ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\iint_0^1 \frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$I=\iint_0^1 \frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=\underbrace{\int_0^1 \frac{yln(y)}{\sqrt{y}+1}dy}_{A}\underbrace{\int_0^1 \frac{xarctan^3(x)}{x^2+1}dx}_{B}$$

$$*\ Working\ on\ A$$

$$A=\int_0^1 \frac{yln(y)}{\sqrt{y}+1}dy=\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{\frac{n}{2}+1}\ln(y)\,dy=-4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+4)^2}=-4\left[\frac{31}{36}-\frac{\pi^2}{12}\right]=$$

$$-\frac{31}{9}+\frac{\pi^2}{3}=-\frac{31}{9}+2\zeta(2)$$

$$*\ Working\ on\ B$$

$$B=\int_0^1 \frac{xarctan^3(x)}{x^2+1}dx\overbrace{\underbrace{=}_{dx=sec^2(y)dy}}^{x=\tan(y)}\int_0^{\frac{\pi}{4}}\frac{\tan(y)arctan^3(\tan(y))}{1+tan^2(y)}.sec^2(y)dy=$$

$$\int_0^{\frac{\pi}{4}}\frac{y^3\tan(y)}{sec^2(y)}.sec^2(y)dy=\int_0^{\frac{\pi}{4}}y^3\tan(y)\overset{I.B.P}{=}-y^3\ln(cos(y))|_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}y^2\ln(cos(y))\,dy=$$

$$\frac{\pi^3}{128}\ln(2)+3\underbrace{\int_0^{\frac{\pi}{4}}y^2\ln(cos(y))\,dy}_{B_1}$$

$$B_1=\int_0^{\frac{\pi}{4}}y^2\ln(cos(y))\,dy\quad\left\{\ln(cos(y))=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2ny)}{n}\right\}$$

$$B_1=-\ln(2)\int_0^{\frac{\pi}{4}}y^2\,dy-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}y^2\cos(2ny)\,dy=-\frac{1}{3}\left(\frac{\pi}{4}\right)^3\ln(2)-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi^2}{32n}\sin\left(\frac{\pi n}{2}\right)-\frac{\sin\left(\frac{\pi n}{2}\right)}{4n^3}+\frac{\pi}{8n^2}\cos\left(\frac{\pi n}{2}\right)\right]=-\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G+$$

$$\frac{1}{8}iLi_4(-i)-\frac{1}{8}iLi_4(i)+\frac{3\pi}{256}\zeta(3)=\frac{\pi^2}{32}G-\frac{\pi}{8}Li_3(-i)-\frac{1}{4}iLi_4(-i)-\frac{187}{46080}i\pi^4-\frac{\pi^3}{192}\ln(2)$$

$$*\,But\ \ Li_4(x)\ +\ Li_4(-x)=\frac{1}{8}\ Li_4\left(x^2\right)$$

$$B=\frac{\pi^3}{128}\ln(2)+3B_1=\frac{\pi^3}{128}\ln(2)+\frac{3\pi^2}{32}G-\frac{3}{8}\pi Li_3(-i)-\frac{3}{4}iLi_4(-i)-\frac{187}{15360}i\pi^4-\frac{\pi^3}{64}\ln(2)$$

$$B=-\frac{\pi^3}{128}\ln(2)+\frac{3\pi^2}{32}G-\frac{3}{8}\pi Li_3(-i)-\frac{3}{4}iLi_4(-i)-\frac{187}{15360}i\pi^4$$

$$I=A.B$$

$$\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$
$$=\left[2\zeta(2)-\frac{31}{9}\right]\left[-\frac{\pi^3}{128}\ln(2)+\frac{3\pi^2}{32}G-\frac{3}{8}\pi Li_3(-i)-\frac{3}{4}iLi_4(-i)-\frac{187}{15360}i\pi^4\right]$$

Solution 2 by Ankush Kumar Parcha-India

$$We\ have\ ,\qquad \Omega=\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=\underbrace{\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy}_{M}\underbrace{\int_0^1\frac{xarctan^3(x)}{x^2+1}dx}_{N}\quad(1)$$

$$M=\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy\overset{y\to x^2}{\Rightarrow}4\int_0^1\frac{x^3\ln(x)}{1+x}dx=4\sum_{n\in N}(-1)^{n-1}\underbrace{\int_0^1x^{n+2}\ln(x)\,dx}_{I.B.P}=$$

$$-4\sum_{n\in N}(-1)^{n-1}[\underbrace{(\frac{\ln(x)}{n+3}\int d(x^{n+3}))\Big|_0^1}_{0}-\frac{1}{n+3}\int_0^1x^{n+2}dx]=-4\sum_{n\in N}\frac{(-1)^{n-1}}{(n+3)^2}\int d\left(x^{n+3}\right)\overset{n\to n-3}{\triangleq}$$

$$-4\left(1-\frac{1}{4}+\frac{1}{9}\right)+4\underbrace{\sum_{n\in N}\frac{(-1)^{n-1}}{n^2}}_{=\eta(2)}=\left(\frac{\pi^2}{3}-\frac{31}{9}\right)$$

