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If $x, y, z > 0$ then:

$$\frac{x}{\sqrt{x^2+3yz}}+\frac{y}{\sqrt{y^2+3xz}}+\frac{z}{\sqrt{z^2+3xy}}\geq\frac{3}{2}$$

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$$\boldsymbol{Lemma}: \forall\, t>0\ :\ f(t)=\frac{1}{\sqrt{1+3t}}+\frac{3}{16}lnt\geq\frac{1}{2}$$

$$f'(t)=-\frac{3}{2(1+3t)\sqrt{1+3t}}+\frac{3}{16t}=3\cdot\frac{(1+3t)\sqrt{1+3t}-8t}{8(1+3t)\sqrt{1+3t}}$$

$$f'(t)=0\to(1+3t)\sqrt{1+3t}=8t\to t=1$$

$$and\ \ f'(t)>0\to\ t>1\ \ and\ \ f'(t)<0\to t<1$$

$$\forall\, t>0\ :\ f(t)\geq Minf(t)=f(1)=\frac{1}{2}$$

$$\frac{1}{\sqrt{1+3t}}+\frac{3}{16}lnt\geq\frac{1}{2}$$

$$t=\frac{xy}{z^2}\to\frac{1}{\sqrt{1+3\cdot\frac{xy}{z^2}}}+\frac{3}{16}ln\frac{xy}{z^2}\geq\frac{1}{2}\ (1)$$

$$t=\frac{yz}{x^2}\to\frac{1}{\sqrt{1+3\cdot\frac{yz}{x^2}}}+\frac{3}{16}ln\frac{yz}{x^2}\geq\frac{1}{2}\ (2)$$

$$t=\frac{zx}{y^2}\to\frac{1}{\sqrt{1+3\cdot\frac{zx}{y^2}}}+\frac{3}{16}ln\frac{zx}{y^2}\geq\frac{1}{2}\ (3)$$

By adding $(1),(2),(3)$:

$$\frac{1}{\sqrt{1+3\cdot\frac{xy}{z^2}}}+\frac{1}{\sqrt{1+3\cdot\frac{yz}{x^2}}}+\frac{1}{\sqrt{1+3\cdot\frac{zx}{y^2}}}+\frac{3}{16}ln\left(\frac{xy}{z^2}\cdot\frac{yz}{x^2}\cdot\frac{zx}{y^2}\right)\geq\frac{3}{2}$$

$$\frac{x}{\sqrt{x^2+3yz}}+\frac{y}{\sqrt{y^2+3xz}}+\frac{z}{\sqrt{z^2+3xy}}+ln1\geq\frac{3}{2}$$

$$\frac{x}{\sqrt{x^2+3yz}}+\frac{y}{\sqrt{y^2+3xz}}+\frac{z}{\sqrt{z^2+3xy}}\geq\frac{3}{2}$$