R M M

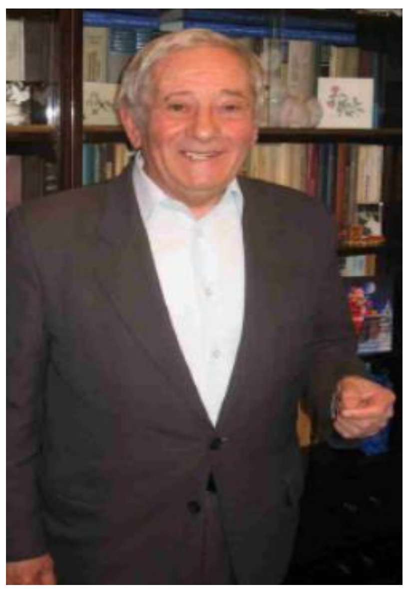
ROMANIAN MATHEMATICAL MAGAZINE

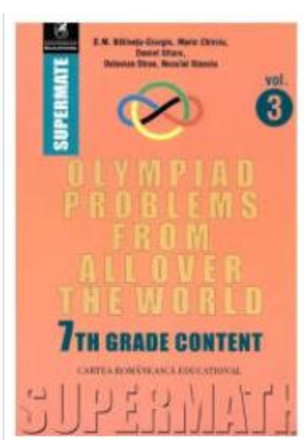
www.ssmrmh.ro

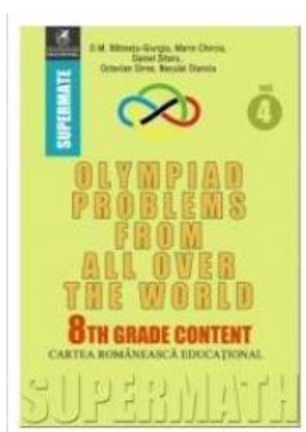
GENERALIZATIONS FOR SOME PUBLISHED PROBLEMS IN THE AMERICAN MATHEMATICAL MONTHLY (AMM) THE PENTAGON MATH JOURNAL AND SCHOOL SCIENCE AND MATHEMATICS JOURNAL (SSMJ)

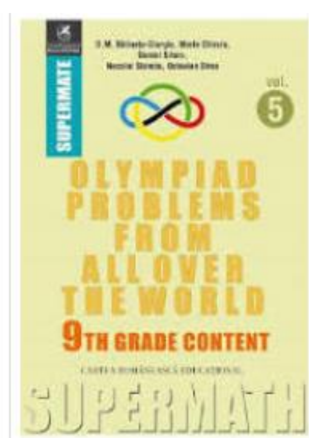
by Daniel Sitaru and Neculai Stanciu-Romania

~ DEDICATED TO 87TH ANIVERSARY OF PROFESSOR D. M. BĂTINEŢU-GIURGIU ~









I. PROBLEM 692 FROM THE PENTAGON, FALL 2011 & PROBLEM 12360 FROM THE AMERICAN MATHEMATICAL MONTHLY, DECEMBER 2022.

Find $\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}\right)$**, where** $x_n=\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$.

Solution without Stirling' approximation.

$$X_n=\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}=\frac{n^2}{x_n}\left(\left(\frac{n+1}{n}\right)^2\cdot\frac{x_n}{x_{n+1}}-1\right)=\frac{n^2}{x_n}\cdot(u_n-1)=$$

$$=\frac{n^2}{x_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{n}{x_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\forall n\ge 2.$$

$$\lim_{n\to\infty}\frac{n}{x_n}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}=$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}\right)=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot e_n\right)=e^2,$$

where $e_n=\left(1+\frac{1}{n}\right)^n$**. Since,** $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{x_n}{x_{n+1}},\forall n\ge 2,$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}\cdot\frac{x_n}{n^2}\right)=\lim_{n\to\infty}\left(\frac{n+1}{x_{n+1}}\cdot\frac{x_n}{n}\cdot\frac{n+1}{n}\right)=e^2\cdot\frac{1}{e^2}\cdot 1=1,\text{ so }\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1;$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}e_n^2\cdot\left(\frac{x_n}{x_{n+1}}\right)^n=e^2\cdot\lim_{n\to\infty}\left(\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\cdot\sqrt[n+1]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\right)=$$

$$=e^2\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{x_{n+1}}{n+1}\right)=e^2\cdot\frac{1}{e^2}\cdot e=e.$$

Hence, $\lim_{n\to\infty}X_n=e^2\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=e^2\cdot 1\cdot\ln e=e^2$.

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

II. PROBLEM 5495 FROM SCHOOL SCIENCE AND MATHEMATICS JOURNAL, APRIL 2018

Let $(x_n)_{n\geq 1}$, $x_1=1$, $x_n=1\cdot\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}$. **Find** $\lim\limits_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}\right)$.

Solution without Stirling' approximation.

