ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz = 1$ then:

$$\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}} \geq \frac{3\sqrt{2}}{2}$$

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Let be $f, g: \mathbb{R}_+^* X \mathbb{R}_+^* X \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$

$$g(x,y,z) = 3\sqrt[3]{\frac{xyz}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}}, f(x,y,z) = \frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}$$

$$f(x,y,z) = \frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{xyz}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}} = g(x,y,z)$$

$$f(x,y,z) \geq g(x,y,z) \ (1)$$

$$\frac{3\sqrt{2}}{2} = \frac{3}{\sqrt{1+\sqrt[3]{xyz}}} = 3\sqrt[3]{\frac{1}{\sqrt{\left(1+\sqrt[3]{xyz}\right)^3}}} \overset{HOLDER}{\geq} 3\sqrt[3]{\frac{1}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}} =$$

$$= 3\sqrt[3]{\frac{xyz}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}} = g(x,y,z) \Rightarrow Max g(x,y,z) = \frac{3\sqrt{2}}{2} (2)$$

By $(1), (2)$:

$$\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}} \geq \frac{3\sqrt{2}}{2}$$

Equality holds for $x = y = z = 1$