ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1 \frac{\ln(x)\,ln^3(x+1)}{x+1}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Kartick Chandra Betal-India

$$I=\int_0^1 \frac{\ln(x)\ln^3(x+1)}{x+1}dx=\int_1^2 \frac{\ln(x-1)\ln^3(x)}{x}dx=\int_{\frac{1}{2}}^1 \frac{-\{\ln(1-x)-\ln(x)\}\ln^3(x)}{x}dx=$$

$$\int_{\frac{1}{2}}^1 \frac{\ln^4(x)}{x}dx-\int_{\frac{1}{2}}^1 \frac{\ln(1-x)\ln^3(x)}{x}dx=\left[\frac{\ln^5(x)}{5}\right]_{\frac{1}{2}}^1+\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^1 x^{n-1}\ln^3(x)dx=$$

$$\frac{\ln^5(2)}{5}+\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{x^n}{n}\ln^3(x)-\frac{3x^n\ln^2(x)}{n^2}+\frac{6x^n\ln(x)}{n^3}-\frac{6x^n}{n^4}\right]_{\frac{1}{2}}^1=\frac{\ln^5(2)}{5}+$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left[-\frac{6}{n^4}-\left\{\frac{-\ln^3(2)}{n.2^n}-\frac{3\ln^2(2)}{n^2.2^n}-\frac{6\ln(2)}{n^3.2^n}-\frac{6}{n^4.2^n}\right\}\right]=\frac{\ln^5(2)}{5}+$$

$$\sum_{n=1}^{\infty}\left\{-\frac{6}{n^5}+\frac{\ln^3(2)}{n^2.2^n}+\frac{3\ln^2(2)}{n^3.2^n}+\frac{6\ln(2)}{n^4.2^n}+\frac{6}{n^5.2^5}\right\}=\frac{\ln^5(2)}{5}-6\zeta(5)+\ln^3(2).\mathrm{Li}_2\left(\frac{1}{2}\right)+$$

$$3\ln^2(2).\mathrm{Li}_3\left(\frac{1}{2}\right)+6\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)=\frac{\ln^5(2)}{5}-6\zeta(5)+\ln^3(2)\left\{\frac{\pi^2}{12}-\frac{\ln^2(2)}{2}\right\}+$$

$$3\ln^2(2)\left\{\frac{\ln^3(2)}{6}-\frac{\pi^2}{12}\ln(2)+\frac{7}{8}\zeta(3)\right\}+6\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)=$$

$$6\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)-6\zeta(5)-\zeta(2)\ln^3(2)+\frac{21}{8}\ln^2(2)\zeta(3)+\frac{\ln^5(2)}{5}$$

Solution 2 by Bui Hong Suc-Vietnam

$$\int_0^1 \frac{\ln(x)\,ln^3(x+1)}{x+1}dx$$

$$\Omega_n=\int_0^1 \frac{\ln(x)\,ln^{n-1}(x+1)}{x+1}dx=\frac{1}{n}\int_0^1 \ln(x)\,d\left(ln^n(x+1)\right)=\frac{1}{n}\left(\ln(x)ln^n(x+1)\Big|_0^1-\int_0^1 \frac{ln^n(x+1)}{x}dx\right)=$$

$$-\frac{1}{n}\int_0^1 \frac{ln^n(x+1)}{x}dx\overset{x+1=\frac{1}{v}}{\cong}\frac{(-1)^{n-1}}{n}\int_{\frac{1}{2}}^1 \frac{ln^n(v)}{v(1-v)}dv=\frac{(-1)^{n-1}}{n}I$$

$$I=\int_{\frac{1}{2}}^{1}\frac{ln^n(v)}{v(1-v)}dv=\int_{\frac{1}{2}}^{1}\frac{ln^n(v)}{v}dv+\int_{\frac{1}{2}}^{1}\frac{ln^n(v)}{1-v}dv=\frac{1}{n+1}ln^{n+1}(v)\Big|_{\frac{1}{2}}^{1}-\int_0^{\frac{1}{2}}\frac{ln^n(v)}{1-v}dv=$$

$$\frac{(-1)^n}{n+1}ln^{n+1}(2)+\sum_{k=1}^{\infty}\int_0^1 v^{k-1}ln^n(v)dv-\sum_{k=1}^{\infty}\int_0^{\frac{1}{2}}v^{k-1}ln^n(v)dv=\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n n!\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}-$$

