Find:

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{\left(1+\sqrt{x}\right)^2 sin^2(x)}{x} dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{\left(1+\sqrt{x}\right)^2 sin^2(x)}{x} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\left(1+2\sqrt{x}+x\right)\left(1-\cos(2x)\right)}{x} dx =$$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos(2x)}{x} dx + 2\int_0^{\frac{\pi}{2}} \frac{1-\cos(2x)}{\sqrt{x}} dx + \int_0^{\frac{\pi}{2}} \left(1-\cos(2x)\right) dx\right) =$$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos(2x)}{2x} d(2x) + 2\int_0^{\frac{\pi}{2}} \frac{1-\cos(2x)}{\sqrt{x}} dx + \int_0^{\frac{\pi}{2}} \left(1-\cos(2x)\right) dx\right) =$$

$$-\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\cos(x)-1}{x} dx + 4\sqrt{x}\Big|_0^{\frac{\pi}{2}} - 2\sqrt{2}\underbrace{\int_0^{\frac{\pi}{2}} \cos(2x)\, d\left(\sqrt{2x}\right)}_{\sqrt{2x}\to x} + \left(x - \frac{\sin(2x)}{2}\right)\Bigg|_0^{\frac{\pi}{2}}\right) =$$

$$\frac{1}{2}\left(\gamma + \ln(\pi) - Ci(\pi) + 2\sqrt{2\pi} - 2\sqrt{2}\int_0^{\sqrt{\pi}} \cos\left(x^2\right) dx + \frac{\pi}{2}\right) =$$

$$= \frac{1}{2}\left(\gamma + \ln(\pi) - Ci(\pi) + 2\sqrt{2\pi} - 2\sqrt{2}C\left(\sqrt{\pi}\right) + \left(\frac{\pi}{2}\right)\right)$$

Solution 2 by Shobhit Jain-India

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(1+\sqrt{x}\right)^2 sin^2(x)}{x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(1+\sqrt{x}\right)^2 sin^2(x)}{x} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1+x+2\sqrt{x}\right)\left(\frac{1-\cos(2x)}{2x}\right) dx \overset{x\to\frac{x}{2}}{=}$$

$$\int_0^{\pi} \left(1+\frac{x}{2}+\sqrt{2x}\right)\left(\frac{1-\cos(x)}{x}\right) dx = \frac{1}{2}\int_0^{\pi} \left(\frac{1-\cos(x)}{x}\right) dx +$$

$$\frac{1}{4}\int_0^{\pi} \left(-\cos(x)\right) dx + \frac{1}{\sqrt{2}}\int_0^{\pi} \left(\frac{1-\cos(x)}{\sqrt{x}}\right) dx$$

$$I=\frac{1}{2}\big(\gamma+\ln(\pi)-Ci(\pi)\big)+\frac{\pi}{4}+\frac{1}{\sqrt{2}}\int_0^{\pi}\left(\frac{1-\cos(x)}{\sqrt{x}}\right)dx$$

$$I=\frac{1}{2}\big(\gamma+\ln(\pi)-Ci(\pi)\big)+\frac{\pi}{4}+\sqrt{2\pi}-\frac{1}{\sqrt{2}}\int_0^{\pi}\left(\frac{\cos(x)}{\sqrt{x}}\right)dx$$

$$I=\frac{1}{2}\big(\gamma+\ln(\pi)-Ci(\pi)\big)+\frac{\pi}{4}+\sqrt{2\pi}-\frac{\sqrt{\pi}}{2}\int_0^{\pi}J_{-\frac{1}{2}}(x)dx$$

$$I=\frac{1}{2}\left(\gamma+\ln(\pi)-Ci(\pi)+\frac{\pi}{2}\right)+\sqrt{2\pi}-\frac{\sqrt{\pi}}{2}C$$

Note section:

$$\gamma\to\ \textit{Euler Mascheroni's constant}$$

$$C=\int_0^{\pi}J_{-\frac{1}{2}}(x)dx\ ,\ \ J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos(x)\to\textit{Numerical constant}$$

$$Ci(x)=-\int_x^{\infty}\frac{\cos(t)}{t}dt=\gamma+\ln(x)\,n-\int_0^{x}\frac{1-\cos(t)}{t}dt$$