ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1\left(\frac{Li_2(-x)}{1+x}+x\arctan^2\left(\frac{1}{x^2}\right)\right)dx$$

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Solution 1 by Pham Duc Nam-Vietnam

$*$ ***Results will be used*** :

1. $\int_0^1\frac{\ln(x)}{1-x}dx=-\frac{\pi^2}{6},\ \text{easy to evaluate}$

2. $\int_0^1\frac{\ln(x)\ln(1+x)}{x}dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}\ln(x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=-\frac{3}{4}\zeta(3)$

3. $\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}dx\overset{I.B.P}{=}$

$$=-\zeta(2)\ln(2)+\int_0^1\frac{Li_2(x)}{1+x}dx=-\zeta(2)\ln(2)-\int_0^1\int_0^1\frac{x\ln(t)}{(1+x)(1-xt)}dxdt=$$

$$=-\zeta(2)\ln(2)+\int_0^1\ln(t)\left(\frac{\ln(2)}{1+t}+\frac{\ln(1-t)}{t}-\frac{\ln(1-t)}{1+t}\right)dt=-\zeta(2)\ln(2)+\ln(2)\int_0^1\frac{\ln(t)}{1+t}dt+$$

$$+\underbrace{\int_0^1\frac{\ln(t)\ln(1-t)}{t}dt}_{I.B.P}-\underbrace{\int_0^1\frac{\ln(t)\ln(1-t)}{1+t}dt}_{I.B.P}=-\frac{3}{2}\zeta(2)\ln(2)+\zeta(3)-\int_0^1\frac{\ln(t)\ln(1+t)}{1-t}dt+$$

$$\int_0^1\frac{\ln(1+t)\ln(1-t)}{t}dt\rightarrow\int_0^1\frac{\ln(t)\ln(1+t)}{1-t}dt=-\frac{3}{2}\zeta(2)\ln(2)+\zeta(3)$$

4. $\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx=-2\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=\frac{\pi}{2}\ln(2)-G\ \therefore\ \int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=\frac{G}{2}-\frac{\pi}{4}\ln(2)$

$$*\,I=\int_0^1\frac{Li_2(-x)}{1+x}dx=\int_0^1\int_0^1\frac{x\ln(t)}{(1+x)(1+xt)}dxdt=\int_0^1\ln(t)\,(\int_0^1\frac{x}{(1+x)(1+xt)}dx)dt=$$

$$\int_0^1\ln(t)\left(\frac{\ln(1+t)}{t(1-t)}-\frac{\ln(2)}{1-t}\right)dt=\zeta(2)\ln(2)+\int_0^1\frac{\ln(t)\ln(1+t)}{t}dt+\int_0^1\frac{\ln(t)\ln(1+t)}{1-t}dt=$$

$$\zeta(2)\ln(2)+\left(-\frac{3}{4}\zeta(3)\right)+\left(-\frac{3}{2}\zeta(2)\ln(2)+\zeta(3)\right)=\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\ln(2)$$

$$*\,J=\int_0^1 x\arctan^2\left(\frac{1}{x^2}\right)dx=\frac{1}{2}\int_0^1\arctan^2\left(\frac{1}{x^2}\right)(x^2)=\frac{1}{2}\int_0^1\arctan^2\left(\frac{1}{x}\right)dx=-\frac{\pi^2}{8}\int_0^1 dx-$$

$$\frac{\pi}{2}\int_0^1 \arctan(x)\,dx+\frac{1}{2}\int_0^1 arctan^2(x)dx=\frac{\pi^2}{8}-\frac{\pi}{2}\left(x\arctan(x)-\frac{1}{2}\ln(1+x^2)\right)\Big|_0^1+\frac{1}{2}\int_0^1 arctan^2(x)dx=$$

$$=\frac{\pi}{4}\ln(2)+\frac{1}{2}\int_0^1 arctan^2(x)dx$$

$$And:\ \int_0^1 arctan^2(x)dx\overset{I.B.P}{=}xarctan^2(x)\Big|_0^1-2\underbrace{\int_0^1\frac{\mathrm{xarctan(x)}}{1+x^2}dx}_{I.B.P}$$

