Refinements of the inequality of means , starting from Radó and Popoviciu's inequalities

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In this note are presented refinements of the inequality of means - obtained by using the monotony of sequences associated with this inequality . In this approach , he plays an essential role using Radó and Popoviciu's inequalities . Some consequences and applications are also presented .

2010 ***Mathematics Subject Classification*** **: 26D15**

If $a_1, a_2, \dots, a_n$ are strictly positive real numbers, they are well known in the literature and math practice the following means :

$$A_n[a] := \frac{1}{n} \cdot \sum_{k=1}^{n} a_k \quad (\textit{arithmetic mean}) , \quad G_n[a] := \sqrt[n]{\prod_{k=1}^{n} a_k} \quad (\textit{geometric mean}) ,$$

$$H_n[a] := \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$ (*harmonic mean*) , as well as the inequality between them :

$$A_n[a] \geqslant G_n[a] \geqslant H_n[a] \quad . \qquad (1)$$

In the following we will present refinements of the classical inequality of means using a method (initiated by us in [5] , [6]) mainly based on the monotony of a string associated with an inequality.

In a first instance we will consider the following string associated with the inequality between the *arithmetic mean* and the *geometric mean* ,

$$D_n[a] := n\left(A_n[a] - G_n[a]\right) . \qquad (2)$$

Regarding this sequence , we will first prove the following ,

1. *Lemma*

The sequence $\left(D_n[a]\right)_n$, defined in relation (2) is monotonically increasing .

Proof

To prove that $D_n[a] \geqslant D_{n-1}[a]$, it comes back to prove the following inequality,

$$n\left(A_n[a] - G_n[a]\right) \geqslant (n-1)\cdot\left(A_{n-1}[a] - G_{n-1}[a]\right) . \qquad (3)$$

But this is nothing than ***Radó's inequality***, (see for example in works [1] , [2] , [4] , [9] , [10]) .

However, for independence from other bibliographic sources, we present here a short demonstration (probably the new one) of it . Indeed - with a convenient arrangement and then with a simple application of of the (*weighted*) *inequality of the means* we have ,

$$(n-1)\cdot G_{n-1}[a] + n\cdot A_n[a] - (n-1)\cdot A_{n-1}[a] = (n-1)\cdot G_{n-1}[a] + a_n \geq$$
$$\geq n\cdot\sqrt[n]{G_{n-1}^{n-1}\cdot a_n} = n\cdot\sqrt[n]{(a_1 a_2 \dots a_{n-1})\cdot a_n} = n\cdot G_n \quad ,$$

equivalent to inequality (3) .

2. *Remark*

Successively using the recurrence (3) , we obtain :

$$n\,(\mathrm{A}_n[\,a\,]-\mathrm{G}_n[\,a\,]) \geqslant (n-1)(\mathrm{A}_{n-1}[\,a\,]-\mathrm{G}_{n-1}[\,a\,]) \geqslant$$

$$\geqslant \;.\;\;.\;\;.\; \geqslant 2(\mathrm{A}_2[\,a\,]-\mathrm{G}_2[\,a\,]) \geqslant \mathrm{A}_1[\,a\,]-\mathrm{G}_2[\,a\,]=0\,. \qquad (4)$$

The inequalities between the extremes constitute themselves the inequality of the means, therefore also taking into account the demonstration from above it follows that the *inequality of means* and *Radó's inequality* are equivalent inequalities .

At the same time, the entire series of inequalities in (4) itself represents a multiple refinement (with $n-2$ intermediates terms) of the inequality between the *arithmetic* and *geometric means* .

Another way of refining the inequality of means also results from the series of successive inequalities (4).

