A PSEUDO-CEBYSHEV'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper it is proved a classical inequality and are given a few applications.

MAIN PROBLEM: If $0 < a_1 \leq a_2 \leq \ldots \leq a_n, n \in \mathbb{N}^*$ then:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_n}{a_1} \geq \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_1}{a_n} \tag{1}$$

Proof.

$$\text{For } n = 2 \Rightarrow \frac{a_1}{a_2} + \frac{a_2}{a_1} \geq \frac{a_2}{a_1} + \frac{a_1}{a_2}. \text{ True.}$$

For $n = 3$:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_1} \geq \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_1}{a_3} \tag{2}$$

$$a_1^2 a_3 + a_2^2 a_1 + a_3^2 a_2 \geq a_2^2 a_3 + a_3^2 a_2 + a_1^2 a_2$$

$$a_1^2(a_3 - a_2) - a_1(a_3^2 - a_2^2) + a_3 a_2(a_3 - a_2) \geq 0$$

$$(a_3 - a_2)(a_1^2 - a_1 a_3 - a_1 a_2 + a_3 a_2) \geq 0$$

$$(a_3 - a_2)(a_2 - a_1)(a_3 - a_1) \geq 0 \tag{3}$$

$$\text{By } 0 < a_1 \leq a_2 \leq a_3 \Rightarrow \begin{cases} a_3 - a_2 \geq 0 \\ a_2 - a_1 \geq 0 \\ a_3 - a_1 \geq 0 \end{cases} \Rightarrow (3)$$

We will use the mathematical induction to prove (1):

$$P(n) : \frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_n}{a_1} \geq \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_1}{a_n} \text{ suppose true}$$

$$P(n+1) : \frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_{n+1}}{a_1} \geq \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_1}{a_{n+1}}$$

$$\text{By } 0 < a_1 \leq a_n \leq a_{n+1} \overset{(2)}{\Rightarrow} \frac{a_1}{a_n} + \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} \geq \frac{a_n}{a_1} + \frac{a_{n+1}}{a_n} + \frac{a_1}{a_{n+1}} \tag{4}$$

By adding $(1), (4)$:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_n}{a_1} + \frac{a_1}{a_n} + \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} \geq$$

$$\geq \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_1}{a_n} + \frac{a_n}{a_1} + \frac{a_{n+1}}{a_n} + \frac{a_1}{a_{n+1}}$$

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_{n+1}}{a_1} \geq \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_1}{a_{n+1}}$$

$$P(n) \Rightarrow P(n+1)$$

□

Application 1: If $0 < x \leq 1$ then:

$$\frac{2x}{x^2+1} + \frac{x^2+1}{2} + \frac{1}{x} \geq \frac{x^2+1}{2x} + \frac{2}{x^2+1} + x$$

Proof.

$$0 < x \leq 1 \Rightarrow 0 < 2x \leq 2 \Rightarrow 0 < 2x < x^2+1 \leq 2$$

We take in $(2) : a_1 = 2x, a_2 = x^2+1, a_3 = 2$.

$$\frac{2x}{x^2+1} + \frac{x^2+1}{2} + \frac{2}{2x} \geq \frac{x^2+1}{2x} + \frac{2}{x^2+1} + \frac{2x}{2}$$

$$\frac{2x}{x^2+1} + \frac{x^2+1}{2} + \frac{1}{x} \geq \frac{x^2+1}{2x} + \frac{2}{x^2+1} + x$$

Equality holds for $x = 1$. □

Application 2: If $0 < a \leq b \leq c$ then in ΔABC holds:

$$\text{a. } \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \tag{5}$$

$$\text{b. } \frac{\sin A}{\sin B} + \frac{\sin B}{\sin C} + \frac{\sin C}{\sin A} \geq \frac{\sin B}{\sin A} + \frac{\sin C}{\sin B} + \frac{\sin A}{\sin C}$$

$$\text{c. } \frac{\mu(A)}{\mu(B)} + \frac{\mu(B)}{\mu(C)} + \frac{\mu(C)}{\mu(A)} \geq \frac{\mu(B)}{\mu(A)} + \frac{\mu(C)}{\mu(B)} + \frac{\mu(A)}{\mu(C)}$$

$$\text{d. } \frac{m_c}{m_b} + \frac{m_a}{m_c} + \frac{m_b}{m_a} \geq \frac{m_b}{m_c} + \frac{m_c}{m_a} + \frac{m_a}{m_b}$$

$$\text{e. } \frac{h_c}{h_b} + \frac{h_a}{h_c} + \frac{h_b}{h_a} \geq \frac{h_b}{h_c} + \frac{h_c}{h_a} + \frac{h_a}{h_b}$$

Proof.

a. We take in $(2) : a_1 = a, a_2 = b, a_3 = c$.

