ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx$$

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Solution 1 by Ankush Kumar Parcha-India

$$We\ \ have\ :\ \int_0^1 \frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx \overset{Partial\ fraction}{\widetilde{=>}} \frac{1}{5}\int_0^1\left(\frac{1+2x}{1+x^2}-\frac{2}{2+x}\right)(ln^2(x)+2\ln(x)+1)dx =$$

$$= -\frac{2}{5}\int_0^1\frac{dx}{2+x}+\frac{2}{5}\int_0^1\frac{x}{1+x^2}dx+\frac{1}{5}\int_0^1\frac{dx}{1+x^2}-\frac{4}{5}\underbrace{\int_0^1\frac{\ln(x)}{2+x}dx}_{I.B.P}+\frac{4}{5}\int_0^1\frac{x\ln(x)}{1+x^2}dx+\frac{2}{5}\int_0^1\frac{\ln(x)}{1+x^2}dx-$$

$$-\frac{2}{5}\underbrace{\int_0^1\frac{ln^2(x)}{2+x}dx}_{I.B.P}+\frac{2}{5}\int_0^1\frac{xln^2(x)}{1+x^2}dx+\frac{1}{5}\int_0^1\frac{ln^2(x)}{1+x^2}dx \underbrace{=}_{|x^2|<1,x\in(0,1)} -\frac{2}{5}\int_0^1 dln(2+x)+\int_0^1\frac{dln(1+x^2)}{5}$$

$$+\int_0^1\frac{darctan(x)}{5}-\frac{4}{5}\underbrace{\left(\ln(x)\int dln\left(1+\frac{x}{2}\right)\right)_0^1}_{0}+\frac{4}{5}\int_0^1 ln\left(1+\frac{x}{2}\right)\frac{dx}{x}+\frac{4}{5}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n-1}\ln(x)\,dx$$

$$+\frac{2}{5}\sum_{n\in N}(-1)^n\int_0^1 x^{2n}\ln(x)dx-\frac{2}{5}\underbrace{\left(ln^2(x)\int dln\left(1+\frac{x}{2}\right)\right)_0^1}_{0}+\frac{4}{5}\underbrace{\int_0^1\frac{\ln(x)}{x}\left(1+\frac{x}{2}\right)dx}_{I.B.P}+$$

$$+\frac{2}{5}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n-1}ln^2(x)dx+\frac{1}{5}\sum_{n\in N_0}(-1)^n\int_0^1 x^{2n}\ln(x)dx$$

$$\left(\therefore \int_0^1 t^m ln^n(t)dt=\frac{(-1)^n.n!}{(m+1)^{n+1}},\ \ n>-1 \bigwedge m\neq 1\right)$$

$$\int_0^1 \frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx=\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)-\frac{4}{5}\int_0^1 dLi_2\left(-\frac{x}{2}\right)-\frac{2}{5}\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^2}}_{=G(Catalan's\ constant)}-$$

$$-\frac{1}{5}\sum_{n\in N}\frac{(-1)^{n-1}}{n^2}-\frac{4}{5}\underbrace{\left(\ln(x)\int dLi_2\left(-\frac{x}{2}\right)\right)_0^1}_{0}+\frac{4}{5}\int_0^1 d\,Li_3\left(-\frac{x}{2}\right)+\frac{1}{10}\sum_{n\in N}\frac{(-1)^{n-1}}{n^3}+\frac{2}{5}\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^3}$$

$$\int_0^1 \frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx=\frac{4}{5}Li_3\left(-\frac{1}{2}\right)-\frac{4}{5}Li_2\left(-\frac{1}{2}\right)-\frac{2G}{5}+\frac{2}{5}\beta(3)+\frac{\eta(3)}{10}-\frac{\eta(2)}{10}+\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)$$

