ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{x^2 \arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$Note\ section: \therefore \ln(cos(t)) = -\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(2nt), \qquad |t| < \frac{\pi}{2}$$

$$Let: -2\Omega = -2\int_0^1 \frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx = \int_0^1 x\arctan(x)\ln(1+x^2)\left(\frac{-2x}{(1+x^2)^2}\right) =$$

$$\int_0^1 x\arctan(x)\ln(1+x^2)d\left(\frac{1}{1+x^2}\right) \overset{I.B.P}{\cong} \frac{x\arctan(x)\ln(1+x^2)}{1+x^2}\Bigg|_0^1 -$$

$$\int_0^1 \frac{1}{1+x^2}\left(\arctan(x)\ln(1+x^2) + \frac{x\ln(1+x^2)}{1+x^2} + 2\frac{x\arctan(x)}{1+x^2}\right)dx$$

$$= \frac{\pi}{8}\ln(2) - \underbrace{\int_0^1 \frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx}_{A} -$$

$$\underbrace{\int_0^1 \frac{x\ln(1+x^2)}{(1+x^2)^2}dx}_{B} - 2\underbrace{\int_0^1 \frac{x\arctan(x)}{(1+x^2)^2}dx}_{C} = \frac{\pi}{8}\ln(2) - A - B - 2C$$

$$a)\ A = \underbrace{\int_0^1 \frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx}_{x=\tan(t)} = \int_0^{\frac{\pi}{4}} t\ln\left(\frac{1}{cos^2(t)}\right)dt = -2\int_0^{\frac{\pi}{4}} t\ln(\cos(t))dt =$$

$$2\int_0^{\frac{\pi}{4}} t\left(\ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(2nt)\right)dt$$

$$= \ln(2)\,t^2\Big|_0^{\frac{\pi}{4}} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\cos(2nt)dt \overset{I.B.P}{\cong} \frac{\pi^2}{16}\ln(2) +$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2nt\sin(2nt) + \cos(2nt)}{4n^2}\Bigg|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{16}\ln(2) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(n\frac{\pi}{2}\sin\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) - 1\right) =$$

$$= \frac{\pi^2}{16}\ln(2) + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(n\frac{\pi}{2}\right) +$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(n\frac{\pi}{2}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{7}{16}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)$$

$$B=\int_0^1\frac{x\ln(1+x^2)}{(1+x^2)^2}dx$$

$$=-\frac{1}{2}\int_0^1\ln(1+x^2)\frac{-2x}{(1+x^2)^2}dx$$

$$=-\frac{1}{2}\int_0^1\ln(1+x^2)\,d\frac{1}{1+x^2}=-\frac{1}{2}\frac{\ln(1+x^2)}{1+x^2}\bigg|_0^1$$

$$+\frac{1}{2}\int_0^1\frac{d(1+x^2)}{(1+x^2)^2}=-\frac{\ln(2)}{4}-\frac{1}{2}\frac{1}{1+x^2}\bigg|_0^1=-\frac{\ln(2)}{4}+\frac{1}{4}$$

$$C=\underbrace{\int_0^1\frac{\mathrm{x}\arctan(\mathrm{x})}{(1+x^2)^2}dx}_{x=\tan(t)}=\int_0^{\frac{\pi}{4}}\frac{\tan(t)\,t\left(1+\tan^2(t)\right)}{\left(1+\tan^2(t)\right)^2}dt=\frac{1}{2}\int_0^{\frac{\pi}{4}}t\sin(2t)dt\overset{I.B.P}{=}\frac{1}{2}\frac{\sin(2t)-2t\cos(2t)}{4}\bigg|_0^{\frac{\pi}{4}}=\frac{1}{8}$$

$$Then: -2\Omega=\frac{\pi}{8}\ln(2)-A-B-2C$$

$$=\frac{\pi}{8}\ln(2)-\left\{\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)\right\}-\left\{-\frac{\ln(2)}{4}+\frac{1}{4}\right\}-2.\frac{1}{8}=$$

$$\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{8}\ln(2)+\frac{\ln(2)}{4}-\frac{1}{2}$$

$$Therefore:\quad \Omega=\frac{21}{128}\zeta(3)+\frac{\pi^2}{32}\ln(2)-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}G-\frac{\ln(2)}{8}+\frac{1}{4}$$

Solution 2 by Exodo Halcalias-Angola

$$\Omega=\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

$$I=\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx-\int_0^1\frac{\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

$$N=\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx$$

$$=-2\int_0^{\frac{\pi}{4}}x\ln(\cos x)dx=-2\int_0^{\frac{\pi}{4}}x\left(\ln\left(\frac{1}{2}\right)-\sum_{k\in N}\frac{(-1)^k}{k}\cos(2kx)\right)dx=$$

$$\frac{\pi^2}{16}\ln(2)+2\sum_{k\in N}\frac{(-1)^k}{k}\left(\frac{\pi}{8k}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{4k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^2}\right)dx=\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}\sum_{k\in N}\frac{(-1)^k}{(2k-1)^2}+$$

$$\frac{1}{2}\sum_{k\in N}\frac{(-1)^k}{(2k)^3}-\frac{1}{2}\sum_{k\in N}\frac{(-1)^k}{k^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)$$

$$M=\int_0^1\frac{\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx=(\arctan(x)\ln(1+x^2)\int d\left(\frac{x}{2(1+x^2)}+\frac{\arctan(x)}{2}\right))\Big|_0^1-$$

$$\frac{1}{2}\int_0^1\frac{\ln(1+x^2)+2x\arctan(x)}{1+x^2}\left(\frac{x}{2(1+x^2)}+\arctan(x)\right)dx=\frac{\pi}{16}\ln(2)+\frac{\pi^2}{32}\ln(2)-$$