$$N=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx\overset{x\to\tan(x)}{\triangleq}\int_0^{\frac{\pi}{4}}x^3\tan(x)\overset{I.B.P}{\triangleq}-(x^3\int dlncos(x))\Big|_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}x^2lncos(x)dx$$

$$(\therefore \sum_{k\in N}\frac{(-1)^k\cos(2kx)}{k}=\ln\left(\frac{sec(x)}{2}\right)\ \ |x|<\frac{\pi}{2})$$

$$N=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx=\frac{\pi^3}{128}\ln(2)+3\ln(2)\int_0^{\frac{\pi}{4}}x^2dx-3\sum_{n\in N}\frac{(-1)^n}{n}\underbrace{\int_0^{\frac{\pi}{4}}x^2\cos(2nx)\,dx}_{I.B.P}=$$

$$\frac{\pi^3}{128}-ln(2)\int_0^{\frac{\pi}{4}}d(x^3)-3\sum_{n\in N}\frac{(-1)^n}{n}\left[\frac{x^2}{2n}\int dsin(2nx))\begin{matrix}\frac{\pi}{4}\\0\end{matrix}-\frac{1}{n}\underbrace{\int_0^{\frac{\pi}{4}}xsin(2nx)}_{I.B.P}\right]=\frac{\pi^3}{128}ln(2)-$$

$$\frac{\pi^3}{64}\ln(2)-\frac{3\pi^2}{32}\sum_{n\in N}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)+3\sum_{n\in N}\frac{(-1)^n}{n^2}\left[-\frac{x}{2n}\int dcos(2nx))\begin{matrix}\frac{\pi}{4}\\0\end{matrix}+\frac{1}{2n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx\right]=$$

$$-\frac{\pi^3}{128}ln(2)-\frac{3\pi^2}{32}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)}_{n\to 2n+1}-\frac{3\pi}{8}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^3}\cos\left(\frac{\pi n}{2}\right)}_{n\to 2n}+\frac{3}{4}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^4}\int dsin(2nx)}_{n\to 2n+1}$$

$$N=\frac{3\pi^2}{32}G-\frac{\pi^3}{128}\ln(2)-\frac{3}{4}\beta(4)+\frac{9}{256}\pi\zeta(3)$$

$$\Omega=M.N$$

$$\Omega=\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=\left[2\zeta(2)-\frac{31}{9}\right]\left[\frac{3\pi^2}{32}G-\frac{\pi^3}{128}\ln(2)-\frac{3}{4}\beta(4)+\frac{9}{256}\pi\zeta(3)\right]$$

Solution 3 by Exodo Halcalias-Angola

$$\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

$$I=\left(\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy\right)\left(\int_0^1\frac{xarctan^3(x)}{x^2+1}dx\right)$$

$$E=\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy=4\int_0^1\frac{y^3\ln(y)}{y+1}dy=4\int_0^1y^2\ln(y)\,dy-4\int_0^1yln(y)dy\ +$$

$$4\int_0^1\ln(y)\,dy-4\int_0^1\frac{\ln(y)}{y+1}dy=-4\left(\frac{1}{3^2}-\frac{1}{2^2}+1\right)-4\left(\ln(y)\int dln(y+1)\right)\Big|_0^1+$$

$$4\int_0^1\frac{\ln(y+1)}{y}dy=-\frac{31}{9}-4\int_0^1dLi_2(-x)=-\frac{31}{9}+4.\frac{\pi^2}{12}=2\zeta(2)-\frac{31}{9}$$

$$H=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx=\int_0^{\frac{\pi}{4}}x^3\tan(x)\,dx=\left(x^3\int dln(\sec(x))\right|_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}x^2\ln(cos(x))\,dx=$$

$$\left(\frac{\pi}{4}\right)^3\ln\left(sec\left(\frac{\pi}{4}\right)\right)+3\int_0^{\frac{\pi}{4}}x^2\left(\ln\left(\frac{1}{2}\right)+\sum_{k\in N}\frac{(-1)^{k-1}\cos(2kx)}{k}dx\right)=\frac{\pi^3}{128}\ln(2)-\left(\frac{\pi}{4}\right)^3\ln(2)+$$

$$3\sum_{k\in N}\frac{(-1)^{k-1}}{k}\int_0^{\frac{\pi}{4}}x^2\cos(2kx)\,dx=\frac{\pi^3}{128}\ln\left(\frac{1}{2}\right)+3\sum_{k\in N}\frac{(-1)^{k-1}}{k}\left(\frac{\pi^2}{32k}-\frac{\sin\left(\frac{\pi k}{2}\right)}{4k^3}+\frac{\pi\cos\left(\frac{\pi k}{2}\right)}{4k^2}\right)$$
$$=$$

$$\frac{\pi^3}{128}\ln\left(\frac{1}{2}\right)+\frac{3\pi^2}{32}\sum_{k\in N}\frac{(-1)^{k-1}}{k^2}-\frac{3}{4}\sum_{k\in N}\frac{(-1)^{k-1}\sin\left(\frac{\pi k}{2}\right)}{k^4}+\frac{3\pi}{8}\sum_{k\in N}\frac{(-1)^{k-1}\cos}{k^3}=\frac{\pi^3}{128}\ln\left(\frac{1}{2}\right)+$$

