11 INDEPENDENT SOLUTIONS FOR A JAPANESSE INEQUALITY

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Abstract. In this paper is presented an inequality proposed by Kunihiko Chikaya-Tokyo-Japan and 11 independent solutions for it from Japan, Greece and Romania.

MAIN RESULT: If $a, b, c > 0$ then:

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc \geq 6$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

Solution 1 by proposer, Solution 2 by Panagiotis Danousis-Greece, Solution 3 by Lazaros Zachariadis-Greece, Solutions 4,5,6,7,8,9,10,11 and generalization by Daniel Sitaru-Romania

Solution 1 by proposer.

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc = \frac{(\frac{1}{a})^2}{a} + \frac{(\frac{1}{b})^2}{b} + \frac{(\frac{1}{c})^2}{c} + 3abc \overset{\text{BERGSTRÖM}}{\geq}$$

$$\geq \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{a+b+c} + 3abc \geq \frac{3(\frac{1}{a} \cdot \frac{1}{b} + \frac{1}{b} \cdot \frac{1}{c} + \frac{1}{c} \cdot \frac{1}{a})}{a+b+c} + 3abc =$$

$$= \frac{3(\frac{a+b+c}{abc})}{a+b+c} + 3abc = 3\left(abc + \frac{1}{abc}\right) \overset{\text{AM-GM}}{\geq} 3 \cdot \sqrt{abc \cdot \frac{1}{abc}} = 6$$

$$\text{Equality holds for: } \frac{1}{a} = \frac{1}{b} = \frac{1}{c}, abc = \frac{1}{abc} \Leftrightarrow a = b = c = 1$$

□

Solution 2 by Panagiotis Danousis - Greece.

Let be $x, y, z > 0$ such that: $a = e^x, b = e^y, c = e^z$. It is known that:

$$e^x \geq 1 + x, \forall x \in \mathbb{R} \text{ with equality for } x = 0.$$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc = e^{-3x} + e^{-3y} + e^{-3z} + 3e^{x+y+z} \geq$$

$$\geq 1 - 3x + 1 - 3y + 1 - 3z + 3(1 + x + y + z) = 6$$

$$\text{Equality holds for: } x = y = z = 0 \Leftrightarrow a = b = c = 1.$$

□

Solution 3 by Lazaros Zachariadis - Greece.

$$\text{Denote: } x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}.$$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc = \sum_{cyc} x^3 + \frac{3}{xyz} \overset{\text{AM-GM}}{\geq} 3xyz + \frac{3}{xyz} =$$

$$= 3\left(xyz + \frac{1}{xyz}\right) \overset{\text{AM-GM}}{\geq} 3 \cdot 2\sqrt{xyz \cdot \frac{1}{xyz}} = 6$$

$$\text{Equality holds for: } x = y = z, xyz = \frac{1}{xyz} \Leftrightarrow x = y = z = 1 \Leftrightarrow a = b = c = 1.$$

□

Solution 4 by Daniel Sitaru - Romania.

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + abc + abc + abc \overset{\text{AM-GM}}{\geq}$$

$$\geq 6\sqrt[6]{\frac{1}{a^3} \cdot \frac{1}{b^3} \cdot \frac{1}{c^3} \cdot abc \cdot abc \cdot abc} = 6$$

$$\text{Equality holds for: } \frac{1}{a^3} = \frac{1}{b^3} = \frac{1}{c^3} = abc \Leftrightarrow a = b = c = 1$$

□

Solution 5 by Daniel Sitaru - Romania.

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\frac{1}{a^3} \cdot \frac{1}{b^3} \cdot \frac{1}{c^3}} + 3abc =$$

$$= \frac{3}{abc} + 3abc = 3\left(\frac{1}{abc} + abc\right) \overset{\text{AM-GM}}{\geq} 3 \cdot 2\sqrt{\frac{1}{abc} \cdot abc} = 6$$

$$\text{Equalithy holds for: } \frac{1}{a^3} = \frac{1}{b^3} = \frac{1}{c^3}, \frac{1}{abc} = abc \Leftrightarrow a = b = c = 1$$

□

Solution 6 by Daniel Sitaru - Romania.

