ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 (Li_3(-x) + x^2 arctan^2(x))dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$I = \int_0^1 (Li_3(-x) + x^2 arctan^2(x))dx$$

$$* S = \int_0^1 Li_3(-x)\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \int_0^1 x^n dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3(n+1)} =$$

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^3} - \frac{1}{n^2} - \frac{1}{n+1} + \frac{1}{n}\right) = -\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - 2\ln(2) + 1$$

$$* Y = \int_0^1 x^2 arctan^2(x) dx \overset{I.B.P}{\cong} \frac{1}{3} x^3 arctan^2(x)\Big|_0^1 - \frac{2}{3}\int_0^1 \frac{x^3 arctan^2(x)\big|}{1+x^2} dx \overset{I.B.P}{\cong}$$

$$\frac{\pi^2}{48} - \frac{2}{3}(\arctan(x)\left(\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)\right)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2 - \ln(1+x^2)}{1+x^2} dx) = \frac{\pi^2}{48} - \frac{2}{3}(\frac{\pi}{8} - \frac{\pi}{8}\ln(2) -$$

$$\frac{1}{2}\left(1 - \frac{\pi}{4}\right) + \underbrace{\frac{1}{2}\int_0^1 \frac{\ln(1+x^2)}{1+x^2} dx)}_{\frac{1}{x} \to x} = \frac{\pi^2}{48} - \frac{2}{3}(-\frac{1}{2} + \frac{\pi}{4} - \frac{\pi}{8}\ln(2) + \frac{1}{2}\int_1^{\infty} \frac{\ln\left(\frac{1+x^2}{x^2}\right)}{1+x^2} dx) =$$

$$\frac{\pi^2}{48} - \frac{2}{3}(-\frac{1}{2} + \frac{\pi}{4} - \frac{\pi}{8}\ln(2) + \frac{1}{4}\int_1^{\infty} \frac{\ln(1+x^2)}{1+x^2} dx - \frac{1}{4}\int_1^{\infty} \frac{\ln(x^2)}{1+x^2} dx) =$$

$$= \frac{\pi^2}{48} - \frac{2}{3}(-\frac{1}{2} + \frac{\pi}{4} - \frac{\pi}{8}\ln(2) -$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(cos(x))\, dx + \frac{1}{2}\int_0^1 \frac{\ln(x)}{1+x^2} dx) = \frac{\pi^2}{48} - \frac{2}{3}\left(-\frac{1}{2} + \frac{\pi}{4} - \frac{\pi}{8}\ln(2) + \frac{\pi}{4}\ln(2) - \frac{G}{2}\right) =$$

$$= \frac{G}{3} + \frac{1}{8}\zeta(2) - \frac{\pi}{12}\ln(2) - \frac{\pi}{6} + \frac{1}{3}$$

$$I = S + Y = -\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - 2\ln(2) + 1 + \frac{G}{3} + \frac{1}{8}\zeta(2) - \frac{\pi}{12}\ln(2) - \frac{\pi}{6} + \frac{1}{3}$$

$$I = \frac{G}{3} + \frac{5}{8}\zeta(2) - \frac{3}{4}\zeta(3) - \frac{\pi}{12}\ln(2) - \frac{\pi}{6} - 2\ln(2) + \frac{4}{3}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1 (Li_3(-x)+x^2 arctan^2(x))dx$$

$$E=\int_0^1 Li_3(-x)dx=\left(Li_3(-x)\int dx\right)\Big|_0^1-\int_0^1 Li_2(-x)\,dx=Li_3(-1)-\left(Li_2(-x)\int dx\right)\Big|_0^1-$$

$$\int_0^1 \ln(1+x)\,dx=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$$E=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$$H=\int_0^1 x^2 arctan^2(x)dx=\left(arctan^2(x)\int d\left(\frac{x^3}{3}\right)\right)\Big|_0^1-\frac{2}{3}\int_0^1\frac{x^3\arctan(x)}{1+x^2}dx=$$

$$\frac{arctan^2(1)}{3}-\frac{2}{3}\int_0^{\frac{\pi}{4}} xtan^3(x)dx=\frac{1}{3}\left(\frac{\pi}{4}\right)^2-\frac{2}{3}\left(x\int d\left(\frac{1}{2cos^2(x)}+\ln(cos(x))\right)\right)\Big|_0^{\frac{\pi}{4}}+$$