$$\frac{3\pi^2}{32}.\frac{1}{2}\sum_{k\in N}\frac{1}{k^2}-\frac{3}{4}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^4}+\frac{3\pi}{8}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k)^3}=$$

$$\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{3\pi^2}{64}.\frac{\pi^2}{6}-\frac{3}{4}\beta(4)+\frac{9}{256}\pi\zeta(2)$$

$$\left(\therefore\ \beta(u)=\frac{(-1)^u2^{1-2u}}{(u-1)!}\psi^{(u-1)}\left(\frac{1}{4}\right)+\left(\frac{1}{2^u}-1\right)\zeta(u)\,,\qquad u\in Z^+\right)$$

$$\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{\zeta(4)}{8}-\frac{3}{4}\left(\frac{1}{768}\psi^{(3)}\left(\frac{1}{4}\right)-\frac{15\zeta(4)}{16}\right)+\frac{9}{128}\pi\zeta(3)$$

$$H=\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{53\zeta(4)}{8}-\frac{1}{1024}\psi^{(3)}\left(\frac{1}{4}\right)+\frac{9}{128}\pi\zeta(3)$$

$$I=\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=E.H$$

$$I=\left(2\zeta(2)-\frac{31}{9}\right)\left(\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{53\zeta(4)}{8}-\frac{1}{1024}\psi^{(3)}\left(\frac{1}{4}\right)+\frac{9}{128}\pi\zeta(3)\right)$$

Solution 4 by Cosghun Memmedov-Azerbaijan

$$\Omega=\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

$$\iint_0^1 \frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy = \left(\int_0^1 \frac{yln(y)}{\sqrt{y}+1}dy\right).\left(\int_0^1 \frac{xarctan^3(x)}{x^2+1}dx\right) = \Psi.\Phi$$

$$\Psi = \int_0^1 \frac{yln(y)}{\sqrt{y}+1}dy \overset{\sqrt{y}\to y}{\cong} 4\int_0^1 \frac{y^3\ln(y)}{y+1}dy = 4\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{n+3}\ln(y)\,dy =$$

$$4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+4)^2} = -\frac{31}{9} + 4.\frac{\pi^2}{12} = 2\zeta(2) - \frac{31}{9}$$

$$\Phi = \int_0^1 \frac{xarctan^3(x)}{x^2+1}dx \overset{\arctan(x)\to x}{\cong} \int_0^{\frac{\pi}{4}} x^3\tan(x)\,dx \overset{I.B.P}{\cong} -\frac{\pi^3}{64}\ln\left(\frac{1}{\sqrt{2}}\right) + 3\underbrace{\int_0^{\frac{\pi}{4}} x^2\ln(cos(x))\,dx}_{\xi} =$$

$$\frac{\pi^3}{128}\ln(2) + 3\xi$$

$$\xi = \int_0^{\frac{\pi}{4}} x^2\ln(cos(x))\,dx = -ln(2)\int_0^{\frac{\pi}{4}} x^2dx - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} x^2\cos(2nx)\,dx = -\frac{\pi^3}{192}\ln(2) -$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}\int_0^{\frac{\pi}{4}} x^2d(\sin(2nx)) \overset{I.B.P}{\cong} -\frac{\pi^3}{192}\ln(2) - \frac{\pi^2}{32}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2} +$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^{\frac{\pi}{4}} xsin(2nx)dx = -\frac{\pi^3}{192}\ln(2) + \frac{\pi^2}{32}G - \sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}\int_0^{\frac{\pi}{4}} xd(\cos(2nx)) =$$

$$-\frac{\pi^3}{192}\ln(2) + \frac{\pi^2}{32}G - \frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx = -\frac{\pi^3}{192}\ln(2) + \frac{\pi^2}{32}G +$$

$$\frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} + \sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{4n^4} = -\frac{\pi^3}{192}\ln(2) + \frac{\pi^2}{32}G + \frac{3}{256}\zeta(3) - \frac{1}{4}\beta(4)$$

$$\xi = -\frac{\pi^3}{192}\ln(2) + \frac{\pi^2}{32}G + \frac{3}{256}\zeta(3) - \frac{1}{4}\left(\frac{\psi^{(3)}\left(\frac{1}{4}\right)}{768} - \frac{\pi^4}{96}\right)$$

$$\Phi = \frac{\pi^3}{128}\ln(2) + 3\xi = \frac{\pi^3}{128}\ln(2) + \frac{9}{256}\pi\zeta(3) + \frac{\pi^4}{128} + \frac{3\pi^2}{32}G - \frac{\psi^{(3)}\left(\frac{1}{4}\right)}{1024}$$

$$\Omega = \Psi.\Phi = \left(2\zeta(2) - \frac{31}{9}\right)\left(\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right) + \frac{53\zeta(4)}{8} - \frac{1}{1024}\psi^{(3)}\left(\frac{1}{4}\right) + \frac{9}{128}\pi\zeta(3)\right)$$