b. We use: $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$ in (5) and recall:

$$0 < \sin A \leq \sin B \leq \sin C$$

c. We take in $(2) : a_1 = \mu(A), a_2 = \mu(B), a_3 = \mu(C)$ and recall:

$$0 < \mu(A) \leq \mu(B) \leq \mu(C)$$

d. We take in $(2) : a_1 = m_c, a_2 = m_b, a_3 = m_a$ and recall:

$$0 < m_c \leq m_b \leq m_a$$

e. We take in $(2) : a_1 = h_c, a_2 = h_b, a_3 = h_a$ and recall:

$$0 < h_c \leq h_b \leq h_a$$

□

Application 3: If $0 < a \leq b \leq c, n \in \mathbb{N}, n \geq 2$ then in ΔABC holds:

$$\text{a. } \frac{a+b}{a+c} + \frac{a+c}{b+c} + \frac{b+c}{a+b} \geq \frac{a+c}{a+b} + \frac{b+c}{a+c} + \frac{a+b}{b+c} \tag{6}$$

$$\text{b. } \sqrt{\frac{a+b}{a+c}} + \sqrt{\frac{a+c}{b+c}} + \sqrt{\frac{b+c}{a+b}} \geq \sqrt{\frac{a+c}{a+b}} + \sqrt{\frac{b+c}{a+c}} + \sqrt{\frac{a+b}{b+c}}$$

$$\text{c. } \sqrt[n]{\frac{a+b}{a+c}} + \sqrt[n]{\frac{a+c}{b+c}} + \sqrt[n]{\frac{b+c}{a+b}} \geq \sqrt[n]{\frac{a+c}{a+b}} + \sqrt[n]{\frac{b+c}{a+c}} + \sqrt[n]{\frac{a+b}{b+c}}$$

$$\text{d. } \sum_{cyc} \frac{\sin A + \sin B}{\sin A + \sin C} \geq \sum_{cyc} \frac{\sin A + \sin C}{\sin A + \sin B}$$

Proof.

$$\text{a. } a \leq b \leq c \Rightarrow \begin{cases} a+c \leq b+c \\ a+b \leq c+b \\ b+a \leq c+a \end{cases} \Rightarrow a+b \leq a+c \leq b+c \tag{7}$$

b. By $(7): a+b \leq a+c \leq b+c \Rightarrow \sqrt{a+b} \leq \sqrt{a+c} \leq \sqrt{b+c}$
We take in $(2): a_1 = \sqrt{a+b}, a_2 = \sqrt{a+c}, a_3 = \sqrt{b+c}$.
c. By $(7): a+b \leq a+c \leq b+c \Rightarrow \sqrt[n]{a+b} \leq \sqrt[n]{a+c} \leq \sqrt[n]{b+c}$
We take in $(2): a_1 = \sqrt{a+b}, a_2 = \sqrt{a+c}, a_3 = \sqrt{b+c}$.
d. We replace in $(6): a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$ □

Application 4: If $0 < a \leq b \leq c, n \in \mathbb{N}, n \geq 2$ then in ΔABC holds:

$$\text{a. } \sqrt[n]{\frac{a}{b}} + \sqrt[n]{\frac{b}{c}} + \sqrt[n]{\frac{c}{a}} \geq \sqrt[n]{\frac{b}{a}} + \sqrt[n]{\frac{c}{b}} + \sqrt[n]{\frac{b}{c}} \tag{8}$$

b. $\sqrt[n]{\frac{\sin A}{\sin B}} + \sqrt[n]{\frac{\sin B}{\sin C}} + \sqrt[n]{\frac{\sin C}{\sin A}} \geq \sqrt[n]{\frac{\sin B}{\sin A}} + \sqrt[n]{\frac{\sin C}{\sin B}} + \sqrt[n]{\frac{\sin B}{\sin C}}$

Proof.
a. $a \leq b \leq c \Rightarrow \sqrt[n]{a} \leq \sqrt[n]{b} \leq \sqrt[n]{c}$
We take in $(2): a_1 = \sqrt[n]{a}, a_2 = \sqrt[n]{b}, a_3 = \sqrt[n]{c}$.
b. We take in $(8): a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$ and recall:

$$0 < \sin A \leq \sin B \leq \sin C \Rightarrow \sqrt[n]{\sin A} \leq \sqrt[n]{\sin B} \leq \sqrt[n]{C}$$

□

Application 5: If $0 < a \leq b \leq c, n \in \mathbb{B}, n \geq 2$ then in acute ΔABC holds:
a. $\frac{\cos C}{\cos B} + \frac{\cos A}{\cos C} + \frac{\cos B}{\cos A} \geq \frac{\cos B}{\cos C} + \frac{\cos C}{\cos A} + \frac{\cos A}{\cos b}$
b. $\sqrt[n]{\frac{\cos C}{\cos B}} + \sqrt[n]{\frac{\cos A}{\cos C}} + \sqrt[n]{\frac{\cos B}{\cos A}} \geq \sqrt[n]{\frac{\cos B}{\cos C}} + \sqrt[n]{\frac{\cos C}{\cos A}} + \sqrt[n]{\frac{\cos A}{\cos B}}$

Proof.
a. $a \leq b \leq c \Rightarrow \cos C \leq \cos B \leq \cos A$.
We take in $(2): a_1 = \cos C, a_2 = \cos B, a_3 = \cos A$.
b. $a \leq b \leq c \Rightarrow \sqrt[n]{\cos C} \leq \sqrt[n]{\cos B} \leq \sqrt[n]{\cos A}$
We take in $(2): a_1 = \sqrt[n]{\cos C}, a_2 = \sqrt[n]{\cos B}, a_3 = \sqrt[n]{\cos A}$.

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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