3. *Theorem* (*Refinement of the AM−GM inequality,* [7])

For $a_k > 0$, $k \in \{1,2,\dots,n\}$, $n \in \mathbb{N}_{\geq 2}$, occur the inequalities :

a)
$$\frac{1}{n}\cdot\sum_{k=1}^{n} a_k \;-\; \sqrt[n]{\prod_{k=1}^{n} a_k} \;\geq\; \frac{1}{n}\cdot \max_{1\leq i<j\leq n}\left(\sqrt{a_i}-\sqrt{a_j}\right)^2 \quad , \qquad (5)$$

b)
$$\frac{1}{n}\cdot\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k} \;\geq \frac{1}{n}\cdot \max_{1\leq i<j<k\leq n}\left(a_i+a_j+a_k-3\cdot\sqrt[3]{a_i\,a_j\,a_k}\right) \quad , \qquad (6)$$

with equality if and only if , $a_1 = a_2 = \ldots = a_n$.

Proof

a) As in the series of inequalities (4) we have $D_1 = 0$, significant remains only the inequality ,

$$D_{\mathrm{n}} \geqslant D_2 \quad\Leftrightarrow\quad n\,(\mathrm{A}_n[\,a\,]-\mathrm{G}_n[\,a\,]) \geqslant 2(\mathrm{A}_2[\,a\,]-\mathrm{G}_2[\,a\,])\,,\quad (\forall)\, n \in \mathbb{N}_{\geq 2}\,.$$

But , $$2\,(\mathrm{A}_2[a]-\mathrm{G}_2[a]) = 2\left(\frac{a_1+a_2}{2}-\sqrt{a_1\,a_2}\right) = \left(\sqrt{a_1}-\sqrt{a_2}\right)^2 \quad ,$$

therefore , $D_{\mathrm{n}} \geq \left(\sqrt{a_1}-\sqrt{a_2}\right)^2$, $(\forall)\, n \in \mathbb{N}_{\geq 2}$. Moreover, due to the symmetry of D_n in relation to the variables a_i , respectively a_j , $(\forall)\; i\,,j \in \{1\,,2\,,\dots,n\}$, it follows that ,

$$D_{\mathrm{n}} \geq \left(\sqrt{a_i}-\sqrt{a_j}\right)^2,\; (\forall)\; n \in \mathbb{N}_{\geq 2}\,,\; ,(\forall)\, i\,,j \in \{1\,,2\,,\dots,n\}\;, \qquad (7)$$

so we will have , $$D_{\mathrm{n}} \geq \max_{1\leq i<j\leq n}\left(\sqrt{a_i}-\sqrt{a_j}\right)^2 \quad , \qquad (8)$$

that is, happens the relationship from the statement. The condition of equality $a_1 = a_2 = \ldots = a_n$ is a necessary condition and sufficient both for equality in the relation (1) and for canceling the quantity $\max_{1\leq i<j\leq n}\left(\sqrt{a_i}-\sqrt{a_j}\right)^2$.

b) Similarly, using the inequality $D_{\mathrm{n}} \geqslant D_3$.

The following result provides a refinement of the *geometric mean* and *harmonic mean* inequality.

4. *Corollary* (*Refinement of the GM−HM inequality*)

If $a_k > 0$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}_{\geq 2}$, then occur the inequality ,

$$\frac{1}{\mathrm{H}_n[a]} - \frac{1}{\mathrm{G}_n[a]} \geq \frac{1}{n} \cdot \max_{1 \leq i < j \leq n} \frac{\left(\sqrt{a_i} - \sqrt{a_j}\right)^2}{a_i a_j} \quad , \tag{9}$$

Proof

In relation (5) , is made the substitution , $a_k \to 1/a_k$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}_{\geq 2}$.

5. *Remark*

In the refinement from Theorem 3 a), in the right-hand member practically intervenes only the pair (a_i, a_j) , namely the one that achieves the maximum of expression $\left(\sqrt{a_i} - \sqrt{a_j}\right)^2$.