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc = \frac{(bc)^3 + (ca)^3 + (ab)^3 + 3(abc)^4}{(abc)^3} =$$

$$= \frac{(bc)^3 + (ca)^3 + (ab)^3 + (abc)^4 + (abc)^4 + (abc)^4}{(abc)^3} \overset{\text{AM-GM}}{\geq}$$

$$\geq \frac{6\sqrt[6]{(bc)^3 \cdot (ca)^3 \cdot (ab)^3 \cdot (abc)^4 \cdot (abc)^4 \cdot (abc)^4}}{(abc)^3} =$$

$$= \frac{6\sqrt[6]{(abc)^{18}}}{(abc)^3} = \frac{6(abc)^3}{(abc)^3} = 6$$

$$\text{Equality holds for: } (bc)^3 = (ca)^3 = (ab)^3 = (abc)^4 \Leftrightarrow a = b = c = 1$$

□

Solution 7 by Daniel Sitaru - Romania.

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=\frac{1}{a^3}+abc+\frac{1}{b^3}+abc+\frac{1}{c^3}+abc\overset{\text{AM-GM}}{\geq}$$

$$\geq 2\sqrt{\frac{1}{a^3}\cdot abc}+2\sqrt{\frac{1}{b^3}\cdot abc}+2\sqrt{\frac{1}{c^3}\cdot abc}=2\sqrt{\frac{bc}{a^2}}+2\sqrt{\frac{ca}{b^2}}+2\sqrt{\frac{ab}{c^2}}\geq$$

$$\overset{\text{AM-GM}}{\geq} 3\sqrt[3]{2\sqrt{\frac{bc}{a^2}}\cdot 2\sqrt{\frac{ca}{b^2}}\cdot 2\sqrt{\frac{ab}{c^2}}}=3\sqrt[6]{\frac{64(abc)^2}{(abc)^2}}=6$$

$$\text{Equality holds for: }\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=abc\Leftrightarrow a=b=c=1$$

□

Solution 8 by Daniel Sitaru - Romania.

$$\frac{1}{a^3}+\frac{2}{b^3}+\frac{3}{c^3}+6abc\overset{\text{AM-GM}}{\geq}12\sqrt[12]{\frac{(abc)^6}{a^3b^6c^9}}=12\sqrt[4]{\frac{a}{c}}$$

$$\frac{2}{a^3}+\frac{3}{b^3}+\frac{1}{c^3}+6abc\overset{\text{AM-GM}}{\geq}12\sqrt[12]{\frac{(abc)^6}{a^6b^9c^3}}=12\sqrt[4]{\frac{c}{b}}$$

$$\frac{3}{a^3}+\frac{1}{b^3}+\frac{2}{c^3}+6abc\overset{\text{AM-GM}}{\geq}12\sqrt[12]{\frac{(abc)^6}{a^9b^3c^6}}=12\sqrt[4]{\frac{b}{a}}$$

By adding:

$$6\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+18abc\geq 12\left(\sqrt[4]{\frac{a}{c}}+\sqrt[4]{\frac{c}{b}}+\sqrt[4]{\frac{b}{a}}\right)\overset{\text{AM-GM}}{\geq}36\sqrt[12]{\frac{a}{c}\cdot\frac{c}{b}\cdot\frac{b}{a}}=36$$

$$6\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+18abc\geq 36$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc\geq 6$$

$$\text{Equality holds for: }\frac{1}{a^3}=\frac{1}{b^3}=\frac{1}{c^3}=abc\Leftrightarrow a=b=c=1$$

□

Solution 9 by Daniel Sitaru - Romania.

