7 OUTSTANDING LIMITS

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Abstract. In this paper we present 7 new limits of sequences and functions.

Theorem 1.

$$\lim_{n\to\infty} \frac{\ln(1+\sqrt[n]{n!})}{\sqrt[n]{(2n-1)!!}} = 0$$

Proof.

$$\lim_{x\to\infty} \ln(1+x)^{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{x}\ln(1+x) \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0;$$

$$\lim_{n\to\infty} \frac{\ln(1+\sqrt[n]{n!})}{\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{(2n-1)!!}} \ln(1+\sqrt[n]{n!})^{\frac{1}{\sqrt[n]{n!}}} = 0\cdot \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{(2n-1)!}} =$$

$$= 0\cdot \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} = 0\cdot\frac{1}{e}\cdot\frac{e}{2} = 0$$

□

Theorem 2.

$$\text{If } a>0, \text{ then } \lim_{n\to\infty} (\sqrt[n]{a}-1)\sqrt[n]{n!} = \frac{\ln a}{e}$$

Proof.

$$(\sqrt[n]{a}-1)\sqrt[n]{n!} = \frac{\sqrt[n]{n!}}{n}\cdot n\cdot(\sqrt[n]{a}-1) = \frac{\sqrt[n]{n!}}{n}\cdot\frac{e^{\frac{1}{n}\cdot\ln a}-1}{\frac{1}{n}\ln a}\cdot\ln a, \forall n\ge 2.$$

$\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ and $\displaystyle\lim_{n\to\infty}\frac{e^{\frac{1}{b}\ln a}-1}{\frac{1}{n}\ln a} = 1$. So, $\displaystyle\lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{n!} = \frac{1}{e}\cdot 1\cdot\ln a = \frac{\ln a}{e}$.

□

Theorem 3.

If $a>0, (b_n)_{n\ge1}>0$ has positive real sequences such that $\displaystyle\lim_{n\to\infty}\frac{b_{n+1}}{nb_n} = b > 0$,

$$\text{then } \lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{b_n} = \frac{b\ln a}{e}.$$

Proof.

$$(\sqrt[n]{a}-1)\sqrt[n]{b_n} = \frac{\sqrt[n]{b_n}}{n}\cdot n\cdot(\sqrt[n]{a}-1) = \frac{\sqrt[n]{b_n}}{n}\cdot\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}\cdot\ln a, \forall n\ge 2.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} = \lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n} = \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{b}{e}$$

$$\lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}=1.$$

$$\lim_{n\to\infty}\left(\sqrt[n]{a}-1\right)\sqrt[n]{b_n}=\frac{b}{e}\cdot 1\cdot\ln a=\frac{b\ln a}{e}$$

□

Theorem 4.

$$\text{If } a>0,\text{ then } \lim_{n\to\infty}\left(\sqrt[n]{a}-1\right)\sqrt[n]{(2n-1)!!}=\frac{2\ln a}{e}.$$

Proof.

$$\left(\sqrt[n]{a}-1\right)\sqrt[n]{(2n-1)!!}=\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot n\cdot\left(\sqrt[n]{a}-1\right)=\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}\cdot\ln a,\forall n\geq 2.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^{n+1}=\frac{2}{e};\lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}=1$$

$$\lim_{n\to\infty}\left(\sqrt[n]{a}-1\right)\sqrt[n]{(2n-1)!!}=\frac{2}{e}\cdot 1\cdot\ln a=\frac{2\ln a}{e}$$

□

Theorem 5.

If $(x_n)_{n\geq 1}, x_n=\sum_{k=1}^{n}\frac{1}{k}$ and $a>0$, then $\lim_{n\to\infty}e^{2x_n}\left(\sqrt[n^4]{a}-1\right)\sqrt[n]{n!\cdot(2n-1)!!}=2e^{2(\gamma-1)}\ln a.$

Proof.

$$e^{2x_n}\left(\sqrt[n^4]{a}-1\right)\sqrt[n]{n!(2n-1)!!}=\frac{e^{2x_n}}{n^2}\cdot n^4\cdot\left(\sqrt[n^4]{a}-1\right)\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}$$

$$=\frac{e^{2x_n}}{e^{2\ln n}}\cdot n^4\cdot\left(e^{\frac{\ln a}{n^4}}-1\right)\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!n}}{}=e^{2(x_n-\ln n)}\cdot\frac{e^{\frac{\ln a}{n^4}}-1}{\frac{\ln a}{n^4}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\ln a$$

$\lim_{n\to\infty}e^{2(x_n-\ln n)}=e^{2\gamma}$, where γ is Euler Mascheroni constant; $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e};$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^{n+1}=\frac{2}{e};\lim_{n\to\infty}\frac{e^{\frac{1}{n^4}\ln a}-1}{\frac{1}{n^4}\ln a}=1.$$

Hence, $\lim_{n\to\infty}e^{2x_n}\left(\sqrt[n^4]{a}-1\right)\sqrt[n]{n!(2n-1)!!}=e^{2\gamma}\cdot 1\cdot\frac{1}{e}\cdot\frac{2}{e}\cdot\ln a=2e^{2(\gamma-1)}\ln a$

□

Theorem 6.

$$\text{If } a>0,\text{ then } \lim_{x\to\infty}\left(a^{\frac{1}{x}-1}\right)\left(\Gamma(x+1)\right)^{\frac{1}{x}}=\frac{\ln a}{e}$$

Proof.

$$f(x) = (a^{\frac{1}{x}} - 1)(\Gamma(x+1))^{\frac{1}{x}} = x(a^{\frac{1}{x}} - 1) \cdot \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = x(e^{\frac{\ln a}{x}} - 1) \cdot \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} =$$
$$= \frac{e^{\frac{\ln a}{x}} - 1}{\frac{\ln a}{x}} \cdot \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \cdot \ln a$$
$$\lim_{x\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{n\to\infty} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} =$$
$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$
$$\text{Hence, } \lim_{x\to\infty} (a^{\frac{1}{x}} - 1)(\Gamma(x+1))^{\frac{1}{x}} = \lim_{x\to\infty} f(x) = 1 \cdot \frac{1}{e} \cdot \ln a = \frac{\ln a}{e}$$

□

Theorem 7.
$$\lim_{x\to\infty} x \sin\frac{1}{x^2} (\Gamma(x+1))^{\frac{1}{x}} = \frac{1}{e}$$

Proof.

$$f(x) = x \sin\frac{1}{x^2} (\Gamma(x+1))^{\frac{1}{x}} = x^2 \sin\frac{1}{x^2} \cdot \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}; \lim_{x\to\infty} x^2 \sin\frac{1}{x^2} = 1;$$
$$\lim_{x\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{n\to\infty} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} =$$
$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$
$$\text{So, } \lim_{x\to\infty} x \sin\frac{1}{x^2} (\Gamma(x+1))^{\frac{1}{x}} = 1 \cdot \frac{1}{e} = \frac{1}{e}.$$

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