$$\sum_{k=1}^{\infty}v^k n!\sum_{i=0}^{n}(-1)^n i!\binom{n}{i}\frac{(-1)^{-i}}{k^{i+1}}ln^{n-i}(v)\Big|_0^{\frac{1}{2}}=\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n.n!.\zeta(n+1)-$$

$$\sum_{k=1}^{\infty}n!\sum_{i=0}^{n}(-1)^n i!\binom{n}{i}ln^{n-i}(2)\frac{\left(\frac{1}{2}\right)^n}{k^{i+1}}=\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n.n!.\zeta(n+1)-$$

$$(-1)^n n!\sum_{i=0}^{n}i!\binom{n}{i}ln^{n-i}(2)Li_{i+1}\left(\frac{1}{2}\right)$$

$$\textit{Then}:$$

$$\Omega_n=\frac{(-1)^{n-1}}{n}\left\{\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n.n!.\zeta(n+1)-(-1)^n n!\sum_{i=0}^{n}i!\binom{n}{i}ln^{n-i}(2)Li_{i+1}\left(\frac{1}{2}\right)\right\}$$

$$iff\ \ n=4\quad \Omega_4=\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx=\frac{120\sum_{i=0}^{\infty}i!\binom{4}{i}ln^{4-i}(2)Li_{i+1}\left(\frac{1}{2}\right)-120\zeta(5)-ln^5\left(\frac{1}{2}\right)}{20}$$

$$\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx=6\ln(2)\,Li_4\left(\frac{1}{2}\right)+6Li_5\left(\frac{1}{2}\right)-6\zeta(5)-\zeta(2)ln^3(2)+\frac{21}{8}ln^2(2)\zeta(3)+\frac{ln^5(2)}{5}$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx$$

$$u=\ln(x)ln^3(x+1)\,,\quad \frac{du}{dx}=\frac{3ln^2(1+x)\ln(x)}{1+x}+\frac{ln^3(x+1)}{x}$$

$$v=\ln(1+x)\,,\qquad dv=\frac{1}{1+x}$$

$$I=\ln(x)\,ln^4(x+1)\Big|_0^1-3\underbrace{\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx}_{I}-\int_0^1\frac{ln^4(x+1)}{x}dx$$

$$I=-3I-\int_0^1\frac{ln^4(x+1)}{x}dx$$

$$I=-\frac{1}{4}\int_0^1\frac{ln^4(x+1)}{x}dx$$

Now the main task, let's calculate this integral …

$$\int_0^1 \frac{ln^4(x+1)}{x}dx$$

Recall that:

$$\int_0^1 \frac{ln^a(x+1)}{x}dx = \frac{ln^{a+1}(2)}{a+1} + a!\,\zeta(a+1) - (-1)^a \int_0^{\frac{1}{2}} \frac{ln^a(x)}{1-x}dx$$

$$\int_0^1 \frac{ln^4(x+1)}{x}dx = \frac{ln^5(2)}{5} + 24\zeta(5) - \int_0^{\frac{1}{2}} \frac{ln^4(x)}{1-x}dx$$

$*$ *Working on A*

$$A = \int_0^{\frac{1}{2}} \frac{ln^4(x)}{1-x}dx \quad x = \frac{p}{2}, \qquad dx = \frac{dp}{2} \quad [0,1]$$

$$A = \int_0^1 \frac{ln^4\left(\frac{p}{2}\right)}{1-\frac{p}{2}}\frac{dp}{2} = \frac{1}{2}\int_0^1 \frac{ln^4\left(\frac{p}{2}\right)}{1-\frac{p}{2}}dp = \frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 \left(\frac{p}{2}\right)^n ln^4\left(\frac{p}{2}\right)dp = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^4\left(\frac{p}{2}\right)dp =$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n[ln^4(p) - 4ln^3(p)\ln(2) + 6ln^2(p)ln^2(2) - 4\ln(p)\,ln^3(2) + ln^4(2)]dp =$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^4(p)dp - 2\ln(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^3(p)dp - 3ln^2(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^2(p)dp -$$

$$2ln^3(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln(p)dp + \frac{ln^4(2)}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n dp = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d^4}{dn^4}\left(\frac{p^{n+1}}{n+1}\bigg|_0^1\right) -$$

$$2\ln(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d^3}{dn^3}\left(\frac{p^{n+1}}{n+1}\bigg|_0^1\right) - 3ln^2(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d^2}{dn^2}\left(\frac{p^{n+1}}{n+1}\bigg|_0^1\right) - 2ln^3(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d}{dn}\left(\frac{p^{n+1}}{n+1}\bigg|_0^1\right) +$$

