ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_1^{\infty}\frac{x^3 ln^2(x)}{(1+x)(1+x^2)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

$$We\ \ have,\quad \int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx \overset{x\to\frac{1}{x}}{\leftrightarrows} \int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x}dx-$$

$$\underbrace{\frac{1}{4}\int_0^1\frac{xln^2(x)}{1+x^2}dx}_{x^2\to x}+\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\underbrace{\frac{1}{2}\int_0^1\frac{xln^2(x)}{(1+x^2)^2}dx}_{I.B.P}+\underbrace{\frac{1}{2}\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx}_{I.B.P}=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx+$$

$$\frac{7}{32}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\underbrace{\left(\frac{ln^2(x)}{4}\int d\left(\frac{x^2}{1+x^2}\right)\right)\Big|_0^1}_{=0}+\underbrace{\frac{1}{2}\int_0^1\frac{xln(x)}{1+x^2}dx}_{x^2\to x}+\underbrace{\left(\frac{ln^2(x)}{4}\int d\left(\frac{x^2}{1+x^2}+\arctan(x)\right)\right)\Big|_0^1}_{=0}$$

$$-\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx-\underbrace{\frac{1}{2}\int_0^1\frac{\ln(x)\arctan(x)}{x}dx}_{I.B.P}=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx+\frac{7}{32}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx+$$

$$\frac{1}{8}\int_0^1\frac{\ln(x)}{1+x}dx-\underbrace{\left(\frac{\arctan(x)}{4}\int d\left(ln^2(x)\right)\right)\Big|_0^1}_{=0}+\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx \overset{|x|<1\ x\in(0,1)}{\leftrightarrows} \frac{1}{2}\sum_{n\in N_0}(-1)^n\int_0^1 x^{2n}\,ln^2(x)dx+$$

$$\frac{7}{32}\sum_{n\in N_0}(-1)^{n-1}\int_0^1 x^{n-1}ln^2(x)dx-\frac{1}{2}\sum_{n\in N_0}(-1)^n\int_0^1 x^{2n}\,ln(x)dx+\frac{1}{8}\sum_{n\in N_0}(-1)^{n-1}\int_0^1 x^{n-1}\ln(x)dx$$

$$\left(\therefore\int_0^1 t^m ln^n(t)\,dt=\frac{(-1)^n.n!}{(m+1)^{n+1}}\quad n>-1\bigwedge m\neq -1\right)$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^3}}_{=\beta(3)}+\frac{7}{16}\underbrace{\sum_{n\in N_0}\frac{(-1)^{n-1}}{(n)^3}}_{=\eta(3)}+\frac{1}{2}\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^2}}_{=G\ (Catalan's\ constant)}$$

$$-\frac{1}{8}\underbrace{\sum_{n\in N}\frac{(-1)^{n-1}}{(n)^2}}_{=\zeta(2)}$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_1^{\infty}\frac{x^3 ln^2(x)}{(1+x)(1+x^2)^2}dx$$

$$\int_1^{\infty}\frac{x^3 ln^2(x)}{(1+x)(1+x^2)^2}dx \overset{x\to\frac{1}{x}}{\cong} \int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)^2}dx =$$

$$\int_0^1 ln^2(x)\left(\frac{1-x}{4(x^2+1)}+\frac{1-x}{2(1+x^2)^2}+\frac{1}{4(x+1)}\right)dx = \frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx - \frac{1}{4}\int_0^1\frac{xln^2(x)}{1+x^2}dx +$$

$$\int_0^1\frac{ln^2(x)}{2(1+x^2)^2}dx - \frac{1}{2}\int_0^1\frac{xln^2(x)}{(1+x^2)^2}dx + \frac{1}{4}\int_0^1\frac{ln^2(x)}{x+1}dx = \frac{1}{4}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-2}\,ln^2(x)dx +$$

$$\frac{7}{32}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{k-1}ln^2(x)dx + \frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-2}\,ln^2(x)dx -$$

$$\frac{1}{16}\sum_{k\in N}(-1)^{k-1}k\int_0^1 x^{k-1}ln^2(x)dx = \frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^3} + \frac{7}{16}\sum_{k\in N}\frac{(-1)^{k-1}}{(k)^3} +$$

$$\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\left(\frac{1}{(2k-1)^2}+\frac{1}{(2k-1)^3}\right) - \frac{1}{8}\sum_{k\in N}\frac{(-1)^{k-1}}{(k)^2}$$