$$-\frac{1}{2}\int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx-\int_0^1\frac{x^2\arctan(x)}{(1+x^2)^2}dx-\int_0^1\frac{x\arctan^2(x)}{1+x^2}dx=$$

$$=\frac{\pi}{16}\ln(2)\left(1+\frac{\pi}{2}\right)-\frac{1}{4}\int_0^1\frac{\ln(1+x)}{(1+x)^2}dx-\frac{N}{2}+\frac{1}{2}\int_0^{\frac{\pi}{4}}xdx+\frac{1}{2}\int_0^{\frac{\pi}{4}}x\cos(2x)dx\,(x^2\int d\ln\left(\frac{1}{\cos(x)}\right))\Big|_0^{\frac{\pi}{4}}+$$

$$\frac{1}{2}\int_0^{\frac{\pi}{4}}x\ln\cos(x)dx\quad M=\frac{\pi}{16}\ln(2)\left(1+\frac{\pi}{2}\right)-\frac{1}{4}\left(\frac{1}{2}-\frac{\ln(2)}{2}\right)-\frac{N}{2}-\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{1}{8}-\frac{\pi^2}{32}\ln(2)+N$$

$$I=N-M$$

$$I=N-\left(-\frac{1}{4}+\frac{\ln(2)}{8}-\frac{N}{2}-\frac{\pi^2}{64}+\frac{\pi}{16}+\frac{\pi}{16}\ln(2)+N\right)=\frac{1}{2}\left(\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)\right)+$$

$$\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)+\frac{1}{4}-\frac{\ln(2)}{8}$$

$$\therefore\ \Omega=\frac{21}{128}\zeta(3)+\frac{\pi^2}{32}\ln(2)-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}G-\frac{\ln(2)}{8}+\frac{1}{4}$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

$$We\ have,\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx\overset{I.B.P}{\cong}-\left(\frac{x\arctan(x)}{2}\int d\left(\frac{1+\ln(1+x^2)}{1+x^2}\right)\right)_0^1+$$

$$\frac{1}{2}\int_0^1\left(\frac{1+\ln(1+x^2)}{1+x^2}\right)\left(\frac{x}{1+x^2}+\arctan(x)\right)dx=-\frac{\pi}{16}\ln(2)-\frac{\pi}{16}+\frac{1}{2}\int_0^1\frac{x}{(1+x^2)^2}dx+$$

$$\frac{1}{2}\int_0^1\frac{x\ln(1+x^2)}{(1+x^2)^2}dx+\frac{1}{2}\int_0^1\frac{\arctan(x)}{1+x^2}dx+\underbrace{\int_0^1\frac{\ln(1+x^2)\arctan(x)}{1+x^2}dx}_{x\to\tan(x)}=-\frac{\pi}{16}\ln(2)-\frac{\pi}{16}-$$

$$-\frac{1}{4}\int_0^1 d\left(\frac{1}{1+x^2}\right)-\frac{1}{4}\int_0^1 d\left(\frac{1+\ln(1+x^2)}{1+x^2}\right)+\int_0^1\frac{d\arctan^2(x)}{4}-\int_0^{\frac{\pi}{4}}x\ln\cos(x)dx$$

$$\left(\therefore\sum_{n\in N}\frac{(-1)^n\cos(2nx)}{n}=\ln\left(\frac{sec|x|}{2}\right)\ ,\ \ |x|<\frac{\pi}{2}\right)$$

$$\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx=\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)+\frac{1}{4}+\ln(2)\int_0^{\frac{\pi}{4}}xdx+\sum_{n\in N}\frac{(-1)^n}{n^2}\underbrace{\int_0^{\frac{\pi}{4}}xcos(2nx)}_{I.B.P}=$$

$$=\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}+$$

$$+\frac{\ln(2)}{2}\int_0^{\frac{\pi}{4}}d(x^2)+\sum_{n\in N}\frac{(-1)^n}{n}\left[\left(\frac{x}{2n}\int dsin(2nx)\right)\begin{matrix}\frac{\pi}{4}\\0\end{matrix}-\frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nx)\,dx\right]$$

$$=\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}+\frac{\pi}{8}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^2}\sin\left(\frac{n\pi}{2}\right)}_{n\to 2n+1}+\frac{1}{4}\sum_{n\in N}\frac{(-1)^n}{n^3}\int_0^{\frac{\pi}{4}}dcos(2nx)=$$

$$=\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}-$$

$$-\frac{\pi}{8}\sum_{n\in N_0}\frac{1}{(2n+1)^2}\sin(n\pi+\frac{\pi}{2})+\frac{1}{4}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^3}\cos\left(\frac{n\pi}{2}\right)}_{n\to 2n}-\frac{1}{4}\sum_{n\in N}\frac{(-1)^n}{n^3}\ \underbrace{\overbrace{->}^{\therefore\cos(n\pi)=(-1)^n\ ,\forall n\in Z}}_{\therefore sin(n\pi)=0,\ \ \forall n\in Z}$$

$$\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx=\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}-\frac{\pi}{8}\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^2}}_{=G(Catalan's\ constant)}+$$

$$+\frac{1}{32}\sum_{n\in N}\frac{(-1)^n}{n^3}-\frac{1}{4}\sum_{n\in N}\frac{(-1)^n}{n^3}=-\frac{\pi}{8}G+\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}+\frac{7}{32}\eta(3)$$

$$\therefore\Omega=\frac{21}{128}\zeta(3)+\frac{\pi^2}{32}\ln(2)-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}G-\frac{\ln(2)}{8}+\frac{1}{4}$$