ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(i)\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^4}{r^4}\geq 16+\sum\frac{m_a}{m_b+m_c}$$

$$(ii)\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^5}{r^5}\geq 32+\sum\frac{w_a}{w_b+w_c}$$

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$$(i)\sum\frac{m_a}{m_b+m_c}\overset{AM-GM}{\leq}\sum\frac{m_a}{2\sqrt{m_bm_c}}\overset{CBS}{\leq}\frac{1}{2}\sqrt{\left(\sum m_a^2\right)\left(\sum\frac{1}{m_bm_c}\right)}\overset{Leibniz}{\leq}$$

$$\leq\frac{1}{2}\sqrt{\left(\frac{3}{4}9R^2\right)\left(\frac{m_a+m_b+m_c}{m_am_bm_c}\right)}\overset{Leunberger\ \&\ m_a\geq\sqrt{s(s-a)}}{\leq}$$

$$\leq\frac{3R}{4}\sqrt{\frac{3(4R+r)}{s^2r}}\overset{Doucet}{\leq}\frac{3R}{4}\sqrt{\frac{3(4R+r)}{3r(4R+r)r}}=\frac{3R}{4r}\ and\ \sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}\overset{Nestbitt}{\geq}\frac{3}{2}$$

$$We\ need\ to\ show:\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^4}{r^4}\geq 16+\sum\frac{m_a}{m_b+m_c}$$

$$\frac{3}{2}+\frac{R^4}{r^4}\geq 16+\frac{3R}{4r}$$

$$4R^4-3Rr^3-58r^4\geq 0$$

$$(R-2r)(4R^3+8R^2r+16Rr^2+29r^3)\geq 0\ true\ (Euler)$$

$$(ii)\sum\frac{w_a}{w_b+w_c}\overset{AM-HM}{\leq}\frac{1}{4}\sum\left(\frac{w_a}{w_b}+\frac{w_a}{w_c}\right)\overset{CBS}{\leq}\frac{1}{4}.2\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{w_a^2}\right)}\overset{h_a\leq w_a\leq\sqrt{s(s-a)}}{\leq}$$

$$\leq\frac{1}{2}\sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{h_a^2}\right)}=\frac{1}{4r}\sqrt{a^2+b^2+c^2}\overset{Leibniz}{\leq}\frac{3R}{2r}$$

$$We\ need\ to\ show:$$

$$\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^5}{r^5}\geq 32+\sum\frac{w_a}{w_b+w_c}$$

$$\frac{3}{2}+\frac{R^5}{r^5}\geq 32+\frac{3R}{4r}$$

$$4R^5-3Rr^4-122r^5\geq 0$$

$$(R-2r)(4R^4+8R^3r+16R^2r^2+32Rr^3+61r^4)\geq 0\ true$$

Equality holds for an equilateral triangle.