ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{x^2(1+\sqrt{x})cos^2(x)}{\sqrt{1-x^2}}dx$$

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$$I=\int_0^1 \frac{x^2(1+\sqrt{x})cos^2(x)}{\sqrt{1-x^2}}dx=\int_0^1 \frac{x^2cos^2(x)}{\sqrt{1-x^2}}dx+\int_0^1 \frac{x^2\sqrt{x}cos^2(x)}{\sqrt{1-x^2}}dx=E+H$$

$$E=\int_0^1 \frac{x^2cos^2(x)}{\sqrt{1-x^2}}dx=\int_0^1 \frac{x^2}{\sqrt{1-x^2}}(1+\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}x^{2k}}{(2k)!})dx=$$

$$=\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx+\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\int_0^1 \frac{x^{2k+2}}{\sqrt{1-x^2}}dx=\frac{1}{2}\int_0^1 \frac{x^{\frac{3}{2}-1}}{\sqrt{1-x}}dx+$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\int_0^1 \frac{x^{k+\frac{3}{2}-1}}{\sqrt{1-x}}dx=\frac{1}{2}(\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right))+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\left(\Gamma\left(\frac{1}{2}\right)\frac{\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(k+1)}\right)=$$

$$=\frac{1}{4}\Gamma^2\left(\frac{1}{2}\right)+\frac{\sqrt{\pi}}{2}(-\frac{\sqrt{\pi}}{4}+\frac{\sqrt{\pi}}{4}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}2^{2k-1}(2(1)_k-(2)_k)}{k!\,(1)_k(2)_k})=$$

$$=\frac{\pi}{4}+\frac{\sqrt{\pi}}{2}\left(-\frac{\sqrt{\pi}}{4}+\frac{\sqrt{\pi}}{4}J_{(0)}(2)-\frac{\sqrt{\pi}}{2}J_{(1)}(2)\right)$$

$$E=\left(\frac{\pi}{8}+\frac{\pi}{8}J_{(0)}(2)-\frac{\pi}{4}J_{(1)}(2)\right)$$

$$H=\int_0^1 \frac{x^2\sqrt{x}cos^2(x)}{\sqrt{1-x^2}}dx=\int_0^1 \frac{x^2\sqrt{x}}{\sqrt{1-x^2}}dx(1+\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}x^{2k}}{(2k)!})dx=$$

$$\int_0^1 \frac{x^2\sqrt{x}}{\sqrt{1-x^2}}dx+\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\int_0^1 \frac{x^{2k+\frac{5}{2}}}{\sqrt{1-x^2}}dx=\frac{1}{2}\int_0^1 \frac{x^{\frac{7}{4}-1}}{\sqrt{1-x}}dx+$$

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\int_0^1 \frac{x^{k+\frac{7}{4}-1}}{\sqrt{1-x}}dx=\frac{1}{2}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{9}{4}\right)}\right)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\left(\Gamma\left(\frac{1}{2}\right)\frac{\Gamma\left(k+\frac{7}{4}\right)}{\Gamma\left(k+\frac{9}{4}\right)}\right)=$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{12\pi\sqrt{2}}{5\Gamma^2\left(\frac{1}{4}\right)}\right) + \frac{\sqrt{\pi}}{2}\left(\frac{6\Gamma\left(\frac{3}{4}\right)\ {}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)}{5\Gamma\left(\frac{1}{4}\right)} - \frac{6\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(\frac{1}{4}\right)}\right) =$$

$$= \frac{3\sqrt{\pi}}{5}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2}}\right)^{-1} + \frac{3\sqrt{\pi}}{5}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2}}\right)^{-1} {}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right) - \frac{3\sqrt{\pi}}{10}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2}}\right)^{-1}$$

$$H = \frac{3\sqrt{\pi}}{10\varpi}\left(\left({}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)+1\right)\right)$$

$$I = E + H = \left(\frac{\pi}{8} + \frac{\pi}{8}J_{(0)}(2) - \frac{\pi}{4}J_{(1)}(2)\right) + \frac{3\sqrt{\pi}}{10\varpi}\left(\left({}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)+1\right)\right)$$

$$J_{(a)}(x) \to \textit{Bassel function of first the kind}$$

$$\varpi \to \textit{Lemniscate constant}$$