Another refinement - it is true, weaker - but in which participate all the components, a_k, $k \in \{1, 2, \ldots, n\}$ $n \in \mathbb{N}^*$, is provided by the following

6. *Proposition* (*other refinement of the AM−GM inequality*)

If $x_k, y_k \in \mathbb{R}$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}_{\geq 2}$, atunci are loc inegalitatea ,

$$\frac{1}{n} \cdot \sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{1}{n \cdot C_n^2} \cdot \sum_{i,j=1}^{n} \left(\sqrt{a_i} - \sqrt{a_j}\right)^2 \quad , \text{ unde } C_n^2 = \binom{n}{2} \quad , \tag{10}$$

with equality if and only if , $a_1 = a_2 = \ldots = a_n$.

Proof

Using inequality (7) from the proof of the previous theorem, by summation after $i, j \in \{1, 2, \ldots, n\}$, we get : $C_n^2 \, D_n \geq \sum_{i,j=1}^{n} \left(\sqrt{a_i} - \sqrt{a_j}\right)^2$, from which the inequality in the statement immediately follows.

Also, for the weighted means , a study similar to the one described above can be undertaken, or - more economical - the following relatively recent and very nice result [3] can be used :

7. *Proposition*

For real numbers $a_1, a_2, \ldots, a_n \geqslant 0$ and $p_1, p_2, \ldots, p_n \geqslant 0$ with $\sum_{k=1}^{n} p_k = 1$, we have

$$\sum_{k=1}^{n} p_k a_k - \prod_{k=1}^{n} a_k^{p_k} \geq n\lambda \cdot \left(\frac{1}{n} \cdot \sum_{k=1}^{n} a_k - \prod_{k=1}^{n} a_k^{1/n} \right) \quad , \tag{11}$$

where $\lambda = \min\{p_1, p_2, \ldots, p_n\}$, with equality if and only if , $a_1 = a_2 = \ldots = a_n$.

With this result, the difference between the *weighted arithmetic* and *geometric means* , according to the difference of the same unweighted means . In the previous notation, the following inequality occurs ,

$$\sum_{k=1}^{n} p_k a_k - \prod_{k=1}^{n} a_k^{p_k} \geq \lambda \cdot D_n \tag{12}$$

Also , taking into account the relation (8) , the following refinement results ,

8. *Theorem* ***(refinement of weighted AM − GM inequality)***

For the real numbers $a_1, a_2, \ldots, a_n \geqslant 0$ and $p_1, p_2, \ldots, p_n \geqslant 0$ with $\sum_{k=1}^{n} p_k = 1$, we have :

$$\sum_{k=1}^{n} p_k a_k - \prod_{k=1}^{n} a_k^{p_k} \geq \min_{1 \leq k \leq n} p_k \cdot \max_{1 \leq i < j \leq n} \left(\sqrt{a_i} - \sqrt{a_j} \right)^2 \quad , \qquad \textbf{(13)}$$

with equality if and only if , $a_1 = a_2 = \ldots = a_n$.

For $p_1 = p_2 = \ldots = p_n = 1/n$, the relationship (5) is regained .

In the same way, the analogs of relations (9) and (10) can be obtained in the version with weighted means.

In a somewhat "dual" way to the string $D_n[a]$, introduced in relation (2), we will also consider the following sequence , associated with the inequality between the *arithmetic mean* and the *geometric mean*

$$C_n[a] := \left(\frac{\mathrm{A}_n[a]}{\mathrm{G}_n[a]} \right)^n . \qquad \textbf{(14)}$$

Also , for this string we will prove the following monotonicity property ,

9. *Lemma*

The sequence $(C_n[a])_n$, defined in relation (14) is monotonically increasing.

Proof

To demonstrate that $C_n[a] \geqslant C_{n-1}[a]$, it comes back to prove the following inequality ,

$$\left(\frac{\mathrm{A}_n[a]}{\mathrm{G}_n[a]} \right)^n \geq \left(\frac{\mathrm{A}_{n-1}[a]}{\mathrm{G}_{n-1}[a]} \right)^{n-1} \quad , \qquad \textbf{(15)}$$

which is nothing but ***Popoviciu's inequality***, (see for example in works [1] , [2] , [9] , [10]) .

