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PROBLEMS FOR JUNIORS

JP.511. Let $n \in \mathbb{N}^*$. Prove that among the numbers $\binom{2n}{1}, \binom{2n}{2}, \ldots, \binom{2n}{n}$ exist at least one number which is not divisible with 16065.

Proposed by Mihály Bencze - Romania

JP.512. Solve the following equation:

$$\log_{a+1}(a^x + 2a + 1) = \log_a((a+1)^x - 2a - 1), a > 1.$$

Proposed by Mihály Bencze - Romania

JP.513. Solve for real numbers:

$$\begin{cases} 2\log_3(2^{x_1}+5)\log_2(3^{x_1}-5) = \log_3(2^{x_2}+5) + \log_2^2(3^{x_3}-5) \\ 2\log_3(2^{x_2}+5)\log_2(3^{x_2}-5) = \log_3(2^{x_3}+5) + \log_2^2(3^{x_4}-5) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 2\log_3(2^{x_n}+5)\log_2(3^{x_n}-5) = \log_3(2^{x_1}+5) + \log_2^2(3^{x_2}-5) \end{cases}$$

Proposed by Mihály Bencze - Romania

JP.514. On the set $M = \{2n+1 | n \in \mathbb{N}^*\}$ define

$$a * b = a + (b-3)2^{[\log_2 a]-1}, \forall a, b \in M,$$

where [.] is the integer part. Prove that $(M, *)$ is a monoid.

Proposed by Mihály Bencze - Romania

JP.515. Solve for real numbers:

$$\sqrt{\left(\frac{3}{5}\sin x + \frac{101}{15}\cos y\right)\left(\frac{5}{3}\sin x + \frac{17}{3}\cos y\right)} +$$
$$+\sqrt{\left(15 - \frac{52}{15}\sin x - \frac{388}{15}\cos y\right)\left(-7 + \frac{6}{5}\sin x + \frac{202}{15}\cos y\right)} = 4$$

Proposed by Mihály Bencze - Romania

JP.516. In ΔABC the following relationship holds:

$$(ab+bc+ca)^2 + n(a^2+b^2+c^2)^2 \geq (n+1)(18Rr)^2, n \in \mathbb{N}$$

Proposed by Marin Chirciu - Romania

JP.517. If $x, y, z \in [0, k]; k > 0$, then $y(x-z) - z(x-k) \le k^2$.

Proposed by Laura Molea and Gheorghe Molea – Romania

JP.518. Let $ABCD$ an convex quadrilateral, $\lambda \in \mathbb{R}$ and M, N be such that:

$$\overrightarrow{AM} = \lambda \cdot \overrightarrow{AB}; \overrightarrow{DN} = \lambda \cdot \overrightarrow{DC}, \overrightarrow{AD} = 3\overrightarrow{BC}.$$

$$\text{Find } \lambda \in \mathbb{R} \text{ such that } \overrightarrow{MN} = 7\overrightarrow{BC}.$$

Proposed by Florică Anastase - Romania

JP.519. In $\Delta ABC, AA', BB', CC'$ - internal bisectors, A'' - symmetric point of A to $BC, N \in (AB), M \in (AN)$ such that $\overrightarrow{CM} = x \cdot \overrightarrow{MN}, \overrightarrow{AB} = x \cdot \overrightarrow{AN}, x \in \mathbb{R}$. Prove that if $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 0$ then A, M, A'' are collinears.

Proposed by Florică Anastase - Romania

JP.520. Prove, that in any ΔABC triangle, the following inequality holds:

$$3\sqrt{\frac{2r}{R}} \le \sin\left(\frac{\widehat{A}}{2} + \widehat{B}\right) + \sin\left(\frac{\widehat{B}}{2} + \widehat{C}\right) + \sin\left(\frac{\widehat{C}}{2} + \widehat{A}\right) \le 3$$

