ln(u)}{1+u}du + \frac{1}{5}\int_0^2 \frac{\ln(u)}{1+u}du = \frac{1}{5}\cdot\frac{\eta(2)}{4} + \frac{2G}{5} - \frac{\ln(2)\ln(3)}{5} + \frac{1}{5}\cdot\left(-\frac{\pi^2}{12}\right) -$$

$$\frac{1}{5}\int_{\frac{1}{2}}^1 \frac{\ln(t)}{t(1+t)}dt = \frac{\pi^2}{240} + \frac{2G}{5} - \frac{\ln(2)\ln(3)}{5} - \frac{\pi^2}{60} - \frac{1}{5}\int_{\frac{1}{2}}^1 \frac{\ln(t)}{t}dt + \frac{1}{5}\int_{\frac{1}{2}}^1 \frac{\ln(t)}{1+t}dt =$$

$$= \frac{2G}{5} - \frac{\ln(2)\ln(3)}{5} - \frac{\pi^2}{80} + \frac{\ln^2(2)}{10} + \frac{P}{5}$$

$$Now\quad P = \int_{\frac{1}{2}}^1 \frac{\ln(t)}{1+t}dt = \sum_{n=1}^{\infty}(-1)^{n+1}\int_{\frac{1}{2}}^1 t^{n-1}\ln(t)\,dt$$

$$Now\quad \int_{\frac{1}{2}}^1 t^{n-1}\ln(t)\,dt = \frac{t^n}{n}\ln(t)\Big|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^1 \frac{t^{n-1}}{n}dt = \frac{\ln(2)}{n2^n} + \frac{1}{n^2 2^n} - \frac{1}{n^2}$$

$$Therefore:\quad P = \sum_{n=1}^{\infty}(-1)^{n+1}\int_{\frac{1}{2}}^1 t^{n-1}\ln(t)\,dt = \sum_{n=1}^{\infty}\left\{\frac{\ln(2)(-1)^{n+1}}{n2^n} - \frac{1}{n^2}\left(-\frac{1}{2}\right)^n + \frac{(-1)^n}{n^2}\right\}$$

$$P=\ln(2)\ln\left(1+\frac{1}{2}\right)-Li_2\left(-\frac{1}{2}\right)-\eta(2)=\ln(2)\ln(3)-ln^2(2)-Li_2\left(-\frac{1}{2}\right)-\frac{\pi^2}{12}$$

$$\int_0^1\frac{xln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx=\frac{2G}{5}-\frac{\ln(2)\ln(3)}{5}-\frac{\pi^2}{80}+\frac{ln^2(2)}{10}+\frac{\ln(2)\ln(3)}{5}-\frac{ln^2(2)}{5}-$$

$$\frac{Li_2\left(-\frac{1}{2}\right)}{5}-\frac{\pi^2}{60}=\frac{2G}{5}-\frac{ln^2(2)}{10}-\frac{7}{40}\zeta(2)-\frac{1}{5}Li_2\left(-\frac{1}{2}\right)$$

$$\int_0^1\frac{xln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx=\frac{2G}{5}-\frac{ln^2(2)}{10}-\frac{7}{40}\zeta(2)-\frac{1}{5}Li_2\left(-\frac{1}{2}\right)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\int_0^1\frac{xln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx$$

$$\therefore\int_0^1\frac{ln^n(x)}{a+x}dx=-\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k\int_0^1 x^{k-1}ln^n(x)dx=-\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k x^k n!\sum_{i=0}^{n}\frac{(-1)^i ln^{n-i}(x)}{(n-i)!\,k^{i+1}}\Bigg|_0^1=$$

$$=(-1)^{n+1}n!\,Li_{n+1}\left(-\frac{1}{a}\right)$$

$$\therefore\int_0^1\frac{ln^n(x)}{a+x^2}dx=-\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k\int_0^1 x^{2k-1-1}ln^n(x)dx=-\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k x^{2k-1}n!\sum_{i=0}^{n}\frac{(-1)^i ln^{n-i}(x)}{(n-i)!\,(2k-1)^{i+1}}\Bigg|_0^1=$$

$$=(-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{a}\right)^k}{(2k-1)^{n+1}}$$

$$1)\psi_n=\int_0^1\frac{xln^n\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx=\frac{(-1)^n}{5}\underbrace{\int_0^1\frac{xln^n(x)}{1+x^2}dx}_{x^2\to x}+\frac{2(-1)^n}{5}\int_0^1\frac{ln^n(x)}{1+x^2}dx-\frac{2(-1)^n}{5}\int_0^1\frac{ln^n(x)}{1+2x}dx=$$

$$\frac{(-1)^n}{5.2^{n+1}}\int_0^1\frac{ln^n(x)}{1+x}dx+\frac{2(-1)^n}{5}\int_0^1\frac{ln^n(x)}{1+x^2}dx-\frac{(-1)^n}{5}\int_0^1\frac{ln^n(x)}{x+\frac{1}{2}}dx=\frac{(-1)^n}{5.2^{n+1}}(-1)^{n+1}n!\,Li_{n+1}(-1)+$$

$$\frac{2(-1)^n}{5}.n!.(-1)^{n+1}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^{n+1}}-\frac{(-1)^n}{5}.n!.(-1)^{n+1}.Li_{n+1}(-2)=$$

$$\frac{n!}{5}\left\{\frac{1}{2^{n+1}}(1-2^{-n})\zeta(n+1)+2\beta(n+1)+Li_{n+1}(-2)\right\}$$

$$iff\quad n=1:\ \psi=\int_0^1\frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx=\frac{1}{5}\left\{\frac{1}{2^2}(1-2^{-1})\zeta(2)+2\beta(2)+Li_2(-2)\right\}=$$

$$\frac{1}{5}\left\{\frac{1}{8}\zeta(2)+2G-Li_2\left(-\frac{1}{2}\right)-\frac{\ln^2(2)}{2}+2Li_2(-1)\right\}=\frac{1}{5}\left\{\frac{1}{8}\zeta(2)+2G-Li_2\left(-\frac{1}{2}\right)-\frac{\ln^2(2)}{2}-\zeta(2)\right\}$$

$$\int_0^1\frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx=\frac{2G}{5}-\frac{\ln^2(2)}{10}-\frac{7}{40}\zeta(2)-\frac{1}{5}Li_2\left(-\frac{1}{2}\right)$$