ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC, the following relationship holds :

$$a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}\geq 2p\sqrt[4]{3}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sin 2A+\sin 2B+\sin 2C=2\sin A\cos A+2\sin A\cos(B-C)$$
$$=2\sin A\left(\cos(B-C)-\cos(B+C)\right)=4\sin A\sin B\sin C=4.\frac{4Rrp}{8R^3}$$
$$\Rightarrow\sum_{cyc}\sin 2A=\frac{2rp}{R^2}\rightarrow(1)$$
$$\text{Now, } a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}=\sum_{cyc}\left((2R\sin A)\left(\sqrt{\frac{\sin A}{\cos A}}\right)\right)$$
$$=2R\sum_{cyc}\frac{\sin^2 A}{\sqrt{\sin A\cos A}}\overset{\text{Bergstrom}}{\geq}2R\frac{\left(\sum_{cyc}\sin A\right)^2}{\sum_{cyc}\sqrt{\sin A\cos A}}\overset{\text{CBS}}{\geq}\frac{2R\left(\frac{p}{R}\right)^2}{\sqrt{3}.\sqrt{\sum_{cyc}(\sin A\cos A)}}$$
$$=\frac{2\sqrt{2}R\left(\frac{p}{R}\right)^2}{\sqrt{3}.\sqrt{\sum_{cyc}\sin 2A}}\overset{\text{via }(1)}{=}\frac{2\sqrt{2}R\left(\frac{p}{R}\right)^2}{\sqrt{3}.\sqrt{\frac{2rp}{R^2}}}=\frac{2p.\sqrt{p}}{\sqrt{3}}\overset{\text{Mitrinovic}}{\geq}\frac{2p.\sqrt{3\sqrt{3}r}}{\sqrt{3r}}=2p\sqrt[4]{3}$$
$$\therefore a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}\geq 2p\sqrt[4]{3}$$
$$\forall \text{ acute } \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$let\ f(x)=\tan x\ ,x\in\left(0,\frac{\pi}{2}\right).f''(x)=2\sec^2 x\tan x>0\ ,$$
$$f\ is\ convex\in\left(0,\frac{\pi}{2}\right).Using\ Jensen\ inequility\ \sum\tan A\geq 3\tan\frac{\pi}{3}=3\sqrt{3}.$$
$$Note: A+B+C=\pi\ ,now\ \ \tan(A+B)=\tan(\pi-C)\ or\ \sum\tan A=\prod\tan A.$$
$$a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}\overset{CHEBYSHEV}{\geq}$$
$$\geq\frac{1}{3}(a+b+c)\left(\sum\sqrt{tanA}\right)\overset{AM-GM}{\geq}\frac{1}{3}.2p.3\left(\prod tanA\right)^{\frac{1}{6}}\geq 2p.\left(3\sqrt{3}\right)^{\frac{1}{6}}=2p\sqrt[4]{3}$$

Equality holds for $a=b=c$.