ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC, the following relationship holds :

$$\frac{1}{a}\sqrt{\cot A}+\frac{1}{b}\sqrt{\cot B}+\frac{1}{c}\sqrt{\cot C}>\frac{3}{p}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}\sqrt{\cot A}+\frac{1}{b}\sqrt{\cot B}+\frac{1}{c}\sqrt{\cot C}>\frac{3}{p}\Leftrightarrow\sum_{cyc}\sqrt{\cot A\,(1+\cot^2 A)}>\frac{6R}{s}$$

$$(\text{For own convenience}, p\equiv s)\Leftrightarrow\sum_{cyc}\cot A+\sum_{cyc}\cot^3 A$$

$$+2\sum_{cyc}\left(\sqrt{\cot A\,(1+\cot^2 A)}.\sqrt{\cot B\,(1+\cot^2 B)}\right)\boxed{\overset{(*)}{>}}\frac{36R^2}{s^2}$$

$$\text{Now, }2\sum_{cyc}\left(\sqrt{\cot A\,(1+\cot^2 A)}.\sqrt{\cot B\,(1+\cot^2 B)}\right)\overset{A-G}{>}2\sum_{cyc}\left(\sqrt{\cot A\,.\,2\cot A}.\sqrt{\cot B\,.\,2\cot B}\right)$$

$$=4\sum_{cyc}\cot A\cot B\Rightarrow 2\sum_{cyc}\left(\sqrt{\cot A\,(1+\cot^2 A)}.\sqrt{\cot B\,(1+\cot^2 B)}\right)\boxed{\overset{(\bullet)}{>}}4$$

$$\text{Again, }\sum_{cyc}\cot A+\sum_{cyc}\cot^3 A\overset{\text{Chebyshev}}{\geq}\sum_{cyc}\cot A+\frac{1}{3}\left(\sum_{cyc}\cot A\right)\left(\sum_{cyc}\cot^2 A\right)\geq$$

$$\sum_{cyc}\cot A+\frac{1}{3}\left(\sum_{cyc}\cot A\right)\left(\sum_{cyc}\cot A\cot B\right)\Rightarrow\sum_{cyc}\cot A+\sum_{cyc}\cot^3 A\boxed{\overset{(\bullet\bullet)}{\geq}}\frac{4}{3}\sum_{cyc}\cot A$$

$$\therefore(\bullet),(\bullet\bullet)\Rightarrow \text{LHS of }(*)\geq\frac{4}{3}\sum_{cyc}\cot A+4=\frac{4}{3}.\frac{s^2-4Rr-r^2}{2rs}+4\overset{?}{>}\frac{36R^2}{s^2}$$

$$\Leftrightarrow\frac{s^2-4Rr-r^2}{6rs}\overset{?}{>}\frac{9R^2-s^2}{s^2}$$

$$\Leftrightarrow\frac{(s^2-4Rr-r^2)^2}{36r^2s^2}\overset{?}{>}\frac{(9R^2-s^2)^2}{s^4}\left(\because 9R^2\overset{\text{Mitrinovic}}{\geq}\frac{4s^2}{3}>s^2\right)$$

$$\Leftrightarrow s^6-(8Rr+38r^2)s^4+r^2s^2(664R^2+8Rr+r^2)-2916R^4r^2\boxed{\underset{(**)}{\underbrace{\overset{?}{>}}}}0\ \textit{and}$$

$$\because(s^2-4R^2-4Rr-r^2)^3>0\left(\because\prod_{cyc}\cos A=\frac{s^2-4R^2-4Rr-r^2}{4R^2}>0\right)$$

$$\therefore \text{in order to prove }(**),\text{it suffices to prove : LHS of }(*)>(s^2-4R^2-4Rr-r^2)^3$$

$$\Leftrightarrow(12R^2+4Rr-35r^2)s^4-(48R^4+96R^3r-592R^2r^2+16Rr^3+2r^4)s^2$$

$$+64R^6+192R^5r-267R^4r^2+160R^3r^3+60R^2r^4+12Rr^5+r^6 \boxed{\overset{(***)}{>}} 0$$

$$\textit{and} \because \left(12R^2+4Rr-35r^2\right)\left(s^2-4R^2-4Rr-r^2\right)^2>0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) > \left(12R^2+4Rr-35r^2\right)\left(s^2-4R^2-4Rr-r^2\right)^2$$

$$\Leftrightarrow \left(48R^4+32R^3r+368R^2r^2-288Rr^3-72r^4\right)s^2 \boxed{\overset{(****)}{>}}$$
$$128R^6+256R^5r+2532R^4r^2-1088R^3r^3-856R^2r^4-288Rr^5-36r^6$$

$$\text{Once } \textit{again}, \text{LHS of } (****) >$$
$$\left(48R^4+32R^3r+368R^2r^2-288Rr^3-72r^4\right)\left(4R^2+4Rr+r^2\right)$$
$$\overset{?}{>} 128R^6+256R^5r+2532R^4r^2-1088R^3r^3-856R^2r^4-288Rr^5-36r^6$$
$$\Leftrightarrow 16t^6+16t^5-221t^4+360t^3-54t^2-72t-9 \overset{?}{>} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)\left(16t^4+80t^3+35t^2+180t+526\right)+1312\right)+511$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{1}{a}\sqrt{\cot A}+\frac{1}{b}\sqrt{\cot B}+\frac{1}{c}\sqrt{\cot C}>\frac{3}{p} \ \forall \ \textit{acute} \ \Delta \ ABC \ (QED)$$