$$\frac{ln^4(2)}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\left(\frac{1}{n+1}\right) = 12\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^5} + 12\ln(2)\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^4} - 6ln^2(2)\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^3} +$$

$$2ln^3(2)\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^2} + \frac{ln^4(2)}{2}\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)} = 24Li_5\left(\frac{1}{2}\right) + 24\ln(2)\,Li_4\left(\frac{1}{2}\right) - 6ln^2(2)[\frac{7}{4}\zeta(3) +$$

$$\frac{1}{3}ln^3(2) - \zeta(2)\ln(2)] + 2ln^3(2)[\zeta(2) - ln^2(2)] + \frac{ln^4(2)}{2}.2\ln(2)$$

$$A = 24Li_5\left(\frac{1}{2}\right) + 24\ln(2)\,Li_4\left(\frac{1}{2}\right) + \frac{21}{2}\zeta(3)ln^2(2) + 2ln^5(2) - 6\zeta(2)ln^3(2) + 2\zeta(2)ln^3(2) - ln^5(2)$$

$$\int_0^1 \frac{ln^4(x+1)}{x}dx = \frac{ln^5(2)}{5} + 24\zeta(5) - A$$

$$\int_0^1 \frac{ln^4(x+1)}{x}dx = \frac{ln^5(2)}{5} + 24\zeta(5) - 24Li_5\left(\frac{1}{2}\right) - 24\ln(2)Li_4\left(\frac{1}{2}\right) - \frac{21}{2}\zeta(3)ln^2(2) -$$

$$-ln^5(2) + 6\zeta(2)ln^3(2) - 2\zeta(2)ln^3(2)$$

$$I = -\frac{1}{4}\int_0^1 \frac{ln^4(x+1)}{x}dx =$$

$$= -\frac{1}{4}\left[24\zeta(5) - 24\ln(2)Li_4\left(\frac{1}{2}\right) - 24Li_5\left(\frac{1}{2}\right) - \frac{21}{2}\zeta(3)ln^2(2) - \frac{4}{5}ln^5(2) + 4\zeta(2)ln^3(2)\right]$$

$$\int_0^1 \frac{\ln(x)\,ln^3(x+1)}{x+1}dx = 6\ln(2)Li_4\left(\frac{1}{2}\right) + 6Li_5\left(\frac{1}{2}\right) - 6\zeta(5) - \zeta(2)ln^3(2) + \frac{21}{8}ln^2(2)\zeta(3) + \frac{ln^5(2)}{5}$$

Solution 4 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^1 \frac{\log(x)\log^3(1+x)}{1+x}dx = \int_0^1 \frac{\left(\log(1+x) + \log\left(\frac{x}{1+x}\right)\right)\log^3(1+x)}{1+x}dx =$$

$$= \int_0^1 \frac{\log^4(1+x)}{1+x}dx + \int_0^1 \frac{\log\left(\frac{x}{1+x}\right)}{1+x}\log^3(1+x)\,dx = \frac{1}{5}\log^2(5) +$$

$$+\int_0^1 \left(\mathrm{Li}_2\left(\frac{1}{1+x}\right)\right)'\log^3(1+x)\,dx = \frac{1}{5}\log^2(5) + \mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2) -$$

$$-3\int_0^1 \frac{\mathrm{Li}_2\left(\frac{1}{1+x}\right)}{1+x}\log^2(1+x)\,dx = \frac{1}{5}\log^2(5) + \mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2) +$$

$$+3\int_0^1 \left(\mathrm{Li}_3\left(\frac{1}{1+x}\right)\right)'\log^2(1+x)\,dx = \frac{1}{5}\log^2(5) + \mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2) +$$

$$+3\left\{\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2) - 2\int_0^1 \frac{\mathrm{Li}_3\left(\frac{1}{1+x}\right)}{1+x}\log(1+x)\,dx\right\} = \frac{1}{5}\log^2(5) +$$

$$+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2) + 3\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2) + 6\int_0^1 \left(\mathrm{Li}_4\left(\frac{1}{1+x}\right)\right)'\log(1+x)\,dx =$$

$$= \frac{1}{5}\log^2(5) + \mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2) + 3\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2) + 6\left\{\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \int_0^1 \frac{\mathrm{Li}_4\left(\frac{1}{1+x}\right)}{1+x}dx\right\} =$$

$$= \frac{1}{5}\log^2(5) + \mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2) + 3\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2) + 6\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + 6\mathrm{Li}_5\left(\frac{1}{2}\right) - 6\zeta(5)$$