ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}},\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}+\frac{R^n}{r^n}\geq 2^n+\sum_{cyc}\frac{a^2}{b^2+c^2}$$

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$$\sum_{cyc}\frac{\cos A}{\cos B+\cos C}=\sum_{cyc}\frac{\sum_{cyc}\cos A-(\cos B+\cos C)}{\cos B+\cos C}$$

$$=\left(\sum_{cyc}\cos A\right)\sum_{cyc}\frac{1}{2\sin\frac{A}{2}\cos\frac{B-C}{2}}-3\overset{0<\cos\frac{B-C}{2}\leq 1\text{ and analogs}}{\geq}\frac{1+\frac{r}{R}}{2}\cdot\sum_{cyc}\frac{1}{\sin\frac{A}{2}}-3$$

$$\overset{A-G}{\geq}\frac{3\left(1+\frac{r}{R}\right)}{2}\cdot\sqrt[3]{\prod_{cyc}\frac{1}{\sin\frac{A}{2}}}-3=\frac{3\left(1+\frac{r}{R}\right)}{2}\cdot\sqrt[3]{\frac{4R}{r}}-3\overset{\text{Euler}}{\geq}\frac{3\left(1+\frac{r}{R}\right)}{2}\cdot\sqrt[3]{8}-3$$

$$\Rightarrow\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\geq\frac{3r}{R}\to(1)$$

$$\text{Now, }\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)=\frac{1}{4}\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right)$$

$$\overset{\text{Bandila}}{\leq}\frac{1}{4}\sum_{cyc}\left(\left(\frac{R}{r}\right)^2-2\right)\Rightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}\leq\frac{3R^2}{4r^2}-\frac{3}{2}\to(2)$$

$$\therefore\text{ via (1) and (2), in order to prove : }\sum_{cyc}\frac{\cos A}{\cos B+\cos C}+\frac{R^2}{r^2}-4\geq\sum_{cyc}\frac{a^2}{b^2+c^2},$$

$$\text{it suffices to prove : }\frac{3r}{R}+\frac{R^2}{r^2}-4-\frac{3R^2}{4r^2}+\frac{3}{2}\geq 0\Leftrightarrow t^3-10t+12\geq 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t^2-4)+2(t-2)+2\right)\geq 0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore\sum_{cyc}\frac{\cos A}{\cos B+\cos C}+\frac{R^2}{r^2}-4\geq\sum_{cyc}\frac{a^2}{b^2+c^2}\to\text{(i)}$$

$$\text{Again, via Nesbitt and (2), in order to prove : }\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}+\frac{R^2}{r^2}-4$$

$$\geq\sum_{cyc}\frac{a^2}{b^2+c^2},\text{ it suffices to prove : }\frac{3}{2}+\frac{R^2}{r^2}-4-\frac{3R^2}{4r^2}+\frac{3}{2}\geq 0\Leftrightarrow R^2\geq 4r^2$$

$$\to\text{true via Euler}\therefore\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}+\frac{R^2}{r^2}-4\geq\sum_{cyc}\frac{a^2}{b^2+c^2}\to\text{(ii)}$$

$$\therefore (i),(ii) \Rightarrow \min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}, \sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}+\frac{R^2}{r^2}-4 \geq \sum_{cyc}\frac{a^2}{b^2+c^2}$$

$$\Rightarrow \frac{R^2}{r^2}-4 \geq \sum_{cyc}\frac{a^2}{b^2+c^2}-\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}, \sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\} \rightarrow (iii)$$

$$\text{Let } F(n) = t^n - 2^n \ \forall\, t = \frac{R}{r} \geq 2 \ (t \rightarrow fixed) \ and \ \forall\, n \geq 2 \ and \ then:$$

$$F'(n) = t^n.(\ln t) - 2^n.(\ln 2) \geq 0$$

$$(\because t^n \geq 2^n \ and \ \ln t \geq \ln 2 \Rightarrow t^n.(\ln t) - 2^n.(\ln 2) \geq 0)$$

$$\therefore F(n) \text{ is } \uparrow \forall\, n \geq 2 \Rightarrow F(n) \geq F(2) \Rightarrow \text{when both } t = \frac{R}{r} \ and \ n \ vary,$$

$$\left(\frac{R}{r}\right)^n - 2^n \geq \frac{R^2}{r^2}-4 \overset{via\,(iii)}{\geq} \sum_{cyc}\frac{a^2}{b^2+c^2}-\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}, \sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}$$

$$\Rightarrow \min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}, \sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}+\frac{R^n}{r^n} \geq 2^n + \sum_{cyc}\frac{a^2}{b^2+c^2} \ \forall\, \Delta ABC,$$

$$'' ='' \ \text{iff } \Delta ABC \text{ is equilateral (QED)}$$