ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_1^\infty \int_0^1 \frac{ln^2(\sqrt{x})ln^2(xy)}{(1+y^2)(2+x)^2}dxdy$$

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$$I=\int_1^\infty \int_0^1 \frac{ln^2(\sqrt{x})ln^2(xy)}{(1+y^2)(2+x)^2}dxdy$$

$*\ But:$

$$ln^2(AB)=ln^2(A)+2\ln(A)\ln(B)+ln^2(B)$$

$$I=\int_1^\infty \int_0^1 \frac{ln^2(\sqrt{x})ln^2(x)}{(1+y^2)(2+x)^2}dxdy+2\int_1^\infty \int_0^1 \frac{ln^2(\sqrt{x})ln(x)\ln(y)}{(1+y^2)(2+x)^2}dxdy+$$

$$\int_1^\infty \int_0^1 \frac{ln^2(\sqrt{x})ln^2(y)}{(1+y^2)(2+x)^2}dxdy=\frac{1}{4}\int_1^\infty \int_0^1 \frac{ln^4(x)}{(1+y^2)(2+x)^2}dxdy+\frac{2}{4}\int_1^\infty \int_0^1 \frac{ln^3(x)ln(y)}{(1+y^2)(2+x)^2}dxdy+$$

$$\frac{1}{4}\int_1^\infty \int_0^1 \frac{ln^2(x)ln^2(y)}{(1+y^2)(2+x)^2}dxdy=\frac{1}{4}\int_1^\infty \frac{dy}{1+y^2}\int_0^1 \frac{ln^4(x)}{(2+x)^2}dx+\frac{1}{2}\int_1^\infty \frac{\ln(y)}{1+y^2}dy\int_0^1 \frac{ln^3(x)}{(2+x)^2}dx+$$

$$+\frac{1}{4}\int_1^\infty \frac{ln^2(y)}{1+y^2}dy\int_0^1 \frac{ln^2(x)}{(2+x)^2}dx$$

By working at the integrals

$$1)\int_1^\infty \frac{dy}{1+y^2}=\tan(y)|_1^\infty=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

$$2)\int_1^\infty \frac{ln(y)}{1+y^2}dy \overset{y\to\frac{1}{y}}{=} \int_1^0 \frac{\ln\left(\frac{1}{y}\right)}{\frac{1+y^2}{y^2}}\frac{-dy}{y^2}=-\int_0^1 \frac{\ln(y)}{1+y^2}dy=-\sum_{n=0}^\infty(-1)^n\int_0^1 y^{2n}\ln(y)\,dy=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^2}=G$$

$$3)\int_1^\infty \frac{ln^2(y)}{1+y^2}dy \overset{y\to\frac{1}{y}}{=} \int_1^0 \frac{ln^2\left(\frac{1}{y}\right)}{\frac{1+y^2}{y^2}}\frac{-dy}{y^2}=\int_0^1 \frac{ln^2(y)}{1+y^2}dy=\sum_{n=0}^\infty(-1)^n\int_0^1 y^{2n}ln^2(y)\,dy=2\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{16}$$

$$4)\ A=\int_0^1 \frac{ln^4(x)}{(2+x)^2}dx=\sum_{n=0}^\infty\binom{-2}{n}2^{-2-n}\int_0^1 x^n\,ln^4(x)dx=\sum_{n=0}^\infty\binom{-2}{n}2^{-2-n}\frac{d}{dn}\int_0^1 x^n\,dx=$$

$$\sum_{n=0}^\infty\binom{-2}{n}\frac{1}{2^{2+n}}\cdot\frac{24}{(n+1)^5}=24\sum_{n=0}^\infty\frac{\binom{-2}{n}}{2^{2+n}.(n+1)^5}=24\left[-\frac{1}{2}Li_4\left(-\frac{1}{2}\right)\right]=-12\,Li_4\left(-\frac{1}{2}\right)$$

$$5)\ B=\int_0^1 \frac{ln^3(x)}{(2+x)^2}dx=\sum_{n=0}^{\infty}\binom{-2}{n}2^{-2-n}\int_0^1 x^n\, ln^3(x)dx=\sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{2^{2+n}}.\int_0^1 x^n\, ln^3(x)dx=$$

$$\sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{2^{2+n}}.\left[-\frac{6}{(n+1)^4}\right]=-6\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{2^{2+n}.(n+1)^4}=-6\left[-\frac{1}{2}Li_3\left(-\frac{1}{2}\right)\right]=3Li_3\left(-\frac{1}{2}\right)$$

$$6)\ C=\int_0^1 \frac{ln^2(x)}{(2+x)^2}dx=\sum_{n=0}^{\infty}\binom{-2}{n}2^{-2-n}\int_0^1 x^n\, ln^2(x)dx=\sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{2^{2+n}}.\int_0^1 x^n\, ln^2(x)dx=$$

$$\sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{2^{2+n}}.\left[\frac{2}{(n+1)^3}\right]=2\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{2^{2+n}.(n+1)^3}=2\left[-\frac{1}{2}Li_2\left(-\frac{1}{2}\right)\right]=-Li_2\left(-\frac{1}{2}\right)$$

This :

$$I=\frac{1}{4}\int_1^{\infty}\frac{dy}{1+y^2}\int_0^1\frac{ln^4(x)}{(2+x)^2}dx+\frac{1}{2}\int_1^{\infty}\frac{\ln(y)}{1+y^2}dy\int_0^1\frac{ln^3(x)}{(2+x)^2}dx+\frac{1}{4}\int_1^{\infty}\frac{ln^2(y)}{1+y^2}dy\int_0^1\frac{ln^2(x)}{(2+x)^2}dx=$$

$$\frac{1}{4}\left(\frac{\pi}{4}\right)\left(-12Li_4\left(-\frac{1}{2}\right)\right)+\frac{1}{2}G\left(3Li_3\left(-\frac{1}{2}\right)\right)+\frac{1}{4}\left(\frac{\pi^3}{16}\right)\left(-Li_2\left(-\frac{1}{2}\right)\right)$$

$$I=-\frac{3}{4}Li_4\left(-\frac{1}{2}\right)+\frac{3}{2}GLi_3\left(-\frac{1}{2}\right)-\frac{\pi^3}{64}Li_2\left(-\frac{1}{2}\right)$$

$$\int_1^{\infty}\int_0^1\frac{ln^2(\sqrt{x})ln^2(xy)}{(1+y^2)(2+x)^2}dxdy=\frac{3}{2}GLi_3\left(-\frac{1}{2}\right)-\frac{3}{4}Li_4\left(-\frac{1}{2}\right)-\frac{\pi^3}{64}Li_2\left(-\frac{1}{2}\right)$$