ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\left(x^2\ln(ln(x))+\frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\therefore\int_0^1\frac{ln^n(x)}{a+x}dx=-\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k\int_0^1 x^{k-1}ln^n(x)dx=-\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k x^k n!\sum_{i=0}^{n}\frac{(-1)^i ln^{n-i}(x)}{(n-i)!\,k^{i+1}}\Bigg|_0^1=(-1)^{n+1}n!\,Li_{n+1}\left(-\frac{1}{a}\right)$$

$$\therefore\int_0^1\frac{ln^n(x)}{1+x^2}dx=-\sum_{k=1}^{\infty}(-1)^k\int_0^1 x^{2k-1-1}ln^n(x)dx=-\sum_{k=1}^{\infty}(-1)^k x^{2k-1}n!\sum_{i=0}^{n}\frac{(-1)^i ln^{n-i}(x)}{(n-i)!\,(2k-1)^{i+1}}\Bigg|_0^1=n!\,(-1)^n\beta(n+1)$$

$$\therefore I_a=\int_0^1 x^{n-1}dx=a^{-1}:\frac{\partial^p}{\partial a^p}\quad I_a=\int_0^1 x^{n-1}ln^p(x)dx=(-1)^p\Gamma(p+1)a^{-1-p}$$

$$K_{a,p}=\int_0^1 x^{a-1}ln^p(x)\ln(ln(x))\,dx=\frac{\partial}{\partial p}\left(\frac{\partial^p}{\partial a^p}I_a\right)=\frac{\partial}{\partial p}\left(\int_0^1 x^{a-1}ln^p(x)dx\right)=$$

$$(-1)^p\Gamma(p+1)a^{-1-p}\{\ln(-1)+\psi(p+1)-\ln(a)\}$$

$$\Omega=\int_0^1\left(x^n\ln(ln(x))+\frac{xln^k\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}\right)dx=\int_0^1 x^n\ln(ln(x))\,dx+(-1)^k\int_0^1\frac{xln^k(x)}{(1+x^2)(x+2)}dx=I+(-1)^kJ$$

$$1)I=\int_0^1 x^n\ln(ln(x))\,dx=K_{n+1,0}=(-1)^0\Gamma(0+1)(n+1)^{-1-0}\{\ln(-1)+\psi(0+1)-\ln(n+1)\}=$$

$$\frac{\ln(-1)+\psi(1)-\ln(n+1)}{n+1}=\frac{\ln\left(e^{i\pi}\right)-\gamma-\ln(n+1)}{n+1}=\frac{i\pi-\gamma-\ln(n+1)}{n+1}$$

$$2)J=\int_0^1\frac{xln^k(x)}{(1+x^2)(x+2)}dx=\frac{2}{5}\int_0^1\frac{xln^k(x)}{(1+x^2)}dx+\frac{1}{5}\int_0^1\frac{ln^k(x)}{(1+x^2)}dx-\frac{2}{5}\int_0^1\frac{ln^k(x)}{(2+x)}dx=\frac{1}{5.2^k}\int_0^1\frac{ln^k(x^2)}{1+x^2}d(x^2)+$$

$$\frac{1}{5}k!\,(-1)^k\beta(k+1)-\frac{2}{5}k!\,(-1)^{k+1}Li_{k+1}\left(-\frac{1}{2}\right)=\frac{1}{5.2^k}k!\,(-1)^{k+1}Li_{k+1}(-1)+\frac{1}{5}k!\,(-1)^k\beta(k+1)+$$

$$\frac{2}{5}k!\,(-1)^kLi_{k+1}\left(-\frac{1}{2}\right)=\frac{(-1)^kk!}{5}\left\{\beta(k+1)+2Li_{k+1}\left(-\frac{1}{2}\right)+\frac{(1-2^{-k})\zeta(k+1)}{2^{2k}}\right\}$$

$$Then\ :\ \Omega=\frac{i\pi-\gamma-\ln(n+1)}{n+1}+\frac{(-1)^kk!}{5}\left\{\beta(k+1)+2Li_{k+1}\left(-\frac{1}{2}\right)+\frac{(1-2^{-k})\zeta(k+1)}{2^{2k}}\right\}=$$

$$\frac{i\pi-\gamma-\ln(n+1)}{n+1}+\frac{k!}{5}\left\{\beta(k+1)+2Li_{k+1}\left(-\frac{1}{2}\right)+\frac{(2^k-1)\zeta(k+1)}{2^{2k}}\right\}$$

$$iff:\ \ n=k=2;\ \Omega=\frac{i\pi-\gamma-\ln(3)}{3}+\frac{2}{5}\left\{\beta(3)+2Li_3\left(-\frac{1}{2}\right)+\frac{3\zeta(3)}{16}\right\}$$

$$\Omega=\frac{i\pi-\gamma-\ln(3)}{3}+\frac{1}{80}\left\{\pi^3+64Li_3\left(-\frac{1}{2}\right)+6\zeta(3)\right\}$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1\left(x^2\ln(ln(x))+\frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}\right)dx=\int_0^1(\underbrace{x^2\ln(ln(x))}_{A}+\underbrace{\frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}}_{B})dx$$

$*$ ***working on A***

$$A=\int_0^1 x^2\ln(ln(x))\,dx\overbrace{\underbrace{=}_{\substack{dx=e^p\\ [-\infty,0]}}}^{\substack{\ln(x)=p\\ x=e^p}}\int_{-\infty}^0 e^{2p}\ln(p).e^p dp=\int_{-\infty}^0 e^{3p}\ln(p)\,dp=\frac{d}{dn}\int_{-\infty}^0\lim_{n\to 0}p^n.e^{3p}dp=$$

$$=\lim_{n\to 0}\frac{d}{dn}\int_{-\infty}^0 p^n.e^{3p}dp=\lim_{n\to 0}\frac{d}{dn}\left[(-1)^n3^{-n-1}\Gamma(n+1)\right]=$$

$$=\lim_{n\to 0}\left[(-1)^n3^{-n-1}\Gamma(n+1)[\psi(n+1)+i\pi-\ln(3)]\right]=\frac{1}{3}[-\gamma+i\pi-\ln(3)]=\frac{i\pi-\gamma-\ln(3)}{3}$$

$*$ ***working on B***

$$B=\int_0^1\frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}dx=\int_0^1\frac{xln^2(x)}{(1+x^2)(x+2)}dx$$

Decompose into partial fraction…

$$B=\int_0^1\frac{(2x+1)ln^2(x)}{5(x^2+1)}dx-\frac{2}{5}\int_0^1\frac{ln^2(x)}{x+2}dx=\frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}ln^2(x)dx+\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}ln^2(x)dx-$$

$$\frac{2}{5}\sum_{n=0}^{\infty}(-1)^n2^{-1-n}\int_0^1 x^nln^2(x)dx=\frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{4(n+1)^3}\right]+\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{(2n+1)^3}\right]-\frac{2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{1+n}}\left[\frac{2}{(n+1)^3}\right]=$$

$$\frac{1}{10}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}+\frac{2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\frac{4}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{1+n}(n+1)^3}=\frac{1}{10}\left[\frac{3}{4}\zeta(3)\right]+\frac{2}{5}\left[\frac{\pi^3}{32}\right]+\frac{4}{5}\left[Li_3\left(-\frac{1}{2}\right)\right]=$$

$$=\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}+\frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$

But $I=A+B$

$$I=\frac{i\pi-\gamma-\ln(3)}{3}+\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}+\frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$