ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{a^4}{a^4+2b^4}+\frac{R^{2024}}{r^{2024}}\ge 2^{2024}+\sum_{cyc}\frac{b^4}{a^4+2c^4}$$

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$$\sum_{cyc}\frac{b^4}{a^4+2c^4}=\frac{b^4}{a^4+2c^4}+\frac{c^4}{b^4+2a^4}+\frac{a^4}{c^4+2b^4}\overset{\text{Reverse Bergstrom}}{\le}$$

$$\frac{b^4}{9}\left(\frac{1}{a^4}+\frac{2}{c^4}\right)+\frac{c^4}{9}\left(\frac{1}{b^4}+\frac{2}{a^4}\right)+\frac{a^4}{9}\left(\frac{1}{c^4}+\frac{2}{b^4}\right)$$

$$\Rightarrow\sum_{cyc}\frac{b^4}{a^4+2c^4}\le\frac{2}{9}\sum_{cyc}\frac{a^4}{b^4}+\frac{1}{9}\sum_{cyc}\frac{b^4}{a^4}\to(1)$$

$$\sum_{cyc}\frac{2b^4}{a^4+2b^4}=\frac{2b^4}{a^4+2b^4}+\frac{2c^4}{b^4+2c^4}+\frac{2a^4}{c^4+2a^4}\overset{\text{Reverse Bergstrom}}{\le}$$

$$\frac{2b^4}{9}\left(\frac{1}{a^4}+\frac{2}{b^4}\right)+\frac{2c^4}{9}\left(\frac{1}{b^4}+\frac{2}{c^4}\right)+\frac{2a^4}{9}\left(\frac{1}{c^4}+\frac{2}{a^4}\right)$$

$$\Rightarrow\sum_{cyc}\frac{2b^4}{a^4+2b^4}\le\frac{2}{9}\sum_{cyc}\frac{b^4}{a^4}+\frac{4}{3}\to(2)$$

$$\therefore\sum_{cyc}\frac{b^4}{a^4+2c^4}-\sum_{cyc}\frac{a^4}{a^4+2b^4}=\sum_{cyc}\frac{b^4}{a^4+2c^4}-\sum_{cyc}\frac{a^4+2b^4-2b^4}{a^4+2b^4}$$

$$=-3+\sum_{cyc}\frac{b^4}{a^4+2c^4}+\sum_{cyc}\frac{2b^4}{a^4+2b^4}\overset{\text{via (1) and (2)}}{\le}$$

$$-3+\frac{2}{9}\sum_{cyc}\frac{a^4}{b^4}+\frac{1}{9}\sum_{cyc}\frac{b^4}{a^4}+\frac{2}{9}\sum_{cyc}\frac{b^4}{a^4}+\frac{4}{3}$$

$$=-3+\frac{4}{3}+\frac{1}{3}\left(\sum_{cyc}\frac{a^4}{b^4}+\sum_{cyc}\frac{b^4}{a^4}\right)-\frac{1}{9}\sum_{cyc}\frac{a^4}{b^4}\overset{A-G}{\le}$$

$$3+\frac{4}{3}+\frac{1}{3}\sum_{cyc}\left(\frac{b^4}{c^4}+\frac{c^4}{b^4}\right)-\frac{1}{9}.3\sqrt[3]{\frac{a^4}{b^4}.\frac{b^4}{c^4}.\frac{c^4}{a^4}}=-2+\frac{1}{3}\sum_{cyc}\left(\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)^2-2\right)$$

$$=-2+\frac{1}{3}\sum_{cyc}\left(\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right)^2-2\right)\overset{\text{Bandila}}{\le}-2+\frac{1}{3}\sum_{cyc}\left(\left(\frac{R^2}{r^2}-2\right)^2-2\right)$$

$$=-2+\frac{1}{3}\sum_{cyc}\left(\frac{R^4}{r^4}-\frac{4R^2}{r^2}+2\right)=-2+\frac{1}{3}\left(\frac{3R^4}{r^4}-\frac{12R^2}{r^2}+6\right)$$

$$\therefore \sum_{\text{cyc}} \frac{b^4}{a^4+2c^4} - \sum_{\text{cyc}} \frac{a^4}{a^4+2b^4} \le \frac{R^4}{r^4} - \frac{4R^2}{r^2} \to (3)$$

Let $F(n) = t^n - 2^n \ \forall\ t = \frac{R}{r} \overset{\text{Euler}}{\ge} 2$ ($t \to$ fixed) and $\forall\ n \ge 2$ and then :

$$F'(n) = t^n.(\ln t) - 2^n.(\ln 2) \ge 0$$

$$(\because t^n \ge 2^n \text{ and } \ln t \ge \ln 2 \Rightarrow t^n.(\ln t) - 2^n.(\ln 2) \ge 0)$$

$\therefore$ for n and t both being variables, $F(n)$ is $\uparrow \forall\ n \ge 2 \Rightarrow F(2024) \ge F(4)$

$$\Rightarrow t^{2024} - 2^{2024} \ge t^4 - 16 \Rightarrow \frac{R^{2024}}{r^{2024}} - 2^{2024} \ge \frac{R^4}{r^4} - 16 \overset{\text{Euler}}{\ge} \frac{R^4}{r^4} - \frac{4R^2}{r^2}$$

$$\overset{\text{via (3)}}{\ge} \sum_{\text{cyc}} \frac{b^4}{a^4+2c^4} - \sum_{\text{cyc}} \frac{a^4}{a^4+2b^4} \Rightarrow \sum_{\text{cyc}} \frac{a^4}{a^4+2b^4} + \frac{R^{2024}}{r^{2024}}$$

$$\ge 2^{2024} + \sum_{\text{cyc}} \frac{b^4}{a^4+2c^4} \ \forall\ \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$