$$\frac{1}{3}\int_0^{\frac{\pi}{4}}\frac{dx}{cos^2(x)}+\frac{2}{3}\int_0^{\frac{\pi}{4}}\ln(cos(x))\,dx$$

$$\left\{\therefore\int_0^{\theta}\ln(cos(z))\,dz=\frac{1}{2}Cl_2(\pi-2\theta)-\theta\ln(2);\ Cl_2\left(\frac{\pi}{2}\right)=\beta(2)=G\right\}$$

$$=\frac{1}{8}\zeta(2)-\frac{2}{3}\left(\frac{\pi}{4}-\frac{\pi}{8}\ln(2)\right)+\frac{1}{3}\int_0^{\frac{\pi}{4}} dtan(x)+\frac{G}{3}-\frac{\pi}{6}\ln(2)$$

$$H=\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$E+H=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1+\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$\int_0^1 (Li_3(-x)+x^2 arctan^2(x))dx=\frac{G}{3}+\frac{5}{8}\zeta(2)-\frac{3}{4}\zeta(3)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}-2\ln(2)+\frac{4}{3}$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1 (Li_3(-x)+x^2 arctan^2(x))dx=\int_0^1 Li_3(-x)dx+\int_0^1 x^2 arctan^2(x)dx=A+B$$

$*\ working\ \ on\ A$

$$A=\int_0^1 Li_3(-x)dx\quad [u=Li_3(-x),\frac{du}{dx}=\frac{Li_2(-x)}{x}\ ;\ \ v=x\,,dv=dx]$$

$$A = xLi_3(-x)\Big|_0^1 - \int_0^1 Li_2(-x) = -\frac{3}{4}\zeta(3) - [xLi_3(-x)\Big|_0^1 + \int_0^1 \frac{xln(1+x)}{x}dx]$$

$$A = -\frac{3}{4}\zeta(3) - [-\frac{1}{2}\zeta(2) + \underbrace{\int_0^1 \ln(1+x)\,dx}_{x+1\to y} = -\frac{3}{4}\zeta(3) - [-\frac{1}{2}\zeta(2) + \int_1^2 \ln(y)\,dy]$$

$$A = -\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - 2\ln(2) + 1$$

$*$ ***working on B***

$$B = \int_0^1 x^2 arctan^2(x)dx \quad [x = \tan(y)\ ; dx = sec^2(y)]\ \left[0, \frac{\pi}{4}\right]$$

$$B = \int_0^{\frac{\pi}{4}} tan^2(y)arctan^2(\tan(y)).sec^2(y)dy = \int_0^{\frac{\pi}{4}} y^2 tan^2(y)\,sec^2(y)dy$$

$$B = \frac{y^2 tan^3(y)}{3}\Bigg|_0^{\frac{\pi}{4}} - \frac{2}{3}\int_0^{\frac{\pi}{4}} ytan^3(y)dy = \frac{1}{3}\left(\frac{\pi}{4}\right)^2 - \frac{2}{3}\int_0^{\frac{\pi}{4}} ytan^3(y)dy$$

$$B = \frac{\pi^2}{48} - \frac{2}{3}\int_0^{\frac{\pi}{4}} ytan^3(y)dy =$$

$$= \frac{\pi^2}{48} - \frac{2}{3}[\left(\frac{sec^2(y)}{2} + \ln(cos(y))\right)y\Big|_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(y)dy - \int_0^{\frac{\pi}{4}} \ln(cos(y))\,dy]$$

$$B = \frac{\pi^2}{48} - \frac{2}{3}[\frac{\pi}{4}\left(1 - \frac{\ln(2)}{2}\right) - \frac{1}{2}\tan\left(\frac{\pi}{4}\right) + \ln(2)\int_0^{\frac{\pi}{4}} dy + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} \cos(2ny)\,dy]$$

$$B = \frac{G}{3} + \frac{1}{8}\zeta(2) - \frac{\pi}{12}\ln(2) - \frac{\pi}{6} + \frac{1}{3}$$

$$A + B = -\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - 2\ln(2) + 1 + \frac{G}{3} + \frac{1}{8}\zeta(2) - \frac{\pi}{12}\ln(2) - \frac{\pi}{6} + \frac{1}{3}$$

$$I = \int_0^1 (Li_3(-x) + x^2 arctan^2(x))dx = \frac{G}{3} + \frac{5}{8}\zeta(2) - \frac{3}{4}\zeta(3) - \frac{\pi}{12}\ln(2) - \frac{\pi}{6} - 2\ln(2) + \frac{4}{3}$$