Proposed by Radu Diaconu - Romania

JP.521. If $a, b, c, d > 0$ such that $(a+b+c)(b+c+d) = 1$, prove that:

$$\sqrt[3]{(a+b)(c+d)} + \sqrt[3]{(b+c)(d+a)} + \sqrt[3]{(c+d)(a+b)} +$$
$$+ \sqrt[3]{(d+a)(b+c)} < \frac{1}{3}\left(\frac{a+b}{b+c} + \frac{b+c}{c+d} + \frac{c+d}{d+a} + \frac{d+a}{a+b} + 4\right)$$

Proposed by Gheorghe Molea - Romania

JP.522. In acute triangle ABC the following relationship holds:

$$\left(\sum \frac{\sin^2 A}{\cos A}\right)\left(\sum \frac{\cos A}{\sin^2 A}\right) \ge 9 + 7\left(\frac{R-2r}{R+r}\right)$$

Proposed by Alexandru Szoros - Romania

JP.523. On the sides AB and AC of a triangle ABC, consider the interior pints E and D, respectively, such that $(\frac{AE}{EB})^2 + (\frac{AD^2}{DC})^2 = 1$. The segments BD and CE intersect at point P. Find the ratio of the areas of quadrilateral $EBCD$ and triangle PBC.

Proposed by George Apostolopoulos – Greece

JP.524. Prove that in any ΔABC the following inequality holds:

$$\frac{\cot\frac{A}{2}}{h_a}+\frac{\cot\frac{B}{2}}{h_b}+\frac{\cot\frac{C}{2}}{h_c}\geq\frac{4R+r}{F}$$

Proposed by Marian Ursărescu - Romania

JP.525. Prove that in any ΔABC the following inequality holds:

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\leq\frac{(2R-r)^2}{r}$$

where n_a, n_b, n_c are Nagel's cevians.

Proposed by Marian Ursărescu - Romania

PROBLEMS FOR SENIORS

SP.511. Let triangle ABC with $\widehat{A}>90^\circ$ and let internal points M_1, M_2, M_3, M_4 on the side BC, such that $BM_1=M_1M_2=M_2M_3=M_3M_4=M_4C$. Also, R_1, R_2 denote the circumradius of triangles AM_1M_2, AM_3M_4, respectively. Prove:

$$BC>\frac{20\sqrt{R_1R_2}}{3(\cot B+\cot C)}$$

Proposed by George Apostolopoulos - Greece

SP.512. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all integers x, y the number

$$f^2(x)+2xf(y)+y^2 \text{ is a perfect square.}$$

Proposed by Baris Koyuncu - Turkiye

SP.513. Given $k\geq 4$. In any triangle ABC prove that:

$$\frac{3}{k}\leq\sum_{cyc}\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}\leq\frac{9k+12}{64}$$

Proposed by George Apostolopoulos – Greece

SP.514. Let be $P(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$ with $n\in\mathbb{N}, n\geq 2, a_i\in\mathbb{R}, (\forall)i=\overline{1,n}$. If the equation $P(x)=0$ has all the roots real, then $(\forall)k>\max\{x_1,x_2,\ldots,x_n\}+1$ we have:

$$(n-1)\cdot P(k)-P'(k)+1>0$$

Proposed by Gheorghe Molea - Romania

SP.515. If $a, b, c, t, k > 0$ such that $(t+a)(t+b)(t+c) = 2k$ and $k > \frac{t^3}{2}$, prove that:

$$\frac{1}{b(t+a)^2} + \frac{1}{c(t+b)^2} + \frac{1}{a(t+c)^2} \ge \frac{3t\sqrt[3]{4k^2}}{k^2}$$

Proposed by Gheorghe Molea - Romania

SP.516.Let be the acuteangled ΔABC and the points $B, A_1, A_2, \dots, A_{n-1}, C$ collinears in this order. Let $R, R_1, R_2, \dots, R_n$ be the circumradies of $\Delta ABC, \Delta ABA_1, \Delta A_1AA_2, \dots, A_{n-1}AC$. Prove that:

$$\max(R_1, R_2, \dots, R_n) \ge \frac{R \sin \widehat{A}}{n \cdot \sin \frac{\widehat{A}}{n}}$$

$$\text{and that } \min(R_1, R_2, \dots, R_n) < \frac{\pi R \cdot \sin \widehat{A}}{2\widehat{A}}$$

