ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz = 1$ then:

$$\frac{x}{(x+2)(y+2)} + \frac{y}{(y+2)(z+2)} + \frac{z}{(z+2)(x+2)} \ge \frac{1}{3}$$

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$$\frac{x}{(x+2)(y+2)} + \frac{y}{(y+2)(z+2)} + \frac{z}{(z+2)(x+2)} \ge \frac{1}{3} \Leftrightarrow$$

$$\frac{x(z+2) + y(x+2) + z(y+2)}{(x+2)(y+2)(z+2)} \ge \frac{1}{3} \Leftrightarrow$$

$$3[xz + xy + yz + 2(x+y+z)] \ge (x+2)(y+2)(z+2) \Leftrightarrow$$

$$3(xy + xz + yz) + 6(x+y+z) \ge \underbrace{xyz}_{1} + 2(xy + xz + yz) + 4(x+y+z) + 8 \Leftrightarrow$$

$$xy + xz + yz + 2(x+y+z) \overset{?}{\overbrace{\ge}} 9$$

$$\frac{1}{z} + \frac{1}{y} + \frac{1}{x} + 2(x+y+z) \overset{?}{\overbrace{\ge}} 9$$

$$\underbrace{\left(x + \frac{1}{x}\right)}_{\ge 2} + \underbrace{\left(y + \frac{1}{y}\right)}_{\ge 2} + \underbrace{\left(x + \frac{1}{x}\right)}_{\ge 2} + x + y + z \ge 6 + x + y + z \overset{A-G}{\overbrace{\ge}}$$

$$\ge 6 + 3 \cdot \sqrt[3]{xyz} = 6 + 3 \cdot 1 = 9$$

$$\textit{Equality holds for } \; x = y = z = 1$$