ROMANIAN MATHEMATICAL MAGAZINE

Prove that for any acute triangle ABC the following inequalities holds

$$\sqrt{\frac{3}{2}} \le \frac{\sin A + \sin B + \sin C}{\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C}} < 2$$

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By CBS inequality, we have

$$\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C} \le \sqrt{(a\cos A + b\cos B + c\cos C)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)},$$

with $a\cos A + b\cos B + c\cos C = \frac{2F}{R}$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc} \le \frac{(a+b+c)^2}{3.4RF}$, then

$$\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C} \le \sqrt{\frac{2}{3}} . \frac{a+b+c}{2R} = \sqrt{\frac{2}{3}} . (\sin A + \sin B + \sin C),$$

which completes the proof of the left side inequality. Equality holds iff ΔABC is equilateral.

Now since $\sqrt{\cos A} \ge \cos A$ (and analogs),

then to prove the right side inequality it suffices to prove that

$$\sin A + \sin B + \sin C < 2(\cos A + \cos B + \cos C) \text{ or } \frac{s}{R} < 2\left(1 + \frac{r}{R}\right) \text{ or } s < 2(R + r)$$

which is true by Gerretsen's inequality, $s \le \sqrt{4R^2 + 4Rr + 3r^2} < 2(R + r)$.