ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$, $xyz=1$ then:

$$\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}\geq\frac{3\sqrt{2}}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Le Thu-Vietnam

$$\therefore\ Lemma\ \left(\sum x\right)^3\geq\frac{27(xyz+\sum x^2y)}{4}\ \forall\, x,y,z\geq 0$$

$$Proof\ :Let\ y>0\overset{WLOG}{=}mid(x,y,z)\ then\ by\ Rearangement\ inequality:$$

$$xyz+\sum x^2y=xyz+x.xy+y.yz+z.zx\leq$$

$$\leq xyz+x.xy+y.xz+z.yz=y(x^2+2xz+z^2)=$$

$$=y(x+z)^2\overset{AM-GM}{\leq}4.\frac{\left(y+\frac{x+z}{2}+\frac{x+z}{2}\right)^3}{27}=\frac{4\left(\sum x\right)^3}{27}$$

$$\therefore\ We\ have\ \left(\sum\frac{x}{\sqrt{y^2+z}}\right)^2\overset{Holder}{\geq}\frac{(\sum x)^3}{\sum x(y^2+z)}\overset{?}{\geq}\left(\frac{3}{\sqrt{2}}\right)^2$$

$$\leftrightarrow 2\left(\sum x\right)^3\overset{?}{\geq}9\left(\sum xy^2+\sum xy\right).\ Using\ above\ lemma$$

$$it\ suffices\ to\ prove\ that\ :2\left(\sum x\right)^3\overset{?}{\geq}27\left(\sum xy-xyz\right)$$

$$By\ AM-GM\ ,\quad 3\sum xy\leq\left(\sum x\right)^2\ since\ xyz=1$$

$$we\ are\ thus\ left\ with:\ 2\left(\sum x\right)^3-9\left(\sum x\right)^2+27\overset{?}{\geq}0$$

$$\overset{u=\sum x}{\leftrightarrow}(u-3)^2(2u+3)\geq 0\ \ which\ completes\ the\ proof$$

$$Equality\ holds\ iff\ x=y=z=1.$$