We also present for this a short demonstration (probably the new one) of it. Really, with a simple applying the (*weighted) inequality of means* we have ,

$$\frac{\mathrm{A}_{n-1}^{n-1}[a]}{\mathrm{G}_{n-1}^{n-1}[a]} \cdot \mathrm{G}_n^n[a] = \mathrm{A}_{n-1}^{n-1}[a] \cdot a_n \leq \left(\frac{(n-1)\mathrm{A}_{n-1}[a] + a_n}{n} \right)^n = \left(\frac{a_1 + a_2 + \ldots + a_n}{n} \right)^n = \mathrm{A}_n^n[a] \quad ,$$

the inequality equivalent to the relationship (15) .

10. *Remark*

Successively using the recurrence relation (15), we obtain :

$$\left(\frac{\mathrm{A}_n[a]}{\mathrm{G}_n[a]} \right)^n \geq \left(\frac{\mathrm{A}_{n-1}[a]}{\mathrm{G}_{n-1}[a]} \right)^{n-1} \geq \ldots \geq \left(\frac{\mathrm{A}_2[a]}{\mathrm{G}_2[a]} \right)^2 \geq \left(\frac{\mathrm{A}_1[a]}{\mathrm{G}_1[a]} \right)^1 = 1 \quad . \qquad \textbf{(16)}$$

And here - the inequality between the extremes is even a demonstration of the inequality of means , and with the proof above shows that the *inequality of means* and *Popoviciu's inequality* are inequalities equivalents. The entire series of inequalities in (16) itself represents a *multiple refinement* (with $n-2$ terms intermediates) of the *inequality* between the *arithmetic* and *geometric means* .

11. *Theorem* **(*Refinement of the AM−GM inequality*)**

For $a_k > 0$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}_{\geq 2}$, , occur the inequalities :

$$a) \qquad \frac{1}{n}\cdot\sum_{k=1}^{n} a_k \Big/ \sqrt[n]{\prod_{k=1}^{n} a_k} \;\geq\; \max_{1\leq i<j\leq n}\left(\frac{a_i + a_j}{2\sqrt{a_i a_j}}\right)^{\frac{2}{n}} , \tag{17}$$

$$b) \qquad \frac{1}{n}\cdot\sum_{k=1}^{n} a_k \Big/ \sqrt[n]{\prod_{k=1}^{n} a_k} \;\geq\; \max_{1\leq i<j<k\leq n}\left(\frac{a_i + a_j + a_k}{3\cdot\sqrt[3]{a_i a_j a_k}}\right)^{\frac{3}{n}} , \tag{18}$$

with equality if and only if , $a_1 = a_2 = \ldots = a_n$.

Proof

From the series of inequalities (16) we extract the inequality ,

$$C_n \geqslant C_2 \;\Leftrightarrow\; \left(\frac{A_n[a]}{G_n[a]}\right)^n \geq \left(\frac{A_2[a]}{G_2[a]}\right)^2 = \left(\frac{a_1 + a_2}{2\sqrt{a_1 a_2}}\right)^2 , \; (\forall)\; n \in \mathbb{N}_{\geq 2} .$$

Moreover, due to the symmetry of C_n in relation to the variables a_i , respectively a_j , $(\forall)\, i, j \in \{1, 2, \cdots, n\}$, it follows that ,

$$C_n \geq \left(\frac{a_i + a_j}{2\sqrt{a_i a_j}}\right)^2 , \; (\forall)\; n \in \mathbb{N}_{\geq 2} ,\; (\forall)\, i, j \in \{1, 2, \ldots, n\} , \tag{19}$$

so we will have ,

$$C_n \geq \max_{1\leq i<j\leq n}\left(\frac{a_i + a_j}{2\sqrt{a_i a_j}}\right)^2 , \tag{20}$$

that is, precisely the relationship in the statement.

The condition of equality $a_1 = a_2 = \ldots = a_n$ is a necessary and sufficient condition for both equality in relation (1), as well as for equality with the quantity unit $\max\limits_{1\leq i<j\leq n}\left(\frac{a_i + a_j}{2\sqrt{a_i a_j}}\right)^2$.