$$\frac{1}{a^3}+\frac{1}{b^3}+2abc\overset{\text{AM-GM}}{\geq}4\sqrt[4]{\frac{(abc)^2}{a^3b^3}}=4\sqrt[4]{\frac{c^2}{ab}}$$

$$\frac{1}{b^3}+\frac{1}{c^3}+2abc\overset{\text{AM-GM}}{\geq}4\sqrt[4]{\frac{(abc)^2}{b^3c^3}}=4\sqrt[4]{\frac{a^2}{bc}}$$

$$\frac{1}{c^3}+\frac{1}{a^3}+2abc\overset{\text{AM-GM}}{\geq}4\sqrt[4]{\frac{(abc)^2}{c^3a^3}}=4\sqrt[4]{\frac{b^2}{ca}}$$

By adding:

$$2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+6abc\geq 4\left(\sqrt[4]{\frac{c^2}{ab}}+\sqrt[4]{\frac{a^2}{bc}}+\sqrt[4]{\frac{b^2}{ca}}\right)\overset{\text{AM-GM}}{\geq}4\cdot 3\sqrt[4]{\frac{c^2}{ab}\cdot\frac{a^2}{bc}\cdot\frac{b^2}{ca}}=12$$

□

Solution 10 by Daniel Sitaru - Romania.

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=$$
$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3-3\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)+3abc\overset{\text{AM-GM}}{\geq}$$
$$\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3-3\cdot\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}+\frac{1}{a}}{3}\right)^3+3abc=$$
$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3-\frac{8}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3+3abc=\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3+3abc\geq$$
$$\overset{\text{AM-GM}}{\geq}\frac{1}{9}\cdot\left(3\sqrt[3]{\frac{1}{abc}}\right)^3+3abc=3\left(\frac{1}{abc}+abc\right)\overset{\text{AM-GM}}{\geq}3\cdot 2\sqrt{\frac{1}{abc}\cdot abc}=6$$
$$\text{Equality holds for: }\frac{1}{a}=\frac{1}{b}=\frac{1}{c}, abc=\frac{1}{abc}\Leftrightarrow a=b=c=1$$

□

Solution 11 by Daniel Sitaru - Romania.

$$f(a,b,c)=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc$$
$$\begin{cases} f'_a=0\\ f'_b=0\\ f'_c=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{a^4}=bc\\ \frac{1}{b^4}=ca\\ \frac{1}{c^4}=ab\end{cases}\Rightarrow(abc)^6=1\Rightarrow abc=1\Rightarrow bc=\frac{1}{a}\Rightarrow\frac{1}{a^4}=\frac{1}{a}$$
$$\Rightarrow a=1.\text{ Analogous: }b=1, c=1$$
$$H_f(1,1,1)=\begin{vmatrix}12&3&3\\3&12&3\\3&3&12\end{vmatrix}\text{ - positive - definite }\Rightarrow(1,1,1)\text{ - minimum point}$$
$$f(a,b,c)\geq f(1,1,1)=6\Rightarrow\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc\geq 6$$

□

Generalization by Daniel Sitaru - Romania

$$\text{If }a_i>0, i\in\overline{1,n}, n\in\mathbb{N}^*\text{ then:}$$
$$\sum_{i=1}^n\frac{1}{a_i^n}+n\prod_{i=1}^n a_i\geq 2n$$

Proof.

$$\sum_{i=1}^n\frac{1}{a_i^n}+n\prod_{i=1}^n a_i\geq\frac{n}{\prod_{i=1}^n a_i}+n\prod_{i=1}^n a_i=$$
$$=n\left(\frac{1}{\prod_{i=1}^n a_i}+\prod_{i=1}^n a_i\right)\overset{\text{AM-GM}}{\geq}n\cdot 2\sqrt{\frac{1}{\prod_{i=1}^n a_i}\cdot\prod_{i=1}^n a_i}=2n$$
$$\text{Equality holds for: }a_i>0, i\in\overline{1,n}, n\in\mathbb{N}^*$$

□

References

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