We have $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{x_n}}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{n^n}{x_n}}\overset{Cauchy-D'Alembert}{=}\lim\limits_{x\to\infty}\frac{(n+1)^{n+1}}{x_{n+1}}\cdot\frac{x_n}{n^n}=\lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{1}{n^n}=$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\lim_{n\to\infty}\frac{(n+1)}{\sqrt[n+1]{(2n+1)!!}}=e\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}$$

$$\overset{C-D'A}{=}e\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=e\cdot\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^n=\frac{e^2}{2}$$ **, (1).**

We have $\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}=\frac{n^2}{\sqrt[n]{x_n}}\cdot(u_n-1)=\frac{n^2}{\sqrt[n]{x_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{n}{\sqrt[n]{x_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$**, (2).**

Above we denote $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}$. **We have** $\lim\limits_{n\to\infty}u_n=1\Rightarrow\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$;

$$\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{2n+1}=\frac{e}{2}.$$

Then, $\lim\limits_{n\to\infty}u_n^n=\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{x_n}{x_{n+1}}\cdot\sqrt[n+1]{x_{n+1}}=e^2\cdot\lim\limits_{n\to\infty}\frac{1}{\sqrt[n+1]{(2n+1)!!}}\cdot\sqrt[n+1]{x_{n+1}}=$

$$=e^2\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{x_{n+1}}}{n+1}=e^2\cdot\frac{e}{2}\cdot\frac{2}{e^2}=e.$$

From (2) and above we obtain that

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}\right)=\frac{e^2}{2}\cdot 1\cdot\ln e=\frac{e^2}{2}$$ **, and we are done!**

III. GENERALIZATION FOR AMM DECEMBER 2022 AND SSMJ APRIL 2018

Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ **be positive real sequences such that** $b_n = a_1 \cdot \sqrt{a_2!} \cdot \sqrt[3]{a_3!} \cdot \ldots \cdot \sqrt[n]{a_n!}$ **and** $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n} = a$. **Find** $\lim\limits_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right)$.

Solution. We have $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^n}{a_n}} \overset{Cauchy-D'Alembert}{=} \lim\limits_{x\to\infty} \frac{(n+1)^{n+1}}{a_{n+1}} \cdot \frac{a_n}{n^n} =$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1} \cdot \lim_{n\to\infty} \frac{a_n \cdot n}{a_{n+1}} = \frac{e}{a} \text{ and}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{b_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{b_n}} \overset{Cauchy-D'Alembert}{=} \lim_{x\to\infty} \frac{(n+1)^{n+1}}{b_{n+1}} \cdot \frac{b_n}{n^n} =$$

$$= \lim_{x\to\infty}\left(\frac{n+1}{n}\right)^n \cdot \frac{b_n(n+1)}{b_{n+1}} = e \cdot \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{a_{n+1}}} = e \cdot \frac{e}{a} = \frac{e^2}{a}.$$

We have $\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} = \frac{n^2}{\sqrt[n]{b_n}} \cdot (u_n - 1) = \frac{n^2}{\sqrt[n]{b_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \frac{n}{\sqrt[n]{b_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$, **(1).**

Above we denote $u_n = \left(\frac{n+1}{n}\right)^2 \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}$. **We have** $\lim\limits_{n\to\infty} u_n = 1 \Rightarrow \lim\limits_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$;

Then, $\lim\limits_{n\to\infty} u_n^n = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n} \cdot \frac{b_n}{b_{n+1}} \cdot \sqrt[n+1]{b_{n+1}} = e^2 \cdot \lim\limits_{n\to\infty}\left(\frac{b_n \cdot (n+1)}{b_{n+1}} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{n+1}\right) =$

$$= e^2 \cdot \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}}}{n+1} = e^2 \cdot \frac{e}{a} \cdot \frac{a}{e^2} = e.$$

From (1) and above we obtain that

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right) = \frac{e^2}{a} \cdot 1 \cdot \ln e = \frac{e^2}{a}$$ **, and we are done!**

IV. GENERALIZATION OF PROBLEM 5710 FROM SCHOOL SCIENCE AND MATHEMATICS JOURNAL (SSMJ), DECEMBER 2022

Let the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$: $a_n = \sum\limits_{k=1}^{n} \arctan \frac{1}{k^2 - k + 1}$ **and** $\lim\limits_{n\to\infty} \frac{b_{n+1}}{nb_n} = b \in R_+^*$.

Compute $\lim\limits_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}$ **.**

Solution. We have:

$$a_n=\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1}=\arctan 1+\sum_{k=2}^{n}\left(\arctan\frac{1}{k+1}-\arctan\frac{1}{k}\right)=$$

$$=\frac{\pi}{4}+\arctan 1-\arctan\frac{1}{n}=\frac{\pi}{2}-\arctan\frac{1}{n}\text{, so }\lim_{n\to\infty}a_n=\frac{\pi}{2}\text{, (1).}$$

From (1) and Cesaro-Stolz theorem we get:

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)n=\lim_{n\to\infty}\frac{\frac{\pi}{2}-a_n}{\frac{1}{n}}\overset{Cesaro-Stolz}{=}\lim_{n\to\infty}\frac{\frac{\pi}{2}-a_{n+1}-\frac{\pi}{2}+a_n}{\frac{1}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{\frac{1}{n}-\frac{1}{n+1}}=$$

$$=\lim_{n\to\infty}(a_{n+1}-a_n)n(n+1)=\lim_{n\to\infty}(n^2+n)\arctan\frac{1}{(n+1)^2-n-1+1}=$$

$$=\lim_{n\to\infty}\frac{n^2+n}{n^2+n+1}(n^2+n+1)\arctan\frac{1}{n^2+n+1}=1\cdot\lim_{n\to\infty}\frac{\arctan\frac{1}{n^2+n+1}}{\frac{1}{n^2+n+1}}=1\cdot 1=1\text{, (2).}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}\overset{C-d'A}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^n}\cdot\frac{n^n}{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{na_n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{b}{e}\text{, (3).}$$

By (2) and (3) we obtain that:

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}=\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)n\cdot\frac{\sqrt[n]{b_n}}{n}=$$

$$=\left(\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)n\right)\cdot\left(\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}\right)=1\cdot\frac{b}{e}=\frac{b}{e}.$$

Remark. For $b=\pi$ **we obtain the problem 5710 from SSMJ.**