The following result provides another refinement of the *geometric mean* and *harmonic mean inequality*.

12. *Corollary* **(*Refinement of the GM−HM inequality*)**

If $a_k > 0$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}_{\geq 2}$, ***then*** occur the inequality :

$$\frac{G_n[a]}{H_n[a]} \geq \max_{1\leq i<j\leq n}\left(\frac{a_i + a_j}{2\sqrt{a_i a_j}}\right)^{\frac{2}{n}} , \tag{21}$$

Proof

In relation (17) we perform the substitution $a_k \to 1/a_k$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}_{\geq 2}$, and it is obtained the inequality in the statement .

13. ***Remark***

In the refinement from Theorem 11 a), in the right-hand member practically only the pair (a_i, a_j) intervenes, namely the one that achieves the maximum expression $\frac{a_i + a_j}{2\sqrt{a_i a_j}}$. Another refinement -it is true, weaker - but in which participate all the components $a_k,\ k \in \{1,2,\ldots,n\},\ n \in \mathbb{N}^*$, is provided by the following ,

14. ***Proposition*** ***(other refinement of the AM−GM inequality)***

If $x_k, y_k \in \mathbb{R}$, $k \in \{1,2,\ldots,n\}$, $n \in \mathbb{N}_{\geq 2}$, ***then*** occur the inequality :

$$\frac{A_n[a]}{G_n[a]} \geq \prod_{i,j=1}^{n} \left(\frac{a_i + a_j}{2\sqrt{a_i a_j}} \right)^{\frac{2}{n C_n^2}} , \text{ where, } C_n^2 = \binom{n}{2} , \tag{22}$$

with equality if and only if , $a_1 = a_2 = \ldots = a_n$,

Proof

Using the inequality (17) from the proof of the previous theorem , by multiplying after indices $i, j \in \{1, 2, \ldots, n\}$, we get : $(C_n)^{C_n^2} \geq \prod_{i,j=1}^{n} \left(\frac{a_i + a_j}{2\sqrt{a_i a_j}} \right)^2$, whence it immediately follows the inequality in the statement.

15. ***Aplication***

We will consider - rather for explanatory purposes - the following simple inequality :

$$log_a b + log_b c + log_c a \geq 3 , \tag{23}$$

where a, b, c are simultaneously in the interval $(0, 1)$ or simultaneously in the interval $(1, \infty)$.

Inequality (23) results immediately with the *inequality of means* , because for $(a) = (a_1, a_2, a_3) =$ $= (log_a b, log_b c, log_c a)$, avem $A_n[a] = \frac{log_a b + log_b c + log_c a}{3}$, $G_n[a] = \sqrt[3]{log_a b \cdot log_b c \cdot log_c a} = 1$.

Applying the refinement from *Theorem* 3 *a*) , we get ,

$$log_a b + log_b c + log_c a \geq 3 + 3 \cdot max\left\{ \left(\sqrt{log_a b} - \sqrt{log_b c}\right)^2, \left(\sqrt{log_b c} - \sqrt{log_c a}\right)^2, \left(\sqrt{log_c a} - \sqrt{log_a b}\right)^2 \right\} . \tag{24}$$

Applying the refinement from *Theorem* 11 *a*) , we get ,

$$log_a b + log_b c + log_c a \geq 3 \cdot max\left\{ \left(\frac{log_a b + log_b c}{2 \cdot \sqrt{log_a b \cdot log_b c}} \right)^{\frac{2}{3}}, \left(\frac{log_b c + log_c a}{2 \cdot \sqrt{log_b c \cdot log_c a}} \right)^{\frac{2}{3}}, \left(\frac{log_c a + log_a b}{2 \cdot \sqrt{log_c a \cdot log_a b}} \right)^{\frac{2}{3}} \right\} . \tag{25}$$

Practically , any problem that uses the *inequality of means* in its solution can benefit from one or more refinements like those in this work .

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