Proposed by Radu Diaconu - Romania

SP.517. In acute $\Delta ABC, BB', CC'$ - altitudes, $C' \in (AB)$, $B' \in (AC), \{H\} = BB' \cap CC', E, F$ middle points of $[BH], [AC]$ respectively. Prove that:

$$4EF^2 \ge (EC' + EB')^2 + (C'F + B'F)^2$$

Proposed by Florică Anastase - Romania

SP.518.Find:

$$\Omega = \lim_{x \to 0}\left(\frac{1}{x} \cdot \lim_{n \to \infty} \sum_{k=1}^{n} 3^{k-1} \sin^3 \frac{x}{3^k}\right), a \in \mathbb{R}$$

Proposed by Florică Anastase - Romania

SP.519. Let $x_i, i = 1, 2, \dots, n$ be positive real numbers such that:

$$\prod_{i=1}^{n} x_i = 1$$

Prove:

$$\sum_{i=1}^{n}\left(\frac{x_i^6 + 1}{x_i + 1}\right)^2 \cdot x_{i+1} \ge n, \text{ where } x_1 = x_{n+1}$$

Proposed by George Apostolopoulos – Greece

SP.520. Prove that in any acute triangle ABC:

$$\sqrt{2}(13k^2-3)\le\sqrt{\pi(\cos^2 A+\cos^2 B)}\le\frac{\sqrt{2}}{2}k$$

where $k\in(0,\frac{1}{2}]$. The product is over all cyclic permutations of (A,B,C).

Proposed by George Apostolopoulos – Greece

SP.521. If $F_0=0,F_1=1,F_{n+2}=F_{n+1}+F_n,\forall n\in\mathbb{N}$, i.e. $\{F_n\}_{n\ge0}$ is Fibonacci's sequence, and $L_0=2,L_1=1$, $L_{n+2}=L_{n+1}+L_n,\forall n\in\mathbb{N}$, i.e. $\{L_n\}_{n\ge0}$ is Lucas' sequence, then prove that:

$$\frac{F_nL_{n+2}^2}{F_{n+3}}+\frac{F_{n+1}L_{n+3}^2}{F_n+F_{n+2}}+(L_n+L_{n+2})^2-2\sqrt{6}\cdot\sqrt{L_nL_{n+1}}\cdot L_{n+2}\ge0,\forall n\in\mathbb{N}^*$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

SP.522. If $\{\varphi\}_{n\ge0}$ is the sequence of Fermat, i.e.
$\varphi_{n+2}-3\varphi_{n+1}-2\varphi_n=0,\varphi_0=0,\varphi_1=1$, then prove that:
$2(\varphi_n^2-\varphi_{n+1}\varphi_{n-1})=2\cdot(-2)^{n-1}$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

SP.523. Let be $\Delta ABC,D,E,F$ the points in which the internal bisectors intersect circumcenter. Prove that:

$$\frac{4}{3}R^2(4R+r)^2\le DE^4+EF^4+FD^4\le4R^2(4R+r)(2R-r)$$

Proposed by Marian Ursărescu - Romania

SP.524. Let be ΔABC and A',B',C' the tangent points of circumcenter with the sides BC,AC, respectively AB. Prove that:

$$\frac{1}{A'B'\cdot A'C'}+\frac{1}{A'B'\cdot B'C'}+\frac{1}{A'C'\cdot B'C'}\le\Big(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\Big)\Big(\frac{R}{r}+1\Big)$$

Proposed by Marian Ursărescu - Romania

SP.525. If $a,b,c\ge0,a+b+c=3$ then:

$$343(ab+bc+ca)^3\le27(5+ab+c)(5+bc+a)(5+ca+b)$$

Proposed by Andrei Ştefan Mihalcea - Romania

UNDERGRADUATE PROBLEMS

UP.511. Prove that:

$$\int_0^{\infty} te^{2t}e^{-e^{-2t}}dt = -\frac{\gamma}{4}$$

where γ is the Euler - Mascheroni constant.

