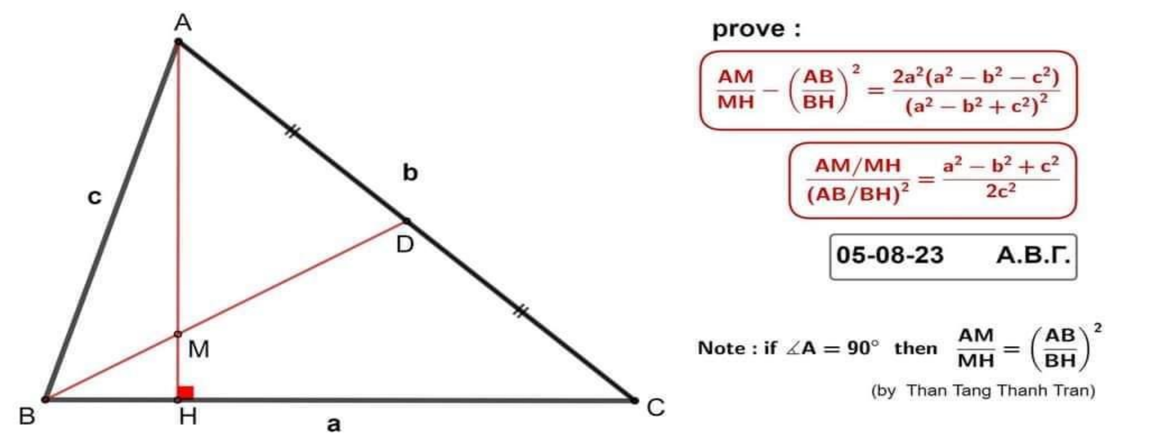
ROMANIAN MATHEMATICAL MAGAZINE



Prove that :

$$\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2=\frac{2a^2(a^2-b^2-c^2)}{(a^2-b^2+c^2)^2} \text{ and } \frac{AM/MH}{(AB/BH)^2}=\frac{a^2-b^2+c^2}{2c^2}$$

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Via Menelaus' theorem on Δ AHC with DMB as transversal,

$$\frac{BC}{BH}\cdot\frac{MH}{AM}\cdot\frac{AD}{CD}=1\Rightarrow\frac{AM}{MH}=\frac{a}{c\cos B}\Rightarrow\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2=\frac{a}{c\cos B}-\frac{c^2}{c^2\cos^2 B}$$

$$\Rightarrow\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2\overset{(*)}{=}\frac{a\cos B-c}{c\cos^2 B}$$

$$\text{Again, }\frac{2a^2(a^2-b^2-c^2)}{(a^2-b^2+c^2)^2}=\frac{2a^2(a^2-b^2-c^2)}{(2ac)^2\cos^2 B}=\frac{a^2-b^2+c^2-2c^2}{2c^2\cos^2 B}$$

$$=\frac{2ac\cos B-2c^2}{2c^2\cos^2 B}=\frac{a\cos B-c}{c\cos^2 B}\overset{\text{via }(*)}{=}\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2$$

$$\therefore\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2=\frac{2a^2(a^2-b^2-c^2)}{(a^2-b^2+c^2)^2}$$

$$\text{Also, }\frac{AM/MH}{(AB/BH)^2}=\frac{\left(\frac{a}{c\cos B}\right)}{\left(\frac{c^2}{c^2\cos^2 B}\right)}=\frac{a\cos B}{c}=\frac{a(a^2-b^2+c^2)}{2ac.c}$$

$$\Rightarrow\frac{AM/MH}{(AB/BH)^2}=\frac{a^2-b^2+c^2}{2c^2}\text{ (QED)}$$