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If $A \in M_n(\mathbb{C}), n \in \mathbb{N}, n \geq 2$ then:

$$rank\ (A^5) + rank\ (A) \geq rank\ (A^4) + rank\ (A^2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hikmat Mammadov-Azerbaijan, Solution 2 by Adrian Popa-Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

If $A \in M_n(\mathbb{C}) \to n \in \mathbb{N} \to n \geq 2$

Then $rank\ (A^5) + rank\ (A) \geq rank\ (A^4) + rank\ (A^2)$

Note: $R(A) \to$ range / column space of A and $N(A) \to$ null spacd / kernel of A

$$rank\ \{PQ\} = rank\ (Q) - \dim\{R(Q) \cap N(P)\}$$

Then: $rank\ \{XYZ\} = rank\ \{YZ\} - \dim\{R(YZ) \cap N(x)\}$ (1)

Also: $rank\ \{XY\} = rank\ \{Y\} - \dim\{R(Y) \cap N(X)\}$ (2)

If vector $\alpha \in R(YZ)$ there exists vector β such that $YZ\beta = \alpha$

$$\Rightarrow Y(Z\beta) = \alpha \Rightarrow \alpha \in R(Y) \Rightarrow R(YZ) \subseteq R(Y) \Rightarrow R(YZ) \cap N(X) \subseteq R(Y) \cap N(X)$$

$\dim\{R(YZ) \cap N(X)\} \leq \dim\{R(Y) \cap N(X)\} - \dim\{R(YZ) \cap N(X)\} \geq -\dim\{R(Y) \cap N(X)\}$ (3)

From (1) and (3) $\to rank\ \{YXZ\} \geq rank\ \{YZ\} - \dim\{R(Y) \cap N(X)\}$

(2) $\to rank\ \{XY\} = -rank\ (Y) + \dim\{R(Y) \cap N(X)\}$

Adding: $rank\,\{XYZ\} - rank\,\{XY\} \geq rank\,\{YZ\} - rank\,\{Y\}$

$$\Rightarrow rank\,\{XYZ\} \geq rank\,\{YZ\} + rank\,\{XY\} - rank\,\{Y\}$$

Let $\rightarrow Z = A^3, Y = A$ and $Z = A$

Then $\rightarrow rank\{A^5\} \geq rank\,\{A^4\} + rank\,\{A^2\} - rank\,\{Y\}$

Therefore $\Rightarrow rank\,\{A^5\} + rank\,\{A\} \geq rank\,\{A^4\} + rank\,\{A^2\}$

Solution 2 by Adrian Popa-Romania

Using Frobenius inequality:

If $A \in M_{n,k}(\mathbb{C}), B \in M_{k,p}(\mathbb{C})$ and $C \in M_{p,n}(\mathbb{C}) \Rightarrow$

$$\Rightarrow rank\,(AB) + rank\,(BC) \leq rank\,(B) + rank\,(ABC)$$

Taking $m = k = p = n$

$$A \rightarrow A^3$$

$$B \rightarrow A$$

$$C \rightarrow A$$

We have: $rank\,(A^3 \cdot A) + rank\,(A \cdot A) \leq rank\,(A) + rank\,(A^3 \cdot A \cdot A)$

$$\Rightarrow rank\,(A^4) + rank\,(A^2) \leq rank\,(A) + rank\,(A^5)$$