$$=\frac{\pi^2}{16}-\left(\frac{1}{2}\ln(1+x^2)\,arctan^2(x)\Big|_0^1-\right.$$

$$\left.\frac{1}{2}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx\right)=\frac{\pi^2}{16}-\frac{\pi}{4}\ln(2)+\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx)=\frac{\pi^2}{16}-\frac{\pi}{4}\ln(2)+\frac{\pi}{2}\ln(2)-G=\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)-G$$

$$J=\int_0^1 xarctan^2\left(\frac{1}{x^2}\right)dx=\frac{\pi}{4}\ln(2)+\frac{1}{2}\left(\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)-G\right)=\frac{\pi^2}{32}+\frac{3\pi}{8}\ln(2)-\frac{G}{2}$$

Combine all results :

$$\Omega=\int_0^1\left(\frac{Li_2(-x)}{1+x}+xarctan^2\left(\frac{1}{x^2}\right)\right)dx=\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\ln(2)+\frac{\pi^2}{32}+\frac{3\pi}{8}\ln(2)-\frac{G}{2}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1\left(\frac{Li_2(-x)}{1+x}+xarctan^2\left(\frac{1}{x^2}\right)\right)dx$$

$$U=\int_0^1\frac{Li_2(-x)}{1+x}dx=\left(Li_2(-x)\int dln((1+x))\right)\Big|_0^1+\int_0^1\frac{ln^2(1+x)}{x}dx=$$

$$Li_2(-1)\ln(2)+\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x}dx+\int_{\frac{1}{2}}^1\frac{ln^2(x)}{1-x}dx=\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)-\frac{2}{3}ln^3(2)+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+$$

$$\int_{\frac{1}{2}}^1\frac{Li_2(x)}{x}dx=\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)-\frac{2}{3}ln^3(2)+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+2\int_{\frac{1}{2}}^1 d\,Li_3(x)=\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)-\frac{2}{3}ln^3(2)+$$

$$2\ln\left(\frac{1}{2}\right)\left(\frac{\pi^2}{12}-\frac{ln^2(2)}{2}\right)+2\zeta(3)-2\left(\frac{7}{8}\zeta(3)+\frac{ln^3(2)}{6}-\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)\right)$$

$$U=\frac{1}{4}\zeta(3)-\frac{\pi^2}{12}\ln(2)$$

$$V=\int_0^1 xarctan^2\left(\frac{1}{x^2}\right)dx=\frac{1}{2}\int_0^1 arctan^2\left(\frac{1}{x}\right)dx=\frac{1}{2}\int_0^1\left(\frac{\pi}{2}-\arctan(x)\right)^2dx=$$

$$-\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}x^2csc^2(x)dx=\frac{1}{2}\left(x^2\int dcot(x)\right)\Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}}-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\frac{x}{\tan(x)}dx=\frac{\pi^2}{32}\cot\left(\frac{\pi}{4}\right)-\left(x\int dln(\sin(x))\right)\Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}}+$$

$$+\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln(\sin(x)\,dx$$

$$\left\{\therefore \int_0^{\theta} \ln(\sin(z)\,dz = -\frac{1}{2}Cl_2(2\theta) - \theta\ln(2)\,;\;\; Cl_2\left(\frac{\pi}{2}\right) = G;\; Cl_2(\pi) = 0\right\}$$

$$V = \frac{\pi^2}{32} + \frac{\pi}{8}\ln(2) + \frac{\pi}{4}\ln(2) - \frac{G}{2} = \frac{\pi^2}{32} + \frac{3\pi}{8}\ln(2) - \frac{G}{2}$$

$$U + V = \frac{1}{4}\zeta(3) - \frac{\pi^2}{12}\ln(2) + \frac{\pi^2}{32} + \frac{3\pi}{8}\ln(2) - \frac{G}{2}$$