Proposed by Said Attaoui - Algerie

UP.512. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.513. Find:

$$\Omega = \lim_{x\to\infty}\left((x+a)\sin\frac{1}{x+a}\sqrt[x+1]{\Gamma(x+2)} - x\sin\frac{1}{x}\sqrt[x]{\Gamma(x+1)}\right); a > 0$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.514. If $f:(0,\infty)\to(0,\infty)$ is a convex function, $0 < a \le b$ then:

$$\frac{1}{4a}\int_0^{4a}f(x)dx - \frac{1}{3a+b}\int_0^{3a+b}f(y)dy \ge$$
$$\ge \frac{1}{a+3b}\int_0^{a+3b}f(z)dz - \frac{1}{4b}\int_0^{4b}f(t)dt$$

Proposed by Daniel Sitaru - Romania

UP.515. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x\right)\right)$$

Proposed by Florică Anastase - Romania

UP.516. Prove that:

$$\int_0^1 \frac{1}{(1-x(1-x))}dx = 2\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}$$

Deduce the value of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}$

Proposed by Said Attaoui – Algeria

UP.517. Prove the equality:

$$\int_0^\infty \frac{\ln x}{x^3 - 3\sqrt{x} + 1} dx = \frac{8\pi^2}{81}\left(5\sin\frac{\pi}{18} - \sqrt{3}\cos\frac{\pi}{18}\right)$$

Proposed by Vasile Mircea Popa – Romania

UP.518. Let $F, f, g : [0,1] \to \mathbb{R}$ such as $g'(x) > 0$ for every $x \in [0,1]$ and $F'(x), \frac{f'(x)}{g'(x)}$ are Riemann integrable. Find:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(F\left(\frac{k}{n}\right) - F\left(\frac{k-1}{n}\right)\right) \frac{f'(\frac{k}{n}) + f'(\frac{k-1}{n})}{g'(\frac{k}{n}) + g'(\frac{k-1}{n})}$$

Proposed by Cristian Miu - Romania

UP.519. In triangle ABC_Δ we note H the orthocentre and O the circumcentre of the triangle. Let D, E, F be the midpoints of $[BC], [AC]$ and $[AB]$ and let A_1, B_1, C_1 be the points symmetric to H with respect to D, E and F, and let H_1 be the orthocentre of the triangle $A_1B_1C_1$. Prove that $HH_1 = 2OH$

Proposed by Pal Orban - Romania

UP.520.If $a_n > 0; n \in \mathbb{N}^*$ is such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{an_n} = a > 0$$

then find:

$$\Omega(a) = \lim_{n\to\infty} (H_n - \log \sqrt[n]{a_n})$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.521. If $a_1 = 1, a_{n+1} = a_n + e^{H_n} \cdot \sin\frac{\pi}{n}; n \in \mathbb{N}^*$ then find:

$$\Omega = \lim_{n\to\infty} \frac{a_n}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.522. Find $x, y > 0$ such that:

$$81x^2 + x + \frac{1}{2x+y} = 16x + 1$$

Proposed by Daniel Sitaru - Romania

UP.523. If $x, y, z > 0, xyz = x + y + z + 2$ then:

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \geq \frac{6}{\sqrt{xyz}}$$

Proposed by Marin Chirciu - Romania

UP.524. In ΔABC then:

$$\sum \frac{(m_b+m_c)^{n+1}}{(m_a+\sqrt{m_b m_c})^n} \geq \frac{12r}{R}(2R-r), n \in \mathbb{N}$$

Proposed by Marin Chirciu - Romania

UP.525. In acute ΔABC the following relationship holds:

$$2s\left(2+\frac{3R}{r}-\frac{R^2}{r^2}\right) \leq \sum \frac{b+c}{\cos A} \leq \frac{4s}{3}\sum \sec A$$

Proposed by Marin Chirciu - Romania

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