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$$\int_0^1 \frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx = \frac{4}{5}\left(Li_3\left(-\frac{1}{2}\right) - Li_2\left(-\frac{1}{2}\right)\right) - \frac{2G}{5} + \frac{3}{40}\zeta(3) + \frac{\pi^3}{80} - \frac{\pi^2}{60} + \frac{\pi}{20} - \frac{2}{5}\ln(3) + \frac{3}{5}\ln(2)$$

$$\textit{Note section}\ :$$

$$\eta(s) = (1 - 2^{1-s}).\zeta(s)$$

$$\beta(3) = \frac{\pi^2}{32}\,, \qquad \zeta(2) = \frac{\pi^2}{6}$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\int_0^1 \frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx = \int_0^1 \frac{x}{(1+x^2)(2+x)}dx + 2\int_0^1 \frac{x\ln(x)}{(1+x^2)(2+x)}dx + \int_0^1 \frac{x^2}{(1+x^2)(2+x)}dx =$$

$$= A + 2B + C$$

$$A = \int_0^1 \frac{x}{(1+x^2)(2+x)}dx = \int_0^1 \left(\frac{2x}{5(1+x^2)} - \frac{1}{5(1+x^2)} - \frac{2}{5(x+2)}\right)dx = \frac{1}{5}\ln(1+x^2)\Big|_0^1 -$$

$$-\frac{2}{5}\ln(2+x)\Big|_0^1 + \frac{1}{5}\arctan(x)\Big|_0^1 = \frac{3}{5}\ln(2) + \frac{\pi}{20} - \frac{2}{5}\ln(3) = \frac{\pi}{20} + \frac{1}{5}\ln\left(\frac{8}{9}\right)$$

$$B = \int_0^1 \frac{x\ln(x)}{(1+x^2)(2+x)}dx = \frac{2}{5}\int_0^1 \frac{x\ln(x)}{1+x^2}dx + \frac{1}{5}\int_0^1 \frac{\ln(x)}{1+x^2}dx - \frac{2}{5}\int_0^1 \frac{\ln(x)}{x+2}dx = \frac{1}{5}(2B_1 + B_2 - 2B_3)$$

$$B_1 = \int_0^1 \frac{x\ln(x)}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln(x)\,dx = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2} = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = -\frac{1}{4}\eta(2) = -\frac{\pi^2}{48}$$

$$B_2 = -G\ , \qquad B_3 = \int_0^1 \frac{\ln(x)}{x+2}dx = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\int_0^1 x^n\ln(x)\,dx = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{(n+1)^2} = -Li_2\left(-\frac{1}{2}\right)$$

$$B = \frac{1}{5}(2B_1 + B_2 - 2B_3) = \frac{1}{5}\left\{-\frac{\pi^2}{24} - G - 2Li_2\left(-\frac{1}{2}\right)\right\} \quad C = \int_0^1 \frac{x\ln^2(x)}{(1+x^2)(2+x)}dx = \frac{2}{5}\int_0^1 \frac{x\ln^2(x)}{1+x^2}dx +$$

$$+\frac{1}{5}\int_0^1 \frac{\ln^2(x)}{1+x^2}dx - \frac{2}{5}\int_0^1 \frac{\ln^2(x)}{x+2}dx = \frac{1}{5}(2C_1 + C_2 - 2C_3) \qquad C_1 = \int_0^1 \frac{x\ln^2(x)}{1+x^2}dx =$$

$$\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln^2(x)dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^3} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} = \frac{1}{4}\eta(3) = \frac{3}{16}\zeta(3)\ \ , C_2 = \int_0^1 \frac{\ln^2(x)}{1+x^2}dx =$$

$$\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln^2(x)dx = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{16}\,, \qquad C_3 = \int_0^1 \frac{\ln^2(x)}{x+2}dx = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\int_0^1 x^n\ln^2(x)dx =$$

$$\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{(n+1)^3}=-2\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{n^3}=2Li_3\left(-\frac{1}{2}\right)\ ,\qquad \frac{1}{5}(2C_1+C_2-2C_3)=\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}+\frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$

$$\Omega=A+2B+C=\frac{\pi}{20}+\frac{1}{5}\ln\left(\frac{8}{9}\right)+\frac{2}{5}\left\{-\frac{\pi^2}{24}-G-2Li_2\left(-\frac{1}{2}\right)\right\}+\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}+\frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$

$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx=\frac{4}{5}\left(Li_3\left(-\frac{1}{2}\right)-Li_2\left(-\frac{1}{2}\right)\right)-\frac{2G}{5}+\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}-\frac{\pi^2}{60}+\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)$$