$$\left\{\therefore \sum_{n\in N}\frac{(-1)^n}{(2k+1)^z} = \beta(z);\ \ \beta(2) = G;\ \ \beta(3) = \frac{3}{16}\pi\zeta(2)\right\}$$

$$= \frac{1}{2}\cdot\frac{3}{16}\pi\zeta(2) + \frac{7}{16}\cdot\frac{\zeta(3)}{4} + \frac{1}{2}\left(G + \frac{3}{16}\pi\zeta(2)\right) - \frac{1}{8}\cdot\frac{\zeta(2)}{2}$$

$$\int_1^{\infty}\frac{x^3 ln^2(x)}{(1+x)(1+x^2)^2}dx = \frac{21}{64}\zeta(3) + \frac{G}{2} + \beta(3) - \frac{\zeta(2)}{16}$$

Solution 3 by Pham Duc Nam-Vietnam

$$*\ A = \int_0^1\frac{ln^2(x)}{1-x^2}dx = \sum_{n=0}^{\infty}\int_0^1 x^{2n}ln^2(x)dx = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = \frac{7}{4}\zeta(3)$$

$$B = \int_0^1\frac{ln^2(x)}{(1+x)^2}dx = -\sum_{n=1}^{\infty}(-1)^n n\int_0^1 x^{n-1}ln^2(x)dx = -2\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2} = \zeta(2)$$

$$a\ge 1:\ C = \int_0^1\frac{ln^2(x)}{a+x^2}dx = \frac{1}{a}\int_0^1\frac{ln^2(x)}{1+\left(\sqrt{\frac{1}{a}}x\right)^2}dx = \frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{1}{a}\right)^n\int_0^1 x^{2n}\,ln^2(x)dx =$$

$$=\frac{2}{a}\sum_{n=0}^{\infty}\left(-\frac{1}{a}\right)^{n}\frac{1}{(2n+1)^{3}}=\frac{1}{4a}\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)$$

$$a=1:C_1=\int_0^1\frac{ln^2(x)}{1+x^2}dx=\frac{1}{4a}\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)\ \therefore\ \Phi\left(-1,s,\frac{1}{2}\right)=2^{s}\beta(s)\to\int_0^1\frac{ln^2(x)}{1+x^2}dx=2\beta(3)$$

$$\frac{d}{da}\bigg|_{a=1}\int_0^1\frac{ln^2(x)}{a+x^2}dx=C_2=-\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=\frac{d}{da}\bigg|_{a=1}\frac{1}{4a}\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)=$$

$$=-\frac{2\Phi\left(-\frac{1}{a},2,\frac{1}{2}\right)+\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)}{8a^2}\Bigg|_{a=1}=-G-\beta(3)\to C_2=G+\beta(3)$$

$$I=\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)^2}dx$$

$$=\int_0^1\frac{ln^2(x)}{(1-x^2)(1+x^2)^2}dx-\int_0^1\frac{xln^2(x)}{(1-x^2)(1+x^2)^2}dx$$

$$*\,E=\int_0^1\frac{ln^2(x)}{(1-x^2)(1+x^2)^2}dx\overset{x\to\frac{1}{x}}{\cong}\frac{1}{8}\int_0^1\frac{ln^2(x)}{(1-x)(1+x)^2}dx=\frac{1}{16}\int_0^1 ln^2(x)\left(\frac{1}{1-x^2}+\frac{1}{(1+x)^2}\right)dx=$$

$$=\frac{1}{16}(A+B)=\frac{7}{64}\zeta(3)+\frac{1}{16}\zeta(2)$$

$$*\,H=\int_0^1\frac{ln^2(x)}{(1-x^2)(1+x^2)^2}dx=\frac{1}{4}\underbrace{\int_0^1\frac{ln^2(x)}{1-x^2}dx}_{A}+\frac{1}{4}\underbrace{\int_0^1\frac{ln^2(x)}{1+x^2}dx}_{C_1}+\frac{1}{2}\underbrace{\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx}_{C_2}=$$

$$=\beta(3)+\frac{7}{16}\zeta(3)+\frac{G}{2}$$

$$I=H-E=\beta(3)+\frac{7}{16}\zeta(3)+\frac{G}{2}-\frac{7}{64}\zeta(3)-\frac{